Results on compositions of deterministic root-to-frontier tree transformations

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Introduction

In this paper we examine the class of deterministic root-to-frontier tree transformations ($\mathcal{DR}$) and some of its usual subclasses such as linear, nondeleting, homomorphism and so on. We present some equalities and inclusions between the compositions of different classes and, as an application, show that $\mathcal{DR}^2=\mathcal{DR}^n$ for each $n\geqq 2$.

We also study all the classes which can be written in the form $\mathcal{K}_1\circ\ldots\circ\mathcal{K}_n$ where each $\mathcal{K}_i$ is $\mathcal{DR}$ or one of its subclasses. We pick out a finite number of these classes and show that every class $\mathcal{K}_1\circ\ldots\circ\mathcal{K}_n$ either equals to one of them or has a rather special form.

1. Notions and notations

For an arbitrary set Y, we denote by $|Y|$ and $\mathcal{P}(Y)$ the cardinality and the power set of Y, respectively. If Y is a singleton, then we identify it with its unique element. Y^* is the free monoid generated by Y with empty word λ.

The set of nonnegative integers is denoted by N. For every $n\in N$, $[n]$ denotes the set $\{1, \ldots, n\}$, especially $[0]=\emptyset$.

By a ranked alphabet we mean an ordered pair (F, ν) where F is a finite set and $\nu: F\to N$ is the arity function. Elements of F are considered as operational symbols, more exactly, if $f\in F$ and $\nu(F)=n$ then f is an n-ary operational symbol. We use the notation $F=\bigcup_{n\in N} F_n$ where the sets $F_n=\nu^{-1}(n)$ are pairwise disjoint.

Now let F be a ranked alphabet and Y a set. The set of all terms or trees over Y of type F is defined as the smallest set $T_F(Y)$ satisfying

(a) $Y\cup F_0\subseteq T_F(Y)$ and,
(b) $f(p_1, \ldots, p_n)\in T_F(Y)$ whenever $f\in F_n$ $(n\in N)$ and $p_i\in T_F(Y)$ $(i\in[n])$.

If $Y=\emptyset$ then $T_F(Y)$ is simply written as T_F.

We define the height $h(p)\in N$, frontier $fr(p)\subseteq Y^*$ and the set of subtrees $sub(p)\subseteq T_F(Y)$ of a tree $p\in T_F(Y)$ by induction:

(a) if $p \in F_0$ then $h(p)=0$, $fr(p)=\lambda$ and $sub(p)=\{p\}$;
(b) if $p \in Y$ then $h(p)=0$, $fr(p)=p$ and $sub(p)=\{p\}$;
(c) if $p=f(p_1, \ldots, p_n)$ then $h(p)=1+max\ \{h(p_i)|i \in [n]\}$, $fr(p)=fr(p_1)\ldots fr(p_n)$ and $sub(p)=\bigcup_{i \in [n]} sub\,(p_i) \cup \{p\}$.

We shall need a countably infinite set $X=\{x_1, x_2, \ldots\}$, elements of which are considered as auxiliary variables. The set of the first n elements $x_1, \ldots, x_n$ of X is denoted by X_n.

Letting $Y=X_n$ we have the set $T_F(X_n)$. Here, the elements of X_n can be used to point out places in the frontier of a tree $p \in T_F(X_n)$. There is a distinguished subset $\hat{T}_F(X_n)$ of $T_F(X_n)$ defined as follows: $p \in \hat{T}_F(X_n)$ iff $p \in T_F(X_n)$ and $fr(p)$ is a permutation of X_n, in other words, each element of X_n appears exactly once in p.

Now let $p \in T_F(X_n)$ and $y_1, \ldots, y_n \in Y$. We denote by $p(y_1, \ldots, y_n)$ the tree obtained by substituting all the occurrences of x_i in p by y_i for each $i \in [n]$. Note that $p(y_1, \ldots, y_n)$ is an element of $T_F(Y)$.

