Loop products and loop-free products

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We introduce loop products of automata and show that, in the presence of input signs inducing the identity state transformation, loop products followed by loop-free products, (i.e. α_0-products) are just as stong as the most general product. See [3] for notations and unexplained concepts. Most recent results on α_0-products can be found in [2].

Take a g^*-product $\mathbf{A}=\mathbf{A}_1\times\ldots\times\mathbf{A}_n(X,\varphi)$ of automata $\mathbf{A}_t=(A_t, X_t, \delta_t)$, $t=1, \ldots, n$, $n\geqq 0$. We call $\mathbf{A}$ an l^*-product (i.e. generalized loop product) if for every $t>1$, $\varphi_t(a_1, \ldots, a_n, x)$ $((a_1, \ldots, a_n)\in A_1\times\ldots\times A_n, x\in X)$ only depends on x and a_{t-1}, and φ_1 only depends on a_n and x. In the special case that $\varphi_t(a_1, \ldots, a_n, x)\in X\cup\{\lambda\}$ $(\varphi_t(a_1, \ldots, a_n, x)\in X)$ we speak about an l^λ-product (l-product, i.e., loop product).

Let $\mathbf{K}$ be a class of automata. We put

$\mathbf{P}_l^*(\mathbf{K})$: all l^*-products of automata from $\mathbf{K}$,
$\mathbf{P}_l^\lambda(\mathbf{K})$: all l^λ-products of automata from $\mathbf{K}$,
$\mathbf{P}_l(\mathbf{K})$: all l-products of automata from $\mathbf{K}$.

Further, we write $\mathbf{P}_{1l}^*(\mathbf{K})$ $(\mathbf{P}_{1l}^\lambda(\mathbf{K}), \mathbf{P}_{1l}(\mathbf{K}))$ for the class of all l^*-products (l^λ-products, l-products) with a single factor of automata from $\mathbf{K}$.

Our result is the following statement.

Theorem. $\mathbf{HSP}_{\alpha_0}\mathbf{P}_l^\lambda(\mathbf{K})=\mathbf{HSP}_{\alpha_0}\mathbf{P}_l^*(\mathbf{K})=\mathbf{HSP}_g^*(\mathbf{K})$ for every class $\mathbf{K}$.

Proof. The inclusions from left to right are obvious. To see that

$$\mathbf{HSP}_g^*(\mathbf{K}) \subseteqq \mathbf{HSP}_{\alpha_0} P_l^\lambda(\mathbf{K}),$$

by $\mathbf{P}_{\alpha_0}^\lambda(\mathbf{K}) = \mathbf{P}_{\alpha_0}^\lambda\mathbf{P}_l^\lambda(\mathbf{K})$, it suffices to show that $\mathbf{HSP}_g^*(\mathbf{K})\subseteqq\mathbf{HSP}_{\alpha_0}^\lambda\mathbf{P}_l^\lambda(\mathbf{K})$.

If $\mathbf{K}$ contains only monotone automata, then $\mathbf{HSP}_g^*(\mathbf{K})=\mathbf{ISP}_{\alpha_0}^\lambda(\mathbf{K})$ by the proof of Theorem 4 in [3] and the inclusion holds. Suppose that $\mathbf{K}$ contains an automaton which is not cycle-free. We claim that $\mathbf{HSP}_{\alpha_0}^\lambda\mathbf{P}_l^\lambda(\mathbf{K})$ is the class of all automata. To this, by Corollary 2 in [3], we have to show the following:

(i) $\mathbf{P}_l^\lambda(\mathbf{K})$ is not counter-free.

(ii) $\mathbf{A}_0\in\mathbf{HSP}_{\alpha_0}^\lambda\mathbf{P}_l^\lambda(\mathbf{K})$.

(iii) For every finite simple group G there exists an automaton $\mathbf{A}\in\mathbf{P}_l^\lambda(\mathbf{K})$ such that G is a homomorphic image of a subgroup of $S(\mathbf{A})$.

Proof of (i). There is an automaton $\mathbf{A}\in\mathbf{K}$ containing a nontrivial cycle, i.e., a cycle with length $n>1$. Obviously, a counter with length n is in $\mathbf{SP}_{1l}(\mathbf{K})$, therefore, $\mathbf{P}_l^\lambda(\mathbf{K})$ is not counter-free.

