A note on α_0^*-products of aperiodic automata

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One of the most celebrated results in the field of compositions of automata is the Krohn—Rhodes decomposition theorem. A detailed presentation can be found e.g. in [1]. It has, sometimes implicitly, inspired a great deal of research on various notions of compositions culminating in a series of interesting papers. For references and most recent results, see [2].

The system given by the Krohn—Rhodes theorem has a peculiar lack of symmetry. While it contains all group-like automata on simple groups, all aperiodic automata are in the meantime realized with cascade compositions of a single aperiodic automaton, the two state identity-reset automaton **U**.

If we want to realize a subclass of permutation automata we need exactly those simple groups which are divisors of characteristic groups of automata from the given subclass. Consequently, there is a continuum of different subclasses of permutation automata closed under cascade compositions, subautomata and homomorphic images. On the other hand, if we are given a subclass of aperiodic automata, we do not need the whole strength of **U** either. The reason is that there are numerous subclasses of aperiodic automata closed under cascades, subautomata and homomorphic images. In this note we are going to show that the exact number of these subclasses is continuum even for α_0^*-products. The notion of the α_0^*-product due to F. Gécseg in [3] is an abstract generalization of the cascade composition.

Although we are using standard automata theoretic concepts we intend to give a very brief account on the notions and notations to be followed throughout the paper.

N and P denote the set of all positive natural numbers and the set of primes $p\neq 1$ in N, respectively. We set $N_0=N\cup\{0\}$.

X^* is the free monoid with identity λ generated by a set X. We write $u<v$ to mean that u is a proper prefix of v, i.e., $u<v$ if and only if there exists a word $u_1\in X^*-\{\lambda\}$ so that $uu_1=v$.

Take an ordinary finite automaton $\mathbf{A}=(A, X, \delta)$ with state set A, input set X and transition function $\delta\colon A\times X\to A$. (We use the same $\delta\colon A\times X^*\to A$ for the usual extension of δ.)

The characteristic semigroup of **A** is the factor semigroup $X^*/\varrho_{\mathbf{A}}$ where the congruence $\varrho_{\mathbf{A}}$ is defined by $u\varrho_{\mathbf{A}}v$ if and only if $\delta(a, u)=\delta(a, v)$ for all $a\in A$. An automaton **A** is said to be aperiodic if $X^*/\varrho_{\mathbf{A}}$ contains only trivial subgroups.

Next we recall the notion of the α_0^*-product from [3]. Let $\mathbf{A}_j=(A_j, X_j, \delta_j)$, $1\leq j\leq n$, be arbitrary automata, X a finite nonvoid set and take a system of feedback functions $\varphi_j\colon A_1\times\ldots\times A_{j-1}\times X\to X_j^*$, $1\leqq j\leqq n$. The α_0^*-product $\mathbf{A}_1\times\ldots\times\mathbf{A}_n[X,\varphi]$ of these automata $\mathbf{A}_j$ with respect to X and φ is $(A_1\times\ldots\times A_n, X, \delta)$, where

$$\delta((a_1, \ldots, a_n), x) = (\delta_1(a_1, u_1), \ldots, \delta_n(a_n, u_n)),$$

$$u_j = \varphi_j(a_1, \ldots, a_{j-1}, x)$$

for all $a_1, \ldots, a_n\in A$, $x\in X$, $1\leqq j\leqq n$. We put for an arbitrary class $\mathscr{K}$ of automata

$\mathbf{P}_{\alpha_0}^*(\mathscr{K})$: all α_0^*-products of automata from $\mathscr{K}$,
$\mathbf{S}(\mathscr{K})$: all subautomata of automata from $\mathscr{K}$,
$\mathbf{H}(\mathscr{K})$: all homomorphic images of automata from $\mathscr{K}$.

Let $\mathscr{K}$ be a class of automata. $\mathscr{K}$ is called closed under $\mathbf{H}$, $\mathbf{S}$ and $\mathbf{P}_{\alpha_0}^*$ if $\mathbf{H}(\mathscr{K})$, $\mathbf{S}(\mathscr{K})$ and $\mathbf{P}_{\alpha_0}^*(\mathscr{K})$ are all subclasses of $\mathscr{K}$. Given $\mathscr{K}$, $\mathbf{HSP}_{\alpha}^*(\mathscr{K})$ is the smallest class containing $\mathscr{K}$ and closed under $\mathbf{H}$, $\mathbf{S}$ and $\mathbf{P}_{\alpha_0}^*$. Thus, $\mathscr{K}$ is closed under $\mathbf{H}$, $\mathbf{S}$ and $\mathbf{P}_{\alpha_0}^*$ if and only if $\mathbf{HSP}_{\alpha_0}^*(\mathscr{K})\subseteqq\mathscr{K}$. The Krohn—Rhodes theorem gives that the class of all aperiodic automata is closed under $\mathbf{H}$, $\mathbf{S}$ and $\mathbf{P}_{\alpha_0}^*$.

We now define a special aperiodic automaton $\mathbf{A}_p$ for every $p\in P$.

$$\mathbf{A}_p = (\{0, 1, \ldots, 2p\},\ \{x, y\},\ \delta_p),$$

$$\delta_p(i, x) = \begin{cases} i+1 & \text{if} \quad 1\leqq i\leqq p,\\ 0 & \text{otherwise,}\end{cases}$$

$$\delta_p(i, y) = \begin{cases} i+1 & \text{if} \quad p+1\leqq i\leqq 2p-1,\\ 1 & \text{if} \quad i=2p,\\ 0 & \text{in all other cases.}\end{cases}$$

The following statement enlists some pecularities of $\mathbf{A}_p$. In this statement u and v denote arbitrary words in $\{x, y\}^*$, i is a natural number $1\leqq i\leqq 2p$, and

$$w_i = \begin{cases} x^{p-i+1}\,y^p\,x^{i-1} & \text{if} \quad 1\leqq i\leqq p,\\ y^{2p-i+1}\,x^p\,y^{i-p-1} & \text{if} \quad p+1\leqq i\leqq 2p.\end{cases}$$

Claim.