By a tree transformation τ we mean a relation from T_F to T_G where F and G are arbitrary ranked alphabets, that is we have $\tau \subseteq F_F \times T_G$. In this way, the identical relation $\iota_F=\{(p, p)|p \in T_F\}$ is clearly a tree transformation. The class of all identical tree transformations is denoted by $\mathscr{I}$. The restriction $\tau|T$ of τ to a subset T of T_F is defined by

$$\tau|T = \{(p, q)|(p, q) \in \tau \quad \text{and} \quad p \in T\}.$$

For any tree transformations $\tau \subseteq T_F \times T_G$ and $\sigma \subseteq T_G \times T_H$ the domain $(dom\ \tau)$ range $(range\ \tau)$ of τ and the composition $(\tau \circ \sigma)$ of τ and σ are defined as usual in the case of relations.

Let $\mathscr{K}_1$ and $\mathscr{K}_2$ be two classes of tree transformations. By their composition $\mathscr{K}_1 \circ \mathscr{K}_2$ we mean $\{\tau_1 \circ \tau_2 | \tau_1 \in \mathscr{K}_1$ and $\tau_2 \in \mathscr{K}_2\}$. For any class $\mathscr{K}$ of tree transformations and $n \in N$ we put $\mathscr{K}^n=\mathscr{K}$ if $n=1$ and $\mathscr{K}^n=\mathscr{K}^{n-1} \circ \mathscr{K}$ if $n>1$. We say that $\mathscr{K}$ is closed under composition if $\mathscr{K}^2 \subseteq \mathscr{K}$ holds. If $\mathscr{I} \subseteq \mathscr{K}$, as with most of the reasonable classes $\mathscr{K}$, then obviously $\mathscr{K}^2 \subseteq \mathscr{K}$ iff $\mathscr{K}^2=\mathscr{K}$.

In this paper we are interested only in tree transformations which can be induced by deterministic root-to-frontier tree transducers.

By a deterministic root-to-frontier tree transducer (or shortly DR transducer) we mean a system

$$\mathfrak{A} = (F, A, G, P, a_0), \quad \text{where} \tag{1}$$

(a) F and G are ranked alphabets;
(b) A is a ranked alphabet — disjoint with F and G — consisting of unary operational symbols, the state set of $\mathfrak{A}$;
(c) a_0 is a distinguished element of A, the initial state;
(d) P is a finite set of productions (or rewriting rules) of the form

$$af(x_1, \ldots, x_m) \to q, \tag{2}$$

where $a \in A$, $m \geq 0$, $f \in F_m$ and $q \in T_G(A \times X_m)$. To guarantee $\mathfrak{A}$ a deterministic behaviour, any two different productions of P are required to have different left-hand sides.

Throughout the paper terms of the form $a(p)$ ($a \in A$ and p is a term) are written simply as ap. If we need to specify a production (2) in a more detailed form, then we can write (2) as

$$af(x_1, \ldots, x_m) \to \bar{q}(a_1 x_{i_1}, \ldots, a_n x_{i_n}) \tag{3}$$

for a suitable $n \geqq 0$, $\bar{q} \in \hat{T}_G(X_n)$, $a_j \in A$, $x_{i_j} \in X_m$ $(j \in [n])$, or as

$$af(x_1, \ldots, x_m) \to \bar{q}(a_{1_1} x_1, \ldots, a_{1_{n_1}} x_1, \ldots, a_{m_1} x_m, \ldots, a_{m_{n_m}} x_m) \tag{4}$$

for some $n_i \geqq 0$, $a_{i_j} \in A$, $(i \in [m], j \in [n_i])$ and $\bar{q} \in \hat{T}_G(X_n)$ where $n = n_1 + \ldots + n_m$.