Proof of (ii). By Lemma 3 in [3], $\mathbf{A}_0\in\mathbf{HSP}_{\alpha_1}^\lambda(\mathbf{K})$. However,

$$\mathbf{HSP}_{\alpha_1}^\lambda(\mathbf{K}) = \mathbf{HSP}_{\alpha_0}^\lambda\mathbf{P}_{1\alpha_1}^\lambda(\mathbf{K}) = \mathbf{HSP}_{\alpha_0}^\lambda P_{1l}^\lambda(\mathbf{K}) \subseteq \mathbf{HSP}_{\alpha_0}^\lambda\mathbf{P}_l^\lambda(\mathbf{K}).$$

Proof of (iii). We show that for every integer $n\geqq 3$ there are an automaton $\mathbf{B}\in(B, Y, \delta')\in\mathbf{P}_l^\lambda(\mathbf{K})$ and a subset $B'=\{b_1, \dots, b_n\}\subseteq B$ so that every such permutation of B' which fixes b_n can be induced by a word in Y^*. Of course, it is enough to prove for transpositions $(b_s b_{s+1})$ with $1\leqq s\leqq n-2$.

Let $\mathbf{A}\in(A, X, \delta)\in\mathbf{K}$ be an automaton containing a nontrivial cycle, i.e. a sequence of states $0, 1, \dots, p-1$ $(p\geqq 2)$ and input signs $x_1, \dots, x_{p-1}, x_0$ with $\delta(0, x_1)=1, \dots, \delta(p-2, x_{p-1})=p-1$, $\delta(p-1, x_0)=0$. In the case that $p=2$ the result follows by the proof of Theorem 2 in [1] (observation due to J. Virágh). Hence we assume $p>2$.

Define $\mathbf{B}$ to be the l^λ-power $\mathbf{A}^n(Y, \varphi)$ with

$$Y = \{y(k, i, j) | 1 \leqq k \leqq n, \quad 0 \leqq i,j \leqq p-1\}$$

and

$$\varphi_t\big(a, y(k, i, j)\big) = \begin{cases} x_j & \text{if} \quad t = k \quad \text{and} \quad a = i, \\ \lambda & \text{otherwise,} \end{cases}$$

$$\text{where} \quad 1 \leqq t \leqq n, \quad a\in A, \quad y(k, i, j)\in Y.$$

Put

$$b_t = 0^{t-1}1\,0^{n-t},$$

$t=1, \dots, n$. (We use the shorthand $a_1 \dots a_n$ for the elements of B.) Fix an integer s, $1\leqq s\leqq n-2$. In five steps we shall construct a word $u=u_1\dots u_5\in Y^*$ such that

$$\delta'(b_t, u) = b_t \quad \text{if} \quad t \neq s,\ s+1,$$

$$\delta'(b_s, u) = b_{s+1},$$

$$\delta'(b_{s+1}, u) = b_s.$$

(The construction is indicated in the Figure for $p=3$, $n=6$ and $s=3$. Blank entries are meant 0.)

Step 1.

$$\begin{aligned} u_1 = {} & y(s+1, 1, 1) \dots y(s+1, 1, p-1)\cdot \\ & y(s+2, p-1, 1) \dots y(s+2, p-1, p-1)\cdot \\ & \vdots \\ & y(n, p-1, 1) \dots y(n, p-1, p-1)\cdot \\ & y(1, p-1, 1) \dots y(n, p-1, p-1)\cdot \\ & \vdots \\ & y(s-1, p-1, 1) \dots y(s-1, p-1, p-1)\cdot \\ & y(s, p-1, 2) \dots y(s, p-1, p-1). \end{aligned}$$

We have

$$\delta(b_t, u_1) = b_t \quad \text{if} \quad t \neq s,$$
$$\delta(b_s, u_1) = (p-1)^n.$$

Step 2.

