On the supplement of sets in functional systems

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Introduction

One of the main problems for functional systems (f.s.) [1] is that of completeness. It consists in indicating all subsets whose functions make a complete set of the functions of a given f.s. by means of f.s. operations. Such subsets are called complete. This problem is closely related to a supplementation problem, i.e. to the question of comparison of representable possibilities of two sets of functions under consideration. The problem is to find out when one of the sets is extended to a complete set "more easily" than the other, and when they behave identically in this sense. The paper consists of three sections. Section 1 deals with the problem on supplement for the systems $\mathcal{P}=(P, I)$ of a general type where P is a set and I is a closure operator determined on the subsets of the set P. In Section 2 the results of Section 1 are applied to the supplementation problem for finite f.s. Section 3 deals with an analysis of two-valued logics. For major notions see [1, 2].

§ 1. Supplementation problem for the system $\mathcal{P}$

Let us consider a pair (P, I) or $\mathcal{P}$ in brief. P is here a nonempty set, I is a closure operator determined on the set $\mathcal{B}(P)$ of all subsets of P i.e. I possesses the properties that $I(Q)\supseteq Q$, $I(I(Q))=I(Q)$ and $I(Q_1)\supseteq I(Q_2)$ if $Q_1\supseteq Q_2$ for all $Q, Q_1, Q_2\in\mathcal{B}(P)$. The set $I(Q)$ is called the closure of Q, the set Q is called closed if $I(Q)=Q$ and is called complete if $I(Q)=P$. The completeness problem for $\mathcal{P}$ consists in finding all complete sets. As mentioned above this problem is the main one for $\mathcal{P}$. It may be interpreted in a broader sense. Namely, to find out Q' for a given Q what supplements Q' make it a complete set. In the case when Q is empty we have a completeness problem. The treatment leads to the following question. Let Q_1 and Q_2 be given, we are to know which of them is "nearer" to being complete, or to be more precise, when with equal supplements Q' the completeness of $Q_1\cup Q'$ will follow from the completeness of $Q_2\cup Q'$. We shall denote this relation by $Q_1\square Q_2$. It is easy to see that it is equivalent to $I(Q_1)\square I(Q_2)$, therefore we can consider Q to be closed sets. Let us denote by $\mathcal{B}(\mathcal{P})$ the set of all closed subsets from P and consider the relation $\square$ on $\mathcal{B}(\mathcal{P})$. It is clear that this relation is reflexive and transitive, and as a relation of preorder it reduces to the equivalence relation $\approx$ on $\mathscr{B}(\mathscr{P})$ determined by both $Q_1 \square Q_2$ and $Q_2 \square Q_1$ and to the partial order relation $\prec$ on the factor set $\tilde{\mathscr{B}}(\mathscr{P})$ of the set $\mathscr{B}(\mathscr{P})$ with respect to this equivalence. The relation $\prec$ is determined as follows. Let $Q_1, Q_2 \in \mathscr{B}(\mathscr{P})$ and $\tilde{Q}_1, \tilde{Q}_2$ be the corresponding equivalence classes. Suppose $\tilde{Q}_1 \prec \tilde{Q}_2$ if $Q_1 \square Q_2$. Thus the study of the relation $\square$ is reduced to one of the relation $\approx$ on $\mathscr{B}(\mathscr{P})$ and $\prec$ on $\tilde{\mathscr{B}}(\mathscr{P})$. We shall call the description of the relation supplementation problem. Its solution enables us to determine which sets are "more complicated" and which are "simpler" by completing them in the same manner, and which sets have similar behaviour under these conditions. In considering the problem it is natural to use the properties of the inclusion lattice formed by $\mathscr{B}(\mathscr{P})$. Let us first recall some facts concerning the completeness problem. From [1] we know that its solution may be obtained by constructing a so-called criterial system. Namely, $\theta \subseteq \mathscr{B}(\mathscr{P})$ is a criterial system, if for any set $Q \subseteq P$ its completeness is equivalent to non-entry of Q as a subset in every set from θ. Criterial systems are known [1] to exist for $I(\emptyset) \neq P$ among them we may choose such a system which can be represented as $\theta_1(\mathscr{P}) \cup \theta_2(\mathscr{P})$ where $\theta_1(\mathscr{P})$ is the set of all precomplete classes in $\mathscr{P}$ and $\theta_2(\mathscr{P})$ is the set of all elements Q, $Q \neq P$, from $\mathscr{B}(\mathscr{P})$ such that no precomplete class has Q as a subset. Let us remind that $Q \in \mathscr{B}(\mathscr{P})$ is called a precomplete class if $I(Q) \neq P$ but $I(Q \cup \{a\}) = P$ holds for any $a \in P \setminus Q$. According to [1] we have in the general case that $\theta_1(\mathscr{P}) \neq \emptyset$ and $\theta_2(\mathscr{P}) = \emptyset$, $\theta_1(\mathscr{P}) \neq \emptyset$ and $\theta_2(\mathscr{P}) \neq \emptyset$, $\theta_1(\mathscr{P}) = \emptyset$ and $\theta_2(\mathscr{P}) \neq \emptyset$, $\theta_1(\mathscr{P}) = \emptyset$ and $\theta_2(\mathscr{P}) = \emptyset$. The last situation holds when $I(\emptyset) = P$. Further we shall assume that this condition is not fulfilled for $\mathscr{P}$ and the pair $\mathscr{P}$ for which the additional condition $\theta_2(\mathscr{P}) = \emptyset$ holds is correct.

Theorem 1. If $Q_1, Q_2 \in \mathscr{B}(\mathscr{P})$ then the relation $Q_1 \square Q_2$ holds if and only if either $Q_2 = P$ or if $Q_2 \neq P$ then $I(Q_1 \cup Q') \in \mathscr{B}(\mathscr{P}) \setminus \{P\}$ is valid for any $Q' \in \mathscr{B}(\mathscr{P}) \setminus \{P\}$ such that $Q' \supseteq Q_2$.

Proof. If $Q_2 = P$ or $Q_2 \neq P$ and the above conditions are fulfilled, then $Q_1 \square Q_2$ is obvious. Let $Q_2 \neq P$, $Q_1 \square Q_2$ and $Q_2 \subseteq Q'$ hold for any $Q' \in \mathscr{B}(\mathscr{P}) \setminus \{P\}$. Consider the set $I(Q_1 \cup Q')$. If $I(Q_1 \cup Q') \neq P$ then our statement is valid. If $I(Q_1 \cup Q') = P$ then by virtue of the relation $Q_1 \square Q_2$ there must be $I(Q_2 \cup Q') = P$ but $I(Q_2 \cup Q') = Q'$, $Q' \neq P$, what disproves the assumed equality $I(Q_1 \cup Q') = P$. The theorem is proved.

Corollary 1.1. Different precomplete classes from $\mathscr{B}(\mathscr{P})$ are not comparable with respect to $\square$.

The theorem, if symmetrically used, gives a criterion of equivalence of two sets. It also demonstrates an obvious sufficient condition of equivalence of two sets. Let $Q_1, Q_2 \in \mathscr{B}(\mathscr{P})$, $Q_1 \subseteq Q_2$ and for any $Q_3 \in \mathscr{B}(\mathscr{P})$ such that $Q_1 \subseteq Q_3$ if $Q_3 \not\subseteq Q_2$ then $Q_3 \supseteq Q_2$. In this case we shall write $Q_1 \boxplus Q_2$. It is clear that this relation will hold for $Q_1 \square Q_2$. The converse does not, generally speaking.