Productions of P can be used to transform (or rewrite) terms of $A \times T_F$ to terms of T_G, by defining the relation $\underset{\mathfrak{A}}{\Rightarrow}$ (called direct derivation) on the set $T_G(A \times T_F(X))$ in the following way: for $p, q \in T_G(A \times T_F(X))$ we say that $p \underset{\mathfrak{A}}{\Rightarrow} q$ iff q can be obtained from p by replacing an occurrence of a subtree $af(p_1, \ldots, p_m)$ of p by the tree $\bar{q}(a_1 p_{i_1}, \ldots, a_n p_{i_n})$ provided the rule (3) is in P. Denoting the reflexive-transitive closure (i.e. the iterated application) of the direct derivation by $\overset{*}{\underset{\mathfrak{A}}{\Rightarrow}}$, the tree transformation $\tau_{\mathfrak{A}(a)}$ induced by $\mathfrak{A}$ with state $a \in A$ is defined by

$$\tau_{\mathfrak{A}(a)} = \{(p, q) \mid p \in T_F,\ q \in T_G \ \text{and}\ ap \overset{*}{\underset{\mathfrak{A}}{\Rightarrow}} q\}.$$

By the tree transformation $\tau_{\mathfrak{A}}$ induced by $\mathfrak{A}$ we mean $\tau_{\mathfrak{A}(a_0)}$, i.e.,

$$\tau_{\mathfrak{A}} = \{(p, q) \mid p \in T_F,\ q \in T_G \ \text{and}\ a_0 p \overset{*}{\underset{\mathfrak{A}}{\Rightarrow}} q\}.$$

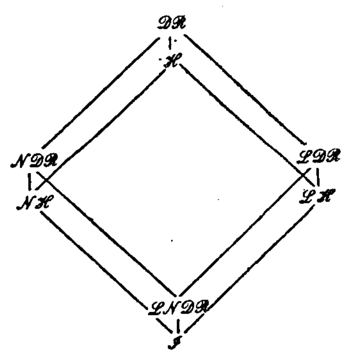
We say that a tree transformation $\tau \subseteq T_F \times T_G$ can be induced by a *DR* transducer if $\tau = \tau_{\mathfrak{A}}$ for some *DR* transducer $\mathfrak{A}$.

Next we introduce some restrictions on *DR* transducers. A *DR* transducer (1) is totally defined if for each $a \in A$ and $f \in F$ there is a rule (2) in P. (1) is called a homomorphism (H) transducer if it is totally defined and has only one state, i.e. $A = \{a_0\}$. Moreover, we say that (1) is

(a) linear (L) if for every rule (4) of P and $i \in [m]$, $n_i \leqq 1$,
(b) nondeleting (N) if for each rule (4) of P and $i \in [m]$, $n_i \geqq 1$,
(c) linear nondeleting (LN) if it is both linear and nondeleting.

The subclasses L, N and LN of H transducers are defined in a similar way.

If K is some subclass of the class of all *DR* transducers defined above, then the class of all tree transformations that can be induced by K transducers is denoted by $\mathcal{K}$. For example $\mathcal{LDR}$ denotes the class of all tree transformations that can be induced by *LDR* transducers. Finally, we present a diagram showing the inclusion relations among the classes $\mathcal{DR}$, $\mathcal{NDR}$, $\mathcal{LDR}$, $\mathcal{LNDR}$, $\mathcal{H}$, $\mathcal{NH}$, $\mathcal{LH}$, $\mathcal{I}$.



2. Equalities and inclusions

By one of the earlier results $\mathcal{DR}$ is not closed under composition (see [4]). This means that, in general, we cannot give a DR transducer $\mathfrak{C}$, for any two DR transducers $\mathfrak{A}$ and $\mathfrak{B}$ such that $\tau_{\mathfrak{C}}=\tau_{\mathfrak{A}}\circ\tau_{\mathfrak{B}}$. However, we can define a DR transducer $\mathfrak{A}\circ\mathfrak{B}$ called the syntactic composition of $\mathfrak{A}$ and $\mathfrak{B}$ with a series of useful properties. This was also stated, in an implicit form, in [3].