(1a) $\delta_p(i, u)\neq 0$ if and only if $u=w_i^k u_1$ where $u_1<w_i$ and $k\in N_0$.
(1b) $\delta_p(i, u)=i$ if and only if $u=w_i^k$ for an integer $k\in N_0$.
(2a) $\delta_p(i, w_j)=0$ if $i\neq j$.
(2b) If u_1 contains both x and y, $u_1<w_j$ then

$$\delta_p(i, u_1^2) = 0 \quad \text{for every } i\neq j.$$

(3) $\delta_p(i, u^k)=i$ implies $\delta_p(i, u)=i$ for every $k\in N$.
(4) If $q\in N$, $q\neq 1$ and $q\neq p$ then $\delta_p(i, u^q v^q)=i$ implies $\delta_p(i, u)=\delta_p(i, v)=i$.

Proof. (1a), (1b), (2a)&(2b): Observe that there exists a single cycle

$$1\xrightarrow{x}2\xrightarrow{x}\ldots\xrightarrow{x}p\xrightarrow{x}p+1\xrightarrow{y}p+2\xrightarrow{y}\ldots\xrightarrow{y}2p\xrightarrow{y}1$$

which does not pass through the 'trapped state' 0. Thus we have to move along this cycle if we want to avoid that state.

(3) By (1a) we have $u=w_i^m u_1$, $u_1<w_i$, $m\in N_0$. Assume that $u_1\neq\lambda$. Since $\delta_p(i, u_1 w_i)=0$ by (2a) we obtain $m=0$. From (1b) it follows that $u_1^k=w_i^l$ for some $l\in N$. However, this is clearly impossible, thus we have $u_1=\lambda$. We can conclude using (1b).

(4) We get $u=w_i^m u_1$ and $v=w_j^l v_1$ from (1a), where $u_1<w_i$, $v_1<w_j$, $m, l\in N_0$ and $j=\delta_p(i, u^q)$. If $i=j$ we are ready by (3). Supposing $i\neq j$ we have $u_1\neq\lambda$, $v_1\neq\lambda$ by (1b). If $m>0$, then by (2a) $\delta_p(i, u_1 w_i)=0$, which would imply $\delta_p(i, u^q v^q)=0$. Thus, $m=0$, and similarly, $l=0$. By (2b), u_1 cannot contain both x's and y's, and the same holds for v_1. By (1b), $u^q v^q=u_1^q v_1^q=w_i^l$ for some $l\in N$, or even, $l=1$, and $i=1$ or $i=p+1$. Suppose that $i=1$, the case $i=p+1$ can be handled likewise. Then $u_1=x^r$ so that $rq=p$. Since p is a prime and $q\neq 1$, $q\neq p$, this gives a contradiction.

As an immediate consequence of Claim (3) we get the following:

Corollary. $\mathbf{A}_p$ is aperiodic for every $p\in P$. (Or even, since we did not use the fact that p is a prime in the proof of Claim (3), $\mathbf{A}_p$ is aperiodic for every natural number $p\in N$)

Theorem. The class of all aperiodic automata contains a continuum of different subclasses closed under **H**, **S** and $\mathbf{P}_{\alpha_0}^*$.

Proof. Let Q be a non-void subset of P. Put $\mathcal{K}_Q=\{\mathbf{A}_q | q\in Q\}$. We show that for $q\in P$, $\mathbf{A}_q\in\mathbf{HSP}_{\alpha_0}^*(\mathcal{K}_Q)$ only if $q\in Q$.

Supposing $\mathbf{A}_q\in\mathbf{HSP}_{\alpha_0}^*(\mathcal{K}_Q)$ there is an α_0^*-product $\mathbf{B}=\mathbf{A}_{p_1}\times\ldots\times\mathbf{A}_{p_l}[\{x, y\}, \varphi]$ of automata from $\mathcal{K}_Q$ such that $\mathbf{A}_q$ is a homomorphic image of a subautomaton $\mathbf{C}=(C, \{x, y\}, \delta)$ of $\mathbf{B}$ under a homomorphism $h: C\to A_q$. We can choose l minimal with this property. Further, it can be assumed that no subautomata of $\mathbf{C}$ other than itself can be mapped homomorphically onto $\mathbf{A}_q$. Let $c\in h^{-1}(1)$. Obviously, c generates $\mathbf{C}$, and $\delta(c, (x^q y^q)^m)=c$ for some $m\in N$. Since the class of all aperiodic automata is closed under α_0^*-products and subautomata we have $\delta(c, x^q y^q)=c$ as well.

Let us now suppose to the contrary $q\notin Q$. Put $c=(i_1, \ldots, i_l)$, $u=\varphi_1(x)$, $v=\varphi_1(y)$. (Observe that $l>0$.) From the definition of the α_0^*-product we then have $\delta_{p_1}(i_1, u^q v^q)=i_1$ which by Claim (4) gives $\delta_{p_1}(i_1, u)=\delta_{p_1}(i_1, v)=i_1$. Since c generates $\mathbf{C}$ it follows that the only state of $\mathbf{A}_{p_1}$ appearing as the first component of a state of $\mathbf{C}$ is i_1. However, this implies that a subautomaton of an α_0^*-product $\mathbf{A}_{p_2}\times\ldots\times\mathbf{A}_{p_l}$ $[\{x, y\}, \psi]$ can be mapped homomorphically onto $\mathbf{A}_q$ contradicting the minimality of l.

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