Proposition 1. If $Q_1, Q_2 \in \mathscr{B}(\mathscr{P})$ and $Q_1 \boxplus Q_2$, $Q_2 \neq P$ then $Q_1 \approx Q_2$.

Proof. Since by definition $Q_1 \subseteq Q_2$ then $Q_1 \square Q_2$. Now we prove that $Q_2 \square Q_1$. Let $Q' \in \mathscr{B}(\mathscr{P}) \setminus \{P\}$ and $Q' \supseteq Q_1$ then, because of $Q_1 \boxplus Q_2$ we have either $Q' \subseteq Q_2$ or $Q' \supseteq Q_2$. In the first case we have $I(Q_2 \cup Q') = Q_2$ in the second case we have

$I(Q_2 \cup Q')=Q'$ i.e. $I(Q_2 \cup Q') \neq P$. Hence by theorem 1 we arrive at $Q_2 \square Q_1$. Consequently, $Q_1 \approx Q_2$ the proposition is proved.

Moreover the class of equivalence with respect to $\approx$ is also characterized by the following obvious proposition.

Proposition 2. If $Q_1, Q_2 \in \mathscr{B}(\mathscr{P})$ and $Q_1 \approx Q_2$ then $I(Q_1 \cup Q_2) \approx Q_1$.

Theorem 1 also permits to describe the relation $\square$ in a different form using the notion of type of a set. Let $Q \in \mathscr{B}(\mathscr{P})$ and $\tau_{\mathscr{P}}(Q)$ be a set of all precomplete classes each of which contains Q as a subset; $\tau_{\mathscr{P}}(Q)$ will be called the type of the set Q. Obviously, $\tau_{\mathscr{P}}(\emptyset)=\theta_1(\mathscr{P})$ and $\tau_{\mathscr{P}}(P)=\emptyset$.
We have

Theorem 2. If $Q_1, Q_2 \in \mathscr{B}(\mathscr{P})$ then for $Q_1 \square Q_2$ we have $\tau_{\mathscr{P}}(Q_1) \supseteq \tau_{\mathscr{P}}(Q_2)$.

Proof. If $Q_2=P$ or $\tau_{\mathscr{P}}(Q_2)=\emptyset$ then the statement is valid. Now let $Q_2 \neq P$ and $\tau_{\mathscr{P}}(Q_2) \neq \emptyset$. Consider $\pi \in \tau_{\mathscr{P}}(Q_2)$. Since $Q_1 \square Q_2$ and $\pi \supseteq Q_2$ we have by theorem 1 that $I(Q_1 \cup \pi) \in \mathscr{B}(\mathscr{P}) \setminus \{P\}$. Hence by virtue of the precompleteness of π we have $Q_1 \subseteq \pi$ and thereby $\tau_{\mathscr{P}}(Q_1) \supseteq \tau_{\mathscr{P}}(Q_2)$. The theorem is proved.

Corollary 2.1. If Q_1, $Q_2 \in \mathscr{B}(\mathscr{P})$ and $Q_1 \approx Q_2$ then $\tau_{\mathscr{P}}(Q_1)=\tau_{\mathscr{P}}(Q_2)$.

Note that the statements reverse to theorem 2 as well as to corollary 2.1 are wrong, generally speaking. They may not hold even for $\mathscr{P}$ such that $\theta_1(\mathscr{P})=\emptyset$. However for correct $\mathscr{P}$ we have

Theorem 3. If Q_1, $Q_2 \in \mathscr{B}(\mathscr{P})$ and $\mathscr{P}$ is a correct system, then $Q_1 \square Q_2$ if and only if $\tau_{\mathscr{P}}(Q_1) \supseteq \tau_{\mathscr{P}}(Q_2)$.

Proof. The "only if" part follows from theorem 2. Now let $\tau_{\mathscr{P}}(Q_1) \supseteq \tau_{\mathscr{P}}(Q_2)$ hold. We shall prove that $Q_1 \square Q_2$. If $\tau_{\mathscr{P}}(Q_2)=\emptyset$ we have in view of $\mathscr{P}$ being correct that $Q_2=P$ and, therefore, $Q_1 \square Q_2$. Let $\tau_{\mathscr{P}}(Q_2) \neq \emptyset$ and suppose that the relation $Q_1 \square Q_2$ does not hold. By theorem 1 it means that for some $Q' \in \mathscr{B}(\mathscr{P}) \setminus \{P\}$ we have $Q' \supseteq Q_2$ and $I(Q_1 \cup Q')=P$. Consider $\tau_{\mathscr{P}}(Q')$. It is obvious that $\tau_{\mathscr{P}}(Q') \subseteq \subseteq \tau_{\mathscr{P}}(Q_2)$ and in view of $\mathscr{P}$ being correct we have $\tau_{\mathscr{P}}(Q') \neq \emptyset$. Let $\pi \in \tau_{\mathscr{P}}(Q')$. Since $\pi \supseteq Q'$ we get $I(Q_1 \cup \pi)=P$.
It follows that $\pi' \notin \tau_{\mathscr{P}}(Q')$ for any $\pi' \in \tau_{\mathscr{P}}(Q_1)$ what is contrary to the relations $\tau_{\mathscr{P}}(Q_1) \supseteq \tau_{\mathscr{P}}(Q')$ and $\tau_{\mathscr{P}}(Q') \neq \emptyset$. So, the assumption concerning the incorrectness of the relation $Q_1 \square Q_2$ is false. The theorem is proved.

Corollary 3.1. If Q_1, $Q_2 \in \mathscr{B}(\mathscr{P})$ and $\mathscr{P}$ is a correct system, then $Q_1 \approx Q_2$ if and only if $\tau_{\mathscr{P}}(Q_1)=\tau_{\mathscr{P}}(Q_2)$.

Theorem 3 and corollary 3.1 permit to describe the relation $\square$ when $\mathscr{P}$ is a correct system. If $\tau(Q)$ is known for $Q \in \mathscr{B}(\mathscr{P})$ then the class of all sets equivalent to the set Q consists of all Q' such that $\tau(Q')=\tau(Q)$ i.e. this class is uniquely determined by the value of τ. We denote it by K_τ. Then the relation $K_\tau \prec K_{\tau'}$ on $\tilde{\mathscr{B}}(\mathscr{P})$ is equivalent to $\tau' \subseteq \tau$.
Let $|A|$ be the cardinality of the set A. Consider $|\tilde{\mathscr{B}}(\mathscr{P})|$. It characterizes the variety of systems in the supplementation problem. By corollary 1.1 we prove the validity of the following statement.

Proposition 3. We have

$$|\theta_1(\mathcal{P})| \leqq |\tilde{\mathcal{B}}(\mathcal{P})| \leqq 2^{|P|}.$$

According to [1] $|\theta_1(\mathcal{P})|$ may take any value $\varkappa \leqq 2^{|P|}$ depending on P and I thus $|\tilde{\mathcal{B}}(\mathcal{P})|$ is majorized from below by the same values. In particular, the equiality $|\tilde{\mathcal{B}}(\mathcal{P})|=2^{|P|}$ is possible which implies extremely great variations of cardinality of the class $\tilde{\mathcal{B}}(\mathcal{P})$.