Definition 1. Let $\mathfrak{A}=(F, A, G, P, a_0)$ and $\mathfrak{B}=(G, B, H, P', b_0)$ be DR transducers. By the syntactic composition of $\mathfrak{A}$ and $\mathfrak{B}$ we mean the DR transducer $\mathfrak{A}\circ\mathfrak{B}=$ $=(F, B\times A, H, P'', (b_0, a_0))$ where P'' is defined in the following manner: the rule

$$(b, a)f(x_1, \ldots, x_m) \to q'\big((b_{1_1}, a_1)\, x_{i_1}, \ldots, (b_{1_{v_1}}, a_1)\, x_{i_1}, \ldots,$$
$$(b_{n_1}, a_n)\, x_{i_n}, \ldots, (b_{n_{v_n}}, a_n)\, x_{i_n}\big)$$

is in P'' for some $v_j\in N$, $b_{j_k}\in B$, $(j\in[n], k\in[v_j])$ and $q'\in T_G(X_v)$ $(v=v_1+\ldots+v_n)$ if and only if there is a rule (3) in P and a state $b\in B$ with

$$b\bar{q} \overset{*}{\underset{\mathfrak{B}}{\Rightarrow}} q'\big(b_{1_1} x_1, \ldots, b_{1_{v_1}} x_1, \ldots, b_{n_1} x_n, \ldots, b_{n_{v_n}} x_n\big).$$

(We let $\mathfrak{B}$ work on $\bar{q}$ as long as it can.)

Lemma 2. Under the notations of the above definition, for any $a\in A$, $b\in B$, $p\in T_F$ and $q\in T_H$

$$(\exists r\in T_G)\big(ap \overset{*}{\underset{\mathfrak{A}}{\Rightarrow}} r \wedge br \overset{*}{\underset{\mathfrak{B}}{\Rightarrow}} q\big) \Rightarrow (b, a)\, p \overset{*}{\underset{\mathfrak{A}\circ\mathfrak{B}}{\Rightarrow}} q. \tag{5}$$

The proof, as usual, can easily be performed by induction on $h(p)$. □

Now we can make the following observations.

(a) We cannot converse (5) because $\mathfrak{B}$ may be deleting, therefore there may be a tree $p\in T_F$ such that $\mathfrak{A}\circ\mathfrak{B}$ can transform p to a tree $q\in T_H$ by deleting some subtree p' of p but $\mathfrak{A}$ can not transform p' with any state $a\in A$. Thus p may be in $dom\, \tau_{\mathfrak{A}\circ\mathfrak{B}}$ but not in $dom\, \tau_{\mathfrak{A}}$ hence not even in $dom\, (\tau_{\mathfrak{A}}\circ\tau_{\mathfrak{B}})$. It can be seen that we can eliminate this problem by requiring $\mathfrak{A}$ to be totally defined (see also in [4]) or $\mathfrak{B}$ to be nondeleting.

(b) Moreover, (5) can also be conversed if p is in $dom\, \tau_{\mathfrak{A}}$ since in this case $\mathfrak{A}$ can always transform p' with some state $a\in A$.

(c) $\mathfrak{A}\circ\mathfrak{B}$ inherits any property $\mathfrak{A}$ and $\mathfrak{B}$ have, where property means one of the following: completely defined, one-state, L, N, LN.