$$u_2 = y(s+3, 1, 1) \dots y(n, 1, 1) \cdot$$
$$y(1, 1, 1) \dots y(s, 1, 1).$$

We have

$$\delta(b_1, u_1 u_2) = 1^{s-1}100^{n-s-1}$$
$$\delta(b_2, u_1 u_2) = 01^{s-2}100^{n-s-1}$$
$$\vdots$$
$$\delta(b_{s-1}, u_1 u_2) = 0^{s-2}1100^{n-s-1}$$
$$\delta(b_s, u_1 u_2) = (p-1)^{s-1}(p-1)(p-1)(p-1)^{n-s-1}$$
$$\delta(b_{s+1}, u_1 u_2) = 0^{s-1}010^{n-s-1}$$
$$\delta(b_{s+2}, u_1 u_2) = 1^{s-1}101^{n-s-1}$$
$$\delta(b_{s+3}, u_1 u_2) = 1^{s-1}1001^{n-s-2}$$
$$\vdots$$
$$\delta(b_n, u_1 u_2) = 1^{s-1}100^{n-s-2}1.$$

Step 3.

$$u_3 = y(s+1, 0, 2) \dots y(s+1, 0, p-1)\, y(s+1, 0, 0) \cdot$$
$$y(s+1, p-1, 0)\, y(s+1, p-1, 1)\, y(s, 0, 1)\, y(s, p-1, 0).$$

We have

$$\delta(b_t, u_1 u_2 u_3) = (b_t, u_1 u_2), \quad t \neq s, s+1,$$
$$\delta(b_s, u_1 u_2 u_3) = (p-1)^{s-1}01(p-1)^{n-s-1}$$
$$\delta(b_{s+1}, u_1 u_2 u_3) = 0^{s-1}100^{n-s-1}.$$

Step 4.

$$u_4 = y(s-1, p-1, 0) \dots y(1, p-1, 0) \cdot$$
$$y(n, p-1, 0) \dots y(s+3, p-1, 0)\, y(s+2, 1, 0).$$

We have

$$\delta(b_t, u_1 u_2 u_3 u_4) = \delta(b_t, u_1 u_2 u_3), \quad t \neq s,$$
$$\delta(b_s, u_1 u_2 u_3 u_4) = 0^{s-1}010^{n-s-1}.$$

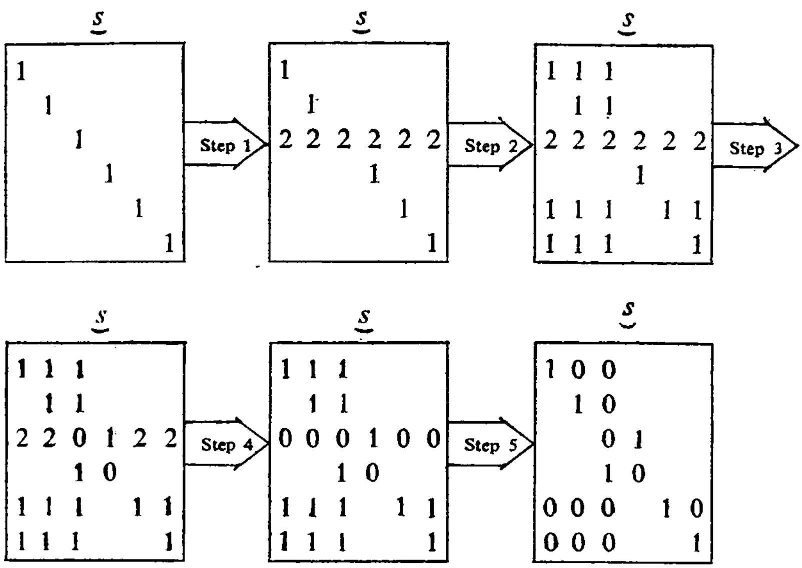
Step 5.

$$\begin{aligned} u_5 = {} & y(s,1,2)\dots y(s,1,p-1)\,y(s,1,0)\cdot \\ & \vdots \\ & y(1,1,2)\dots y(1,1,p-1)\,y(1,1,0)\cdot \\ & y(n,1,2)\dots y(n,1,p-1)\,y(n,1,0)\cdot \\ & \vdots \\ & y(s+3,1,2)\dots y(s+3,1,p-1)\,y(s+3,1,0). \end{aligned}$$

We obtained:

$$\delta(b_t, u) = b_t, \quad t \neq s,\ s+1,$$
$$\delta(b_s, u) = b_{s+1},$$
$$\delta(b_{s+1}, u) = b_s.$$

This ends the proof.



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