§ 2. Supplementation problem for functional systems

A functional system (f.s.) is such a system $\mathcal{P}=(P, I)$ in which the set P is a set of functions, and I is the closure operator given by the automation. If P consists of functions defined on the collections from the subsets of a natural series with values of the very functions taken from the natural series, then the f.s. is called a truth functional system (t.f.s.). If P consists of lexicographic functions, then the functional system is called a sequential functional system (s.f.s.). Typical examples of f.s. are many-valued logics (examples of t.f.s.) and algebras of automata (examples of s.f.s.). An important class of t.f.s. is formed by finite t.f.s; (f.t.f.s.). They are defined as follows. Let $E_k=\{0, 1, \dots, k-1\}$, $k>1$, $U=\{u_1, u_2, \dots\}$ be the alphabet of the variables u_m whose values are the elements from E_k, let P_k be a set of all functions $f(u_{i_1}, \dots, u_{i_n})$ with values from E_k, $M_k \subseteqq P_k$ and let I_Φ be a special closure operator called a finite automaton-given operator. I_Φ is specified by a collection Ω of finite-place operations ω given by the automaton over the elements from M_k which falls into two parts Ω_1 and Ω_2. The collection Ω_1 gives the closure operator $I_{\mathcal{P}P}$ corresponding to the closure of the subsets $M' \subseteqq M_k$ with respeet to taking all the superpositions of functions from M'. The collection Ω_2 is finite. The system $\mathcal{M}_k=(M_k, I_\Phi)$ is called a finite t.f.s. Consider the partial order $\prec$ on the factor set $\tilde{\mathcal{B}}(\mathcal{M}_k)$ in the form of an oriented graph. The elements of $\tilde{\mathcal{B}}(\mathcal{M}_k)$ will be points in space. Any two points a and b are connected with an oriented edge from a to b if $b \prec a$ and there is no point c distinct from a and b such that $b \prec c \prec a$. The graph obtained is denoted by $G(\mathcal{M}_k)$ and the number of its vertices is denoted by $|G(\mathcal{M}_k)|$. It is to the description of this graph that the supplementation problem is reduced for f.t.s.f. We introduce some notions to characterize f.t.f.s. $\mathcal{M}_k$. Let $M \subseteqq P_k$ and $M^{(n)}$ be a set of all functions from M which depend only on variables from the alphabet $U_n=\{u_1, u_2, \dots; u_n\}$, let $p_k^{(n)}$ be a number of elements in $P_k^{(n)}$. It is clear that $p_k^{(n)}=\sum_{i=1}^{n} C_n^i \cdot k^{k^i}$. Let $S(P_k^{(n)})$ be a set of functions from $P_k^{(n)}$ each of them is equal to u_i, $i=1, 2, \dots, n$ for some i. For the finite set $M' \subseteqq P_k$; we shall use $m(M')$ for the greatest index of the variable of the functions from M'. Let $\Omega_2=\{\omega_1, \omega_2, \dots, \omega_r\}$ hold in the f.t.f.s. $\mathcal{M}_k$. Let the value of $\omega_j(f_1; f_2, \dots, f_{s_j})$ be defined and

$$m_j = m(\{f_1, f_2, \dots, f_{s_j}, \omega_j(f_1, f_2, \dots, f_{s_j})\}), \quad j = 1, 2, \dots, r.$$

Since m_j depends only on ω_j then we can introduce the notation $|\omega_j|$ for m_j. Let $m(\Omega_2)=\max\{|\omega_1|, |\omega_2|, \dots, |\omega_r|\}$. If $\mathcal{M}_k$ is finitely generated and M_k^* is a set of finite $M' \subseteqq M_k$ such that $I_\Phi(M')=M_k$ then let $m_0=\inf_{M' \in M_k^*} m(M')$ and $s=\max(m_0, m(\Omega_2))$.

Let the nonempty set $M\subseteqq P_k^{(s)}\cap M_k$ be called R-set if $I_\Phi(M)\cap P_k^{(s)}=M$ and $M\neq \neq P_k^{(s)}\cap M_k$. We denote it by R. Let $\mathscr{R}=(R\cup S(P_k^{(s)}), R)$. We shall say that the function $f(x_1, x_2, \dots, x_n)$ from P_k retains $\mathscr{R}$ if $f(g_1, g_2, \dots, g_n)\in R$ holds for any collection of functions $g_1, g_2, \dots, g_n$ from $R\cup S(P_k^{(s)})$. The class of all functions from M_k with $\mathscr{R}$ will be denoted by $\cup(\mathscr{R})$. We will call the R-set R maximal unless there exists an R-set R' such that $\cup(R')\supseteqq\cup(R)$, $R\neq R'$. Let, for f.t.f.s. $\mathscr{M}_k$, $\mathbf{R}$ be the set of all maximal R-sets, and let $\mathbf{R}$ be a set of all pairs $\mathscr{R}$ for which $R\in\mathbf{R}$.

Theorem 4. If an f.t.f.s. $\mathscr{M}_k=(M_k, I_\Phi)$ is finitely generated, then the following statements are true:

1) $$|G(\mathscr{M}_k)|\leqq 2^{2^{p_k(s)}}$$

holds for the graph $G(\mathscr{M}_k)$;

2) the graph $G(\mathscr{M}_k)$ can be constructed effectively.

Proof. We start from some given finite set $M\subseteqq M_k$ such that $I_\Phi(M)=M_k$. According to [1] the finitely generated f.t.f.s. $\mathscr{M}_k$ is correct, s and $\mathbf{R}$ can be found by M effectively, $\cup(\mathbf{R})$ coincides with the set of all precomplete classes in $\mathscr{M}_k$ and $|\cup(\mathbf{R})|\leqq 2^{p_k^{(s)}}$. Let $\mathbf{R}=\{\mathscr{R}_1, \mathscr{R}_2, \dots, \mathscr{R}_l\}$ and $\cup_i=\cup(\mathscr{R}_i)$, $i=1, 2, \dots, l$. Consider the set Σ whose elements are expressions of the form $\mathfrak{A}=\mathop{\&}_{j=1}^{t}\cup_{i_j}$ where $i_{j'}<i_{j''}$ and $\cup_{i_{j'}}\neq\cup_{i_{j''}}$ for $j'<j''$, $t\geqq 0$. The formula of $\mathfrak{A}$ is interpreted as a set of functions which is equal to the intersection of the sets $\cup_{i_j}$ which form $\mathfrak{A}$. This set is denoted by $\hat{\mathfrak{A}}$. For $t=0$ we have an empty conjunction $\mathfrak{A}_0$ which by definition generates the whole set M_k. Let $\tau(\mathfrak{A})=\tau(\hat{\mathfrak{A}})$. We introduce a partial preorder relation on Σ putting $\mathfrak{A}\lesssim\mathfrak{B}$ if and only if $\tau(\mathfrak{A})\supseteqq\tau(\mathfrak{B})$, $\tau(\mathscr{M}_k)$ being an empty set by definition. It is obvious that this preorder coincides on Σ (by the above interpretation) with the relation $\Box$, and is reduced to the equivalence relation $\approx$ and the relation of partial order $\prec$ on the factor set $\tilde{\Sigma}$ of the set Σ with respect to this equivalence. Represent this partial order as a graph $\mathfrak{G}(\mathscr{M}_k)$ in the same way as it was done in constructing the graph $G(\mathscr{M}_k)$. Now establish a connection between these graphs. We see that for any $\tau\subseteqq\theta_1(\mathscr{M}_k)$ the following holds: if, in the graph $G(\mathscr{M}_k)$ and $\mathfrak{G}(\mathscr{M}_k)$ there exists a vertex which denotes a class of sets of a given type τ (a vertex of type τ), then this vertex is unique, and both graphs have such vertices simultaneously. Let us establish the correspondence between the vertices of the graphs $G(\mathscr{M}_k)$ and $\mathfrak{G}(\mathscr{M}_k)$ by the property of coincidence of their types τ. Since it is a one-to-one correspondence, then $l\leqq \leqq|G(\mathscr{M}_k)|\leqq 2^{2^{p_k(s)}}$ holds and thereby relation 1) is established. Now, if we extend the correspondence between the graphs to the correspondence between the edges connecting the corresponding vertices, we can see that these graphs are isomorphic, so, to establish property 2) it is suffices to show that the graph $\mathfrak{G}(\mathscr{M}_k)$ can effectively be constructed and the types of its vertices can effectively be determined. For this purpose we first establish that the relations: (1) $\hat{\mathfrak{A}}_1\subseteqq\hat{\mathfrak{A}}_2$ and (2) $\hat{\mathfrak{A}}_1^{(q)}\subseteqq\hat{\mathfrak{A}}_2^{(q)}$ are equivalent if $\mathfrak{A}_1$; $\mathfrak{A}_2\in\Sigma$ and $q=k^{k^s\cdot 2^{p_k^{(s)}}}$. This results from relation (1) being a consequence of (2). Now let us prove it. Suppose $M\subseteqq P_k$. Let $\overline{M}^{(n)}$ denote a set of all functions from $M^{(n)}$ depending exactly on all the variables from U_n. We construct a pair $\overline{\mathscr{R}}_i=(\overline{R}_i^{(s)}\cup\overline{S}(P_k^{(s)})^{(s)}, \overline{R}_i^{(s)})$ corresponding to $\mathscr{R}_i=(R_i\cup S(P_k^{(s)}), R_i)$ and