We give a summary of the above observations:

Lemma 3. For any DR transducers $\mathfrak{A}$ and $\mathfrak{B}$ the following hold:

(a) if $\mathfrak{A}$ is totally defined or $\mathfrak{B}$ is nondeleting then $\tau_{\mathfrak{A}\circ\mathfrak{B}}=\tau_{\mathfrak{A}}\circ\tau_{\mathfrak{B}}$,

(b) $\tau_{\mathfrak{A}\circ\mathfrak{B}}|\mathrm{dom}\, \tau_{\mathfrak{A}}=\tau_{\mathfrak{A}}\circ\tau_{\mathfrak{B}}$,

(c) if $\mathfrak{A}$ and $\mathfrak{B}$ are x then $\mathfrak{A}\circ\mathfrak{B}$ is also x where $x=$completely defined, one-state, L, N, LN. □

From the above lemma we have a series of equalities some of which were stated in different works.

$$\mathcal{H}\circ\mathcal{DR}=\mathcal{DR} \qquad (6)$$

$$\mathcal{LH}\circ\mathcal{DR}=\mathcal{DR} \qquad (7)$$

$$\mathcal{NH}\circ\mathcal{DR}=\mathcal{DR} \qquad (8)$$

$$\mathcal{LH}\circ\mathcal{LDR}=\mathcal{LDR} \qquad (9)$$

$$\mathcal{NH}\circ\mathcal{NDR}=\mathcal{NDR} \qquad (10)$$

$$\mathcal{H}\circ\mathcal{H}=\mathcal{H} \qquad (11)$$

$$\mathcal{H}\circ\mathcal{NH}=\mathcal{H} \qquad (12)$$

$$\mathcal{H}\circ\mathcal{LH}=\mathcal{H} \qquad (13)$$

$$\mathcal{NH}\circ\mathcal{H}=\mathcal{H} \qquad (14)$$

$$\mathcal{LH}\circ\mathcal{H}=\mathcal{H} \qquad (15)$$

$$\mathcal{NH}\circ\mathcal{NH}=\mathcal{NH} \qquad (16)$$

$$\mathcal{LH}\circ\mathcal{LH}=\mathcal{LH} \qquad (17)$$

These follow from the fact that an $H(LH, NH)$ transducer is always completely defined.

Moreover, we also obtain

$$\mathcal{DR}\circ\mathcal{NDR}=\mathcal{DR} \qquad (18)$$

$$\mathcal{DR}\circ\mathcal{LNDR}=\mathcal{DR} \qquad (19)$$

$$\mathcal{DR}\circ\mathcal{NH}=\mathcal{DR} \qquad (20)$$

$$\mathcal{NDR}\circ\mathcal{NDR}=\mathcal{NDR} \qquad (21)$$

$$\mathcal{NDR}\circ\mathcal{LNDR}=\mathcal{NDR} \qquad (22)$$

$$\mathcal{NDR}\circ\mathcal{NH}=\mathcal{NDR} \qquad (23)$$

$$\mathcal{LDR}\circ\mathcal{LNDR}=\mathcal{LDR} \qquad (24)$$

$$\mathcal{LNDR}\circ\mathcal{NDR}=\mathcal{NDR} \qquad (25)$$

$$\mathcal{LNDR}\circ\mathcal{LNDR}=\mathcal{LNDR} \qquad (26)$$

Here we used that the second components are all nondeleting.

A frequently quoted equality is

$$\mathcal{H}\circ\mathcal{LDR}=\mathcal{DR} \qquad (27)$$

which can be found in [1] and [2]. From the proof of (27), it turns out that we may also declare it in the form

$$\mathcal{NH}\circ\mathcal{LDR}=\mathcal{DR} \qquad (28)$$

moreover, if we consider only H transducers we get

$$\mathcal{NH}\circ\mathcal{LH}=\mathcal{H}. \qquad (29)$$

By lemma 3, $\mathcal{LH}\circ\mathcal{NH}\subseteq\mathcal{H}$ and it is not difficult to see the conversed inclusion shown by the following lemma.

Lemma 4. $\mathcal{H}\subseteq\mathcal{LH}\circ\mathcal{NH}$.