introduce a set $\cup(\bar{\mathscr{R}}_i)$ by analogy with the set $\cup(\mathscr{R}_i)$. We see that $\cup(\mathscr{R}_i)=\cup(\bar{\mathscr{R}}_i)$ holds. For a set $\bar{R}_i^{(s)}\cup\bar{S}(P_k^{(s)})^{(s)}$ we construct a matrix pair T_i as follows. Let all $h_1, h_2, \ldots, h_r$ be the functions from $\bar{R}_i^{(s)}$ and all $\lambda_1, \lambda_2, \ldots, \lambda_s$ be the functions from $\bar{S}(P_k^{(s)})^{(s)}$. Their choice can be represented by the summary table

$$\begin{array}{ccc|cccccc}
u_1 & u_2 \ldots & u_s & h_1 & h_2 \ldots & h_r & \lambda_1 & \lambda_2 \ldots & \lambda_s \\
\hline
a_{11} & a_{12} \ldots & a_{1s} & b_{11} & b_{12} \ldots & b_{1r} & c_{11} & c_{12} \ldots & c_{1s} \\
 & \ldots & & & \ldots & & & \ldots & \\
a_{v1} & a_{v2} \ldots & a_{vs} & b_{v1} & b_{v2} \ldots & b_{vr} & c_{v1} & c_{v2} \ldots & c_{vs} \\
 & \ldots & & & \ldots & & & \ldots & \\
a_{k^s1} & a_{k^s2} \ldots & a_{k^ss} & b_{k^s1} & b_{k^s2} \ldots & b_{k^sr} & c_{k^s1} & c_{k^s2} \ldots & c_{k^ss}
\end{array}$$

Let us single out two of its matrices

$$T_i' = \begin{pmatrix} b_{11} & b_{12} \ldots b_{1r} & c_{11} & c_{12} \ldots c_{1s} \\ & \ldots & & \ldots \\ b_{k^s1} & b_{k^s2} \ldots b_{k^sr} & c_{k^s1} & c_{k^s2} \ldots c_{k^ss} \end{pmatrix},$$

$$T_i'' = \begin{pmatrix} b_{11} & b_{12} \ldots b_{kr} \\ & \ldots \\ b_{k^s1} & b_{k^s2} \ldots b_{k^sr} \end{pmatrix}$$

and consider the pair $T_i=(T_i', T_i'')$. We shall say that $f(x_1, x_2, \ldots, x_m)$ from P_k retains T_i if for any matrix

$$T = \begin{pmatrix} d_{11} & d_{12} \ldots d_{1m} \\ & \ldots \\ d_{k^s1} & d_{k^s2} \ldots d_{k^sm} \end{pmatrix}$$

whose columns are all taken from the matrix T_i' the column

$$f(T) = \begin{pmatrix} f(d_{11}, d_{12}, \ldots, d_{1m}) \\ \ldots \\ f(d_{k^s1}, d_{k^s2}, \ldots, d_{k^sm}) \end{pmatrix}$$

will be the column of the matrix T_i''. Let $\cup(T_i)$ be the set of all functions from M_k retaining T. We see that $\cup(T_i)=\cup(\bar{\mathscr{R}}_i)$. Now we introduce an operation with respect to the matrices A and B. $C=A\cdot B$, if C consists exactly of all such columns γ which result from placing the column α of A on top of the column β of B.

We denote by A^r the result of multiplication of A by itself r times. For the expression $\mathfrak{A}=\mathop{\&}\limits_{j=1}^{t} \cup_{i_j}$, $t>0$ we shall construct a pair $T(\mathfrak{A})=(T('\mathfrak{A}), T''(\mathfrak{A}))$ where

$$T'(\mathfrak{A}) = T_{i_1}'^{r_1}\cdot T_{i_2}'^{r_2}\cdot \ldots \cdot T_{i_t}^{r_t},$$

$$T''(\mathfrak{A}) = T_{i_1}''^{r_1}\cdot T_{i_2}''^{r_2}\cdot \ldots \cdot T_{i_t}''^{r_t}, \quad \sum_{w=1}^{t} r_w = 2^{p_k(s)}$$

and $r_w > 0$ for all w. By introducing a set $\cup(T(\mathfrak{A}))$ by analogy with the set $\cup(T_i)$ we can see that $\cup(T(\mathfrak{A})) = \bigcap_{j=1}^{t} \cup(T_{i_j})$ and hence $\cup(T(\mathfrak{A})) = \bar{\mathfrak{A}}$.

As established in [1] the relation $\cup_i^{(q)} \neq M_k^{(q)}$ holds. Therefore in the course of proving relation (1) to be a consequence of (2) we may assume that $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are distinct from $\mathfrak{A}_0$.

Now suppose that relation (2) holds whereas relation (1) does not. It means that in $\hat{\mathfrak{A}}_1$ there is a function f such that $f \notin \hat{\mathfrak{A}}_2$. It is clear that it must depend on v variables and $v > q$. Notice that by construction the matrices $T'(\mathfrak{A})$ and $T''(\mathfrak{A})$ have columns of the same length equal to $k^s \cdot 2^{p_k(s)}$ for any $\mathfrak{A}$ from Σ, $\mathfrak{A} \neq \mathfrak{A}_0$. Thus, these matrices have not more than q different columns. By assumption f does not retain $T(\mathfrak{A}_2)$. This means that there is a matrix T consisting of the columns of the matrix $T'(\mathfrak{A}_2)$ such that $f(T) \notin T''(\mathfrak{A}_2)$. The matrix T has not more than q different columns, so, we may assume without loss of generality that it is formed by successive groups of equal columns. In accordance with these groups we divide the variables of f into the same groups and in f replace every variable of the j-th group by the variable x_j, $1 \leq j \leq q$. As a result we get f' from $\hat{\mathfrak{A}}_1^{(q)}$ not retaining $T(\mathfrak{A}_2)$ either, what is at variance with relation (2). Thus, relation (1) is a consequence of relation (2).

Let now $\mathfrak{A} \in \Sigma \setminus \{\mathfrak{A}_0\}$. Construct $T(\mathfrak{A})$. According to [1] we can effectively construct the set $M_k^{(q)}$ and, consequently, the set $\cup(T(\mathfrak{A}))^{(q)} = (\hat{\mathfrak{A}})^{(q)}$. Since (1) is a consequence of (2) we can effectively define all precomplete classes $\cup_i$ such that $\cup_i \supseteq \hat{\mathfrak{A}}^{(q)}$ and thereby estimate $\tau(\mathfrak{A})$. Knowing these values and the value of $\tau(\mathfrak{A}_0)$ we can construct the graph $\mathfrak{G}(\mathcal{M}_k)$ and, consequently $G(\mathcal{M}_k)$. The theorem is proved.

It is known from [1] that f.t.f.s. $\mathcal{P}_k = (P_k, I_{sp})$ is finitely generated and for the number $|\theta_1(\mathcal{P}_k)|$ of precomplete classes in it we have

$$|\theta_1(\mathcal{P}_k)| \sim \delta(k) \cdot k \cdot 2^{C_{k-1}^{[k-1/2]}} \quad \text{for} \quad k \to \infty$$

where $\delta(k) = 2$ if k is even, and $\delta(k) = 1$ if k is odd. By this we arrive at

Corollary 4.1. The graph $G(\mathcal{P}_k)$ can effectively be constructed and

$$\delta(k) \cdot k \cdot 2^{C_{k-1}^{[k-1/2]}} \lesssim |G(\mathcal{P}_k)| \lesssim 2^{\delta(k) \cdot k \cdot 2^{C_{k-1}^{[k-1/2]}}} \quad \text{for} \quad k \to \infty.$$

§ 3. Supplementation problem for Post classes

Let us consider the supplementation problem for the f.t.f.s. $\mathcal{M}_2 = (M_2, I_{sp})$ where $M_2 \subseteq P_2$. E. Post is known to have described all the closed classes M_2 [2]. He established that the set of these classes is countable and that they are finite-generated. He constructed an inclusion lattice formed by these classes. The set of the classes in question is reduced to the following: C_i, A_i, D_j, $L_{k'}$, O_l, S_r, P'_r, F_s^n, F_s^∞ where $i = 1, 2, 3, 4$, $j = 1, 2, 3$, $k' = 1, 2, 3, 4, 5$, $l = 1, 2, \ldots, 9$, $r = 1, 3, 5, 6$, $s = 1, 2, \ldots, 8$, $n = 2, 3, \ldots$.

The class C_1 contains all the functions of the algebra of logic and coincides with P_2. C_2 consists of all $f(x_1, x_2, \ldots, x_n)$ such that C_3 consists of all $f(x_1, x_2, \ldots, x_n)$ such that $C_4 = C_2 \cap C_3$. The class A_1 comprises all monotone functions; $A_2 = C_2 \cap A_1$; $A_3 = C_3 \cap A_1$; $A_4 = A_2 \cap A_3$. The class D_3 consists of all functions

$f(x_1, x_2, \ldots, x_n)$ such that $f(x_1, x_2, \ldots, x_n)=\bar{f}(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$. The function $f^*(x_1, x_2, \ldots, x_n)=\bar{f}(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$ being called dual with respect to f and the set $\mathfrak{M}^*$ consisting of all functions dual with respect to the functions of $\mathfrak{M}$ is called dual with respect to $\mathfrak{M}$; the class $\mathfrak{M}$ is called self-dual if $\mathfrak{M}=\mathfrak{M}^*$; $D_1=C_4\cap D_3$; $D_2=A_1\cap D_3$.

The class L_1 consists of all functions $f(x_1, x_2, \ldots, x_n)$ such that $f(x_1, x_2, \ldots, x_n)=\sum_{i=1}^{n} x_i+d \pmod 2$; $L_2=C_2\cap L_1$; $L_3=C_3\cap L_1$; $L_4=L_2\cap L_3$; $L_5=D_3\cap L_1$. O_9 comprises all the functions essentially depending on not more than one variable; $O_8=A_1\cap O_9$; $O_4=D_3\cap O_9$; $O_5=C_2\cap O_9$; $O_6=C_3\cap O_9$; $O_1=O_5\cap O_6$; O_7 consists of all constant functions; $O_2=O_5\cap O_7$; $O_3=O_6\cap O_7$. The class S_6 consists of all functions $f(x_1, x_2, \ldots, x_n)=x_1\vee x_2\vee\ldots\vee x_n$ and all constants; $S_3=C_2\cap S_6$; $S_5=C_3\cap S_6$; $S_1=S_3\cap S_5$. The class P_6' consists of all functions $f(x_1, x_2, \ldots, x_n)=x_1\& x_2\&\ldots\& x_n$ and all constants; $P_5'=C_2\cap P_6'$; $P_3'=C_3\cap P_6'$; $P_1'=P_5'\cap P_3'$. A function is said to satisfy the condition a^n, $n\geqq 2$ if any n collections in which it is equal to 0 have a common coordinate equal to 0. Analogously, with the replacement of 0 by 1 we introduce the condition A^n. The class F_4^n consists of all functions with property a^n; $F_1^n=C_4\cap F_4^n$; $F_3^n=A_1\cap F_4^n$; $F_2^n=F_1^n\cap F_3^n$; F_8^n consists of all functions with property A^n; $F_5^n=C_4\cap F_8^n$; $F_7^n=A_3\cap F_8^n$; $F_6^n=F_5^n\cap F_7^n$. A function is said to satisfy the condition a^∞ if all the collections in which it is equal to zero have a common coordinate equal to zero. Again by replacing 0 by 1 we introduce the property A^∞. The class F_4^∞ consists of all functions with property a^∞; $F_1^\infty=C_4\cap F_4^\infty$; $F_3^\infty=A_1\cap F_4^\infty$; $F_2^\infty=F_1^\infty\cap F_3^\infty$; F_8^∞ consists of all functions with property A^∞; $F_5^\infty=C_4\cap F_8^\infty$; $F_7^\infty=A_3\cap F_8^\infty$; $F_6^\infty=F_5^\infty\cap F_7^\infty$. The above inclusion lattice formed by these classes is given in Fig. 1. In this figure classes are represented by points. Two points are connected by an arc if the underlying point denotes a class contained immediately in the top class (i.e. there are no intermediate classes between them). The lattice has an axis of symmetry. Self-dual classes are represented by points on the axis; classes dual with respect to each other are represented symmetrically with respect to the axis. The self-dual classes are $C_1, C_4, A_1, A_4, D_1, D_2, D_3, L_1, L_4, L_5, O_1, O_4, O_7, O_8, O_9$. For other classes we have

$$C_2 = C_3^*,\quad A_2 = A_3^*,\quad L_2 = L_3^*,\quad P_1' = S_1^*,\quad P_3' = S_3^*,\quad P_5' = S_5^*,$$
$$P_6' = S_6^*,\quad O_5 = O_6^*,\quad O_2 = O_3^*,\quad F_1^n = (F_5^n)^*,\quad F_2^n = (F_6^n)^*,$$
$$F_3^n = (F_7^n)^*,\quad F_4^n = (F_8^n)^*,\quad F_1^\infty = (F_5^\infty)^*,\quad F_2^\infty = (F_6^\infty)^*,$$
$$F_3^\infty = (F_7^\infty)^*,\quad F_4^\infty = (F_8^\infty)^*.$$