Proof. Let us be given an H transducer $\mathfrak{A}=(F, a, G, P, a)$. Obviously, each production can be taken in the following form:

$$af(x_1, \ldots, x_m) \to q(ax_{i_1}, \ldots, ax_{i_n}) \tag{30}$$

where $m \geqq 0$, $f \in F_m$, $1 \leqq i_1 < \ldots < i_n \leqq m$ and q is a suitable tree from $T_G(X_n)$ containing at least one occurrence of x_j for every $j \in [n]$. For each production (30) take a new operational symbol $\bar{f}$ with arity n and put $\bar{F} = \{\bar{f} \mid f \in F\}$. Now we can introduce the H transducers $\mathfrak{B}=(F, b, \bar{F}, P', b)$ and $\mathfrak{C}=(\bar{F}, c, G, P'', c)$ as follows: whenever a production (30) is in P let the productions $bf(x_1, \ldots, x_m) \to \bar{f}(bx_{i_1}, \ldots, bx_{i_n})$ and $c\bar{f}(x_1, \ldots, x_n) \to q(cx_1, \ldots, cx_n)$ be in P' and P'', respectively. Clearly, $\mathfrak{B}$ is an LH and $\mathfrak{C}$ is an NH transducer, moreover the equivalence

$$ap \overset{*}{\underset{\mathfrak{A}}{\Rightarrow}} q \Leftrightarrow (\exists p \in T_F)\left(bp \overset{*}{\underset{\mathfrak{B}}{\Rightarrow}} r \wedge cr \overset{*}{\underset{\mathfrak{C}}{\Rightarrow}} q\right)$$

can be proved, for each $p \in T_F$ and $q \in T_G$, by induction on $h(p)$. Hence we have our lemma and the equality:

$$\mathscr{L}\mathscr{H} \circ \mathscr{N}\mathscr{H} = \mathscr{H}. \tag{31}$$

Our next lemma follows from exercise 2 on p. 213 of [3]. This states that $dom\ \tau_{\mathfrak{A}}$ can always be recognized by some DR recognizer (for definition see also [3]) for any DR transducer $\mathfrak{A}$. However, we mention that the following correction is needed in the definition of the DR recognizer in [3]: the realisation of an operational symbol of arity 0 must be considered as a subset of the state set and not as an element of it.

Lemma 5. For any given DR transducer $\mathfrak{A}=(F, A, G, P, a_0)$ there exists an $LNDR$ transducer $\mathfrak{A}'=(F, \mathscr{P}(A), F, P', \{a_0\})$ such that $\tau_{\mathfrak{A}'} \cdot = \iota_F | \mathrm{dom}\ \tau_{\mathfrak{A}}$.

Proof. Let P' be constructed as follows: for any $B=\{a_1, \ldots, a_k\} \in \mathscr{P}(A)$, $m \in N$ and $f \in F_m$ the rule $Bf(x_1, \ldots, x_m) \to f(B_1 x_1, \ldots, B_m x_m)$ is in P' if and only if the next conditions hold:

(a) for each $i \in [k]$ there is a production

$$a_i f(x_1, \ldots, x_m) \to q_i(a^i_{1_1} x_1, \ldots, a^i_{1_{n_1}} x_1, \ldots, a^i_{m_1} x_m, \ldots, a^i_{m_{n_m}} x_m)$$

in P where $n_1, \ldots, n_m \geqq 0$ (depend on i), $a^i_{j_k} \in A$,

$$(j \in [m], k \in [n_j]), \; q_i \in \hat{T}_G(X_n), \quad (n = n_1 + \ldots + n_m);$$

(b) $$B_j = \bigcup_{i \in [n]} \{a^i_{j_1}, \ldots, a^i_{j_{n_j}}\}, \quad j \in [m].$$

Then we can verify the following statement: for any

$$B = \{a_1, \ldots, a_k\} \in \mathscr{P}(A) \quad \text{and} \quad p \in T_F$$

$$Bp \overset{*}{\underset{\mathfrak{A}}{\Rightarrow}} p \Leftrightarrow (\forall j \in [k])(\exists q \in T_G)(a_j p \overset{*}{\underset{\mathfrak{A}}{\Rightarrow}} q). \quad \Box$$

Now let $\mathfrak{A}$ and $\mathfrak{B}$ be two arbitrary DR transducers. Then, by lemmas 3 and 5 we have

$$\tau_{\mathfrak{A}} \circ \tau_{\mathfrak{B}} = \tau_{\mathfrak{A} \circ \mathfrak{B}} | dom\ \tau_{\mathfrak{A}} = (\iota_F | dom\ \tau_{\mathfrak{A}}) \circ \tau_{\mathfrak{A} \circ \mathfrak{B}} = \tau_{\mathfrak{A}'} \circ \tau_{\mathfrak{A} \circ \mathfrak{B}}$$

from where

$$\mathcal{DR}^2 = \mathcal{LNDR} \circ \mathcal{DR} \tag{32}$$

and, if $\mathfrak{A}$ and $\mathfrak{B}$ are *LDR* transducers then

$$\mathcal{LDR}^2 = \mathcal{LNDR} \circ \mathcal{LDR}. \tag{33}$$

We are ready to prove one of our main results.

Theorem 6. For any $n \geqq 2$

$$\mathcal{DR}^n = \mathcal{LNDR} \circ \mathcal{DR} \quad \text{and} \tag{34}$$

$$\mathcal{LDR}^n = \mathcal{LNDR} \circ \mathcal{LDR} \tag{35}$$

Proof. We follow an induction on n. The case $n=2$ is already proved, the induction step of (34) (and similarly that of (35)) is shown by the following computation:

$$\mathcal{DR}^{n+1} \overset{(32)}{=} \mathcal{LNDR} \circ \mathcal{DR} \circ \mathcal{DR}^{n-1} = \mathcal{LNDR} \circ \mathcal{DR}^n \overset{\text{i.h.}}{=} \mathcal{LNDR} \circ \mathcal{LNDR} \circ \mathcal{DR} \overset{(26)}{=}$$

$$\mathcal{LNDR} \circ \mathcal{DR}. \quad \square$$

Consequence 7. For every $n \geqq 2$

$$\mathcal{DR}^n = \mathcal{DR}^2 \quad \text{and} \tag{36}$$

$$\mathcal{LDR}^n = \mathcal{LDR}^2. \tag{37}$$

We shall also need the following result.

Lemma 8.

$$\mathcal{DR} \subseteq \mathcal{NDR} \circ \mathcal{LH} \tag{38}$$

Proof. Let $\mathfrak{A}=(F, A, G, P, a_0)$ be a *DR* transducer. We construct an *NDR* transducer $\mathfrak{B}$ and an *LH* transducer $\mathfrak{C}$ such that $\tau_{\mathfrak{A}} = \tau_{\mathfrak{B}} \circ \tau_{\mathfrak{C}}$. To this end, consider an arbitrary but fixed order of the productions from P and number them from 1 to $|P|$ in the following form

$$i\colon\ af(x_1, \ldots, x_m) \to \bar{q}(a_{1_1} x_1, \ldots, a_{1_{n_1}} x_1, \ldots, a_{m_1} x_m, \ldots, a_{m_{n_m}} x_m), \tag{39}$$

where

$$n_j \geqq 0,\ a_{j_k} \in A,\ (j \in [m],\ k \in [n_j]) \quad \text{and} \quad \bar{q} \in \hat{T}_G(X_n),$$

$(n = n_1 + \ldots + n_m)$. We mention that the symbols used in the specification of the i-th production depend on i. Now, for each $i \in [|P|]$ and $j \in [m]$ define u_j by

$$u_j = \begin{cases} n_j & \text{if} \quad n_j > 0 \\ 1 & \text{if} \quad