Thus, the supplementation problem for any f.t.f.s. $\mathcal{M}_2$ is reduced to considering such f.t.f.s. $\mathcal{M}_2$ for which M_2 coincides with one of the classes of the set

$$Z = \{C_1, C_3, C_4, A_1, A_3, A_4, D_1, D_2, D_3, L_1, L_3, L_4, L_5,$$
$$F_5^n, F_6^n, F_7^n, F_8^n, F_5^\infty, F_6^\infty, F_7^\infty, F_8^\infty, P_1', P_3', P_5', P_6', O_1, O_3, O_4, O_6, O_7, O_8, O_9\}.$$

By virtue of theorem 3 and corollary 3.1, the solution of the supplementation problem, under condition that $M_2\in Z$, is as follows. By the aid of a Post lattice we calculate $\tau_{\mathcal{M}}(M')$ for every closed class $M'\subseteqq M_2$. The set of all classes of the same type of τ is declared the class K_τ. On the set of these classes we introduce the relation of

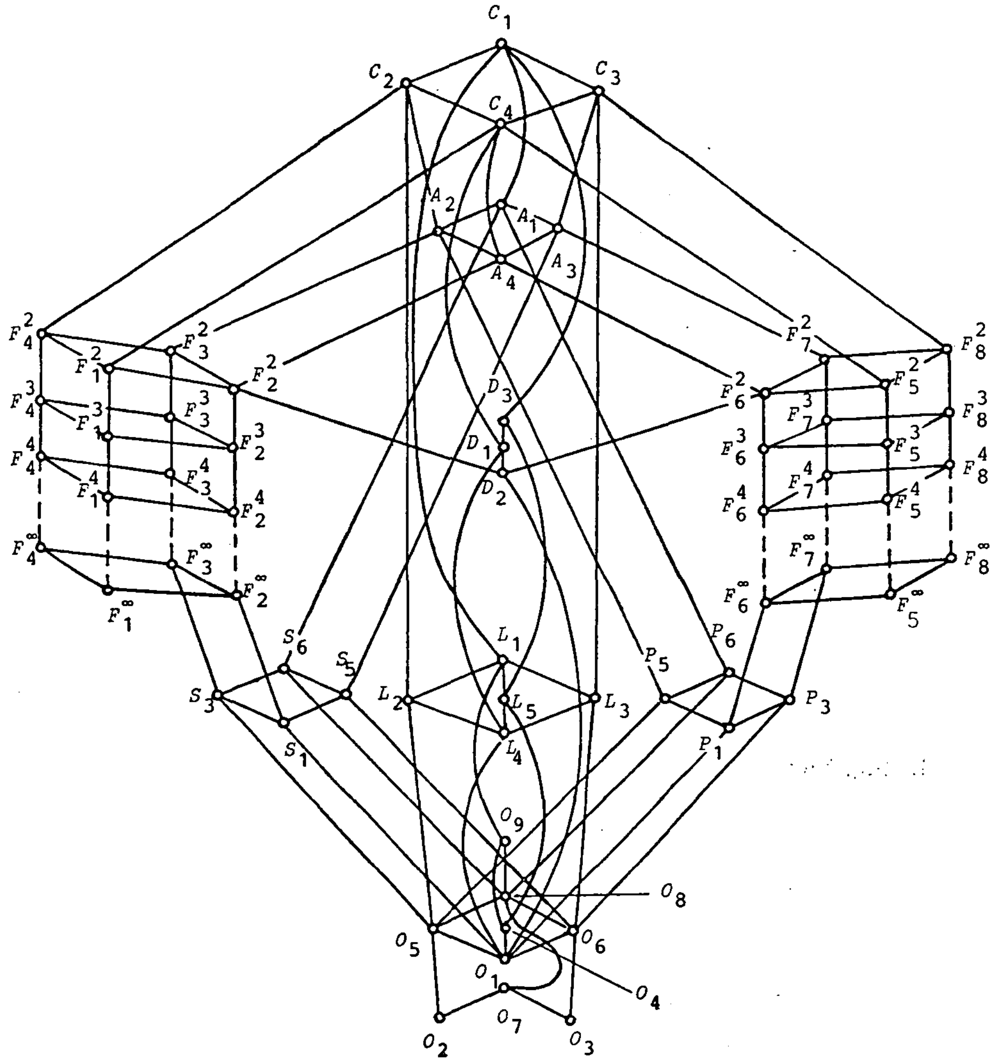

Fig. 1

partial order which coincides with the relation of inclusion for types of classes. As already mentioned the attributing of closed classes to a single class K_τ corresponds to $\approx$. In this way we construct the relation $\square$ over $\mathscr{B}(\mathscr{M}_2)$. Now we give the Post lattice and the results of calculations for the graph $G(\mathscr{M}_2)$. There turned out to be eleven graphs of this kind accurate to isomorphism. They are given in Fig. 2—8 with the edges oriented from top to bottom. Now let us describe the values of the parameters K_i for different graphs and classes M_2.

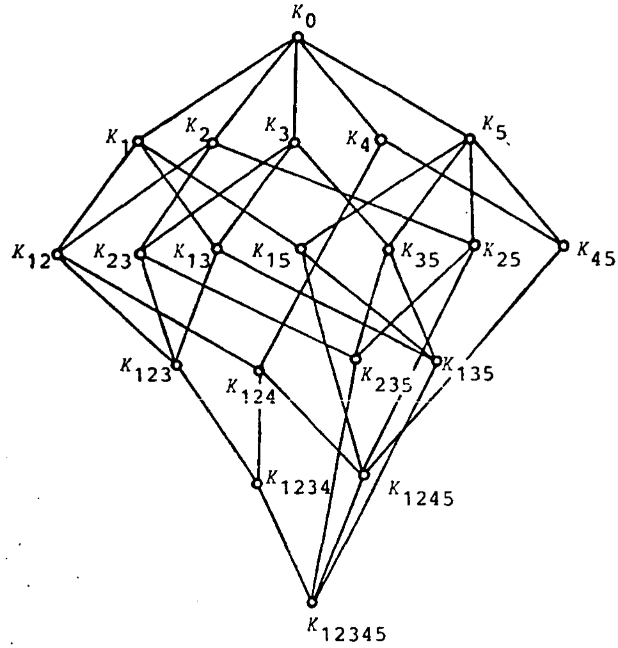

Fig. 2

In Fig. 2 we have the graph $G(C_1)$. Here

$$K_0 = \{C_1\}, \quad K_1 = \{C_2, F_4^2, F_4^3, \ldots, F_4^n, \ldots, F_4^\infty\},$$

$$K_2 = \{C_3, F_8^2, F_8^3, \ldots, F_8^n, \ldots, F_8^\infty\}, \quad K_3 = \{A_1, P_6', S_6'\},$$

$$K_4 = \{D_3\}, \quad K_5 = \{L_1, O_9\},$$

$$K_{12} = \{C_4, F_5^2, F_5^3, \ldots, F_5^n, \ldots, F_5^\infty, F_1^2, F_1^3, \ldots, F_1^n, \ldots, F_1^\infty\},$$

$$K_{23} = \{A_3, F_7^2, F_7^3, \ldots, F_7^n, \ldots, F_7^\infty, S_5, P_3'\},$$

$$K_{13} = \{A_2, F_3^2, \ldots, F_3^\infty, F_5', S_3\}, \quad K_{15} = \{L_2\},$$

$$K_{35} = \{O_8, O_7\}, \quad K_{25} = \{L_3\}, \quad K_{45} = \{L_5, O_4\},$$

$$K_{123} = \{A_4, F_6^2, F_6^3, \ldots, F_6^\infty, F_2^2, F_2^3, \ldots, F_2^\infty; P_1', S_1\},$$

$$K_{124} = \{D_1\}, \quad K_{135} = \{O_5, O_2\}, \quad K_{235} = \{O_6, O_3\},$$

$$K_{1234} = \{D_2\}, \quad K_{1245} = \{L_4\}, \quad K_{12345} = \{O_1\}.$$

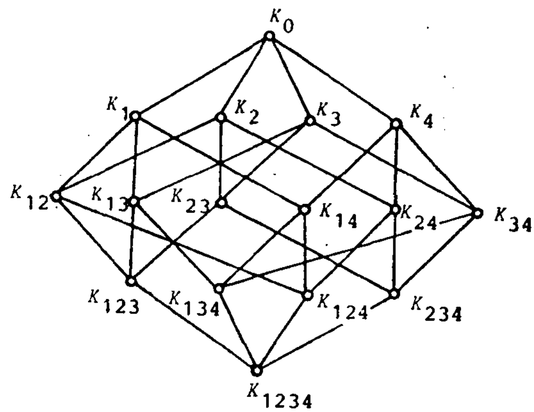

Fig. 3

In Fig. 3 we have the graph $G(\mathscr{A}_1)$. Here

$$K_0 = \{A_1\}, \quad K_1 = \{A_2, F_3^2, F_3^3, \ldots, F_3^\infty\},$$
$$K_2 = \{A_3, F_7^2, F_7^3, \ldots, F_7^\infty\}, \quad K_3 = \{S_6\}, \quad K_4 = \{P_6'\},$$
$$K_{12} = \{A_4, F_6^2, F_6^3, \ldots, F_6^\infty, F_2^2, F_2^3, \ldots, F_2^\infty, D_2\},$$
$$K_{13} = \{S_3\}, \quad K_{23} = \{S_5\}, \quad K_{14} = \{P_5'\},$$
$$K_{24} = \{P_3\}, \quad K_{34} = \{O_8, O_7\}, \quad K_{123} = \{S_1\},$$
$$K_{134} = \{P_1'\}, \quad K_{124} = \{O_5, O_2\}, \quad K_{234} = \{O_6, O_3\},$$
$$K_{1234} = \{O_1\}.$$

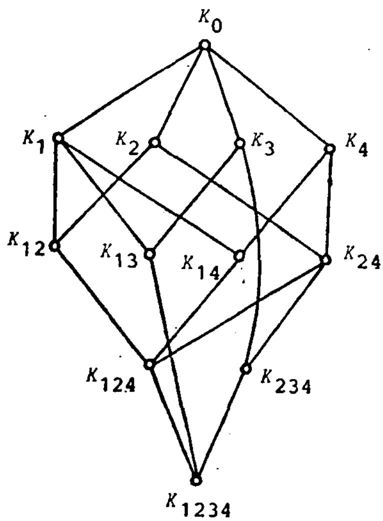

Fig. 4

In Fig. 4 we have the graph $G(C_3)$. Here

$$K_0 = \{C_3\}, \quad K_1 = \{C_4, F_1^2, F_1^3, \ldots, F_1^\infty, D_1\},$$
$$K_2 = \{A_3, S_5\}, \quad K_3 = \{L_3\}, \quad K_4 = \{F_8^2, F_8^3, \ldots, F_8^\infty\},$$
$$K_{12} = \{A_4, F_2^2, F_2^3, \ldots, F_2^\infty, S_1\} \quad K_{13} = \{L_4\},$$
$$K_{14} = \{F_5^2, F_5^3, \ldots, F_5^\infty\}, \quad K_{24} = \{F_7^2, F_7^3, \ldots, F_7^\infty, P_3'\},$$
$$K_{124} = \{F_6^2, F_6^3, \ldots, F_6^\infty, P_1', D_2\}, \quad K_{234} = \{O_6, O_3\}, \quad K_{1234} = \{O_1\}.$$

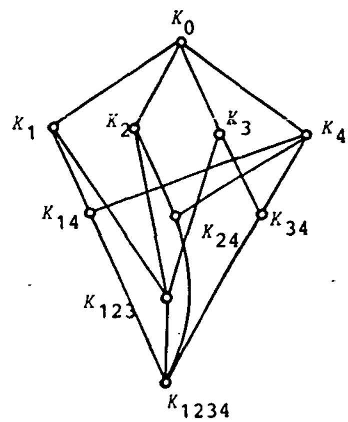

Fig. 5

In Fig. 5 we have the graph $G(\mathscr{L}_1)$. Here

$$K_0 = \{L_1\}, \quad K_1 = \{L_2\}, \quad K_2 = \{L_3\}, \quad K_3 = \{L_5\}, \quad K_4 = \{O_9, O_8, O_7\},$$
$$K_{14} = \{O_5, O_2\}, \quad K_{24} = \{O_6, O_3\}, \quad K_{34} = \{O_4\}, \quad K_{123} = \{L_4\}, \quad K_{1234} = \{O_1\}.$$

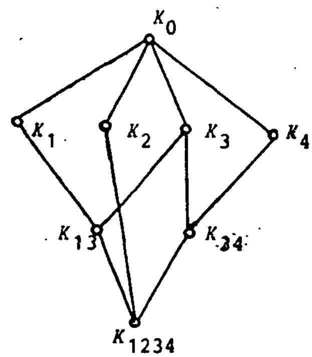

Fig. 6

In Fig. 6 we have the graph $G(C_4)$. Here

$$K_0 = \{C_4\},\quad K_1 = \{F_1^2, F_1^3, \ldots, F_1^\infty\},\quad K_2 = \{D_1, L_4\},\quad K_3 = \{A_4\},$$
$$K_4 = \{F_5^2, F_5^3, \ldots, F_5^\infty\},\quad K_{13} = \{F_2^2, F_2^3, \ldots, F_2^\infty, S_1\},$$
$$K_{34} = \{F_6^2, F_6^3, \ldots, F_6^\infty, P_1'\},\quad K_{1234} = \{D_2, O_1\}.$$

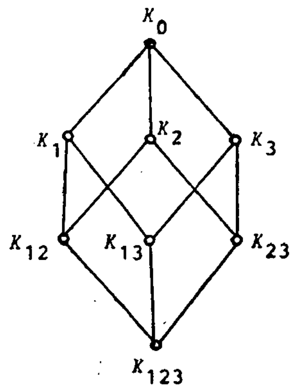

Fig. 7

In Fig. 7 we have the graph $G(\mathscr{F}_8^n)$. Here

$$K_0 = \{F_8^n\},\quad K_1 = \{F_8^{n+1}, F_8^{n+2}, \ldots, F_8^\infty\},\quad K_2 = \{F_5^n\},\quad K_3 = \{F_7^n\},$$
$$K_{12} = \{F_5^{n+1}, F_5^{n+2}, \ldots, F_5^\infty\},\quad K_{13} = \{F_7^{n+1}, F_7^{n+2}, \ldots, F_7^\infty, P_3', O_6, O_3\},$$
$$K_{23} = \{F_6^n\}\quad (n > 2),\quad K_{23} = \{F_6^2, D_2\}\quad (n = 2),$$
$$K_{123} = \{F_6^{n+1}, F_6^{n+2}, \ldots, F_6^\infty, P_1', O_1\}.$$

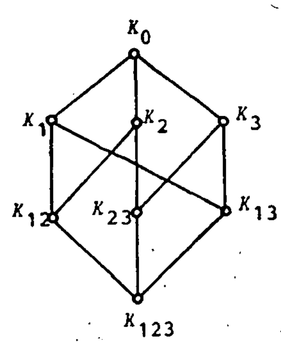

Fig. 8

In Fig. 8 we have the graph $G(\mathscr{M}_2)$ where

$$M_2 \in \{A_3, P_6'\}.$$

For $M_2 = A_3$ we have $K_0 = \{A_3\}$, $K_1 = \{A_4, F_2^2, F_2^3, \ldots, F_2^\infty\}$, $K_2 = \{S_5\}$, $K_3 = \{F_7^2, F_7^3, \ldots, F_7^\infty, P_3'\}$, $K_{12} = \{S_1\}$, $K_{13} = \{F_6^2, F_6^3, \ldots, F_6^\infty, P_1', D_2\}$, $K_{23} = \{O_6\}$, $K_{123} = \{O_1\}$.

For $M_2 = P_6'$ we have $K_0 = \{P_6'\}$, $K_1 = \{P_5'\}$, $K_2 = \{O_7, O_8\}$, $K_3 = \{P_3'\}$, $K_{12} = \{O_2, O_5\}$, $K_{23} = \{O_3, O_6\}$, $K_{13} = \{P_1'\}$, $K_{123} = \{O_1\}$.

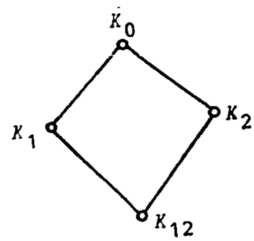

Fig. 9

In Fig. 9 we have the graph $G(\mathscr{M}_2)$ where

$$M_2 \in \{F_7^n, F_7^\infty, F_6^2, F_5^n, F_8^\infty, P_3', P_5', D_1, D_3, A_4, L_3, L_5, O_8\}.$$

For $M_2 = F_7^n$ we have $K_0 = \{F_7^n\}$, $K_1 = \{F_6^2, D_2\}$ for $n=2$, and $K_1 = \{F_6^n\}$ for $n>2$, $K_2 = \{F_7^{n+1}, F_7^{n+2}, \ldots, F_7^\infty, P_3', O_6, O_3\}$, $K_{12} = \{F_6^{n+1}, \ldots, F_6^\infty, P_1, O_1\}$.

For $M_2 = F_7^\infty$ we have $K_0 = \{F_7^\infty\}$, $K_1 = \{F_6^\infty\}$, $K_2 = \{P_3', O_6, O_3\}$, $K_{12} = \{P_1', O_1\}$.

For $M_2 = F_6^2$ we have $K_0 = \{F_6^2\}$, $K_1 = \{D_2\}$, $K_2 = \{F_6^3, F_6^4, \ldots, F_6^\infty, P_1'\}$, $K_{12} = \{O_1\}$.

For $M_2 = F_5^n$ we have $K_0 = \{F_5^n\}$, $K_1 = \{F_5^{n+1}, F_5^{n+2}, \ldots, F_5^\infty\}$, $K_2 = \{F_6^2, D_2\}$ for $n=2$ and $K_2 = \{F_6^n\}$ for $n>2$, $K_{12} = \{F_6^{n+1}, F_6^{n+2}, \ldots, F_6^\infty, P_1', O_1\}$.

For $M_2 = F_8^\infty$ we have $K_0 = \{F_8^\infty\}$, $K_1 = \{F_7^\infty, P_3', O_6, O_3\}$, $K_2 = \{F_5^\infty\}$, $K_{12} = \{F_6^\infty, P_1', O_1\}$.

For $M_2 = P_3'$ we have $K_0 = \{P_3'\}$, $K_1 = \{O_6, O_3\}$, $K_2 = \{P_1'\}$, $K_{12} = \{O_1\}$.

For $M_2 = P_5'$ we have $K_0 = \{P_5'\}$, $K_1 = \{P_1'\}$, $K_2 = \{O_5\}$, $K_{12} = \{O_1\}$.

For $M_2 = D_1$ we have $K_0 = \{D_1\}$, $K_1 = \{L_4\}$, $K_2 = \{D_2\}$, $K_{12} = \{O_1\}$.

For $M_2 = D_3$ we have $K_0 = \{D_3\}$, $K_1 = \{D_1, D_2\}$, $K_2 = \{L_5, O_4\}$, $K_{12} = \{L_4, O_1\}$.

For $M_2=A_4$ we have $K_0=\{A_4\}$, $K_1=\{F_2^2, F_2^3, \ldots, F_2^\infty, S_1\}$, $K_2=\{F_6^2, F_6^3, \ldots$ $\ldots, F_6^\infty, P_1'\}$, $K_{12}=\{D_2, O_1\}$.

For $M_2=L_3$ we have $K_0=\{L_3\}$, $K_1=\{L_4\}$, $K_2=\{O_3, O_6\}$, $K_{12}=\{O_1\}$.

For $M_2=L_5$ we have $K_0=\{L_5\}$, $K_1=\{L_4\}$, $K_2=\{O_4\}$, $K_{12}=\{O_1\}$.

For $M_2=O_8$ we have $K_0=\{O_8\}$, $K_1=\{O_2, O_5\}$, $K_2=\{O_3, O_6\}$, $K_{12}=\{O_1\}$.

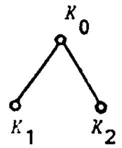

Fig. 10

In Fig. 10 we have the graph $G(\mathcal{M}_2)$ where

$$M_2 \in \{O_6, O_7, O_9\}.$$

For $M_2=O_6$ we have $K_0=\{O_6\}$, $K_1=\{O_1\}$, $K_2=\{O_3\}$.

For $M_2=O_7$ we have $K_0=\{O_7\}$, $K_1=\{O_2\}$, $K_2=\{O_3\}$.

For $M_2=O_9$ we have $K_0=\{O_9\}$, $K_1=\{O_1, O_4\}$, $K_2=\{O_2, O_3, O_7, O_8\}$.

Fig. 11

In Fig. 11 we have the graph $G(\mathcal{M}_2)$ where

$$M_2 \in \{P_1', D_2, L_4, F_5^\infty, F_6^n, F_6^\infty, O_4\} \text{ where } n>2.$$

For $M_2=P_1'$ we have $K_0=\{P_1'\}$, $K_1=\{O_1\}$.

For $M_2=D_2$ we have $K_0=\{D_2\}$, $K_1=\{O_1\}$.

For $M_2=L_4$ we have $K_0=\{L_4\}$, $K_1=\{O_1\}$.

For $M_2=F_5^\infty$ we have $K_0=\{F_5^\infty\}$, $K_1=\{F_6^\infty, P_1', O_1\}$.

For $M_2=F_6^n$ we have $K_0=\{F_6^n\}$, $K_1=\{F_6^{n+1}, F_6^{n+2}, \ldots, F_6^\infty, P_1', O_1\}$.

For $M_2=F_6^\infty$ we have $K_0=\{F_6^\infty\}$, $K_1=\{P_1', O_1\}$.

For $M_2=O_4$ we have $K_0=\{O_4\}$, $K_1=\{O_1\}$.

In Fig. 12 we have the graph $G(\mathcal{M}_2)$ where $M_2 \in \{O_1, O_3\}$. It has one vertex corresponding to K_0 which coincides with $\{O_1\}$ or $\{O_3\}$.

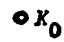

Fig. 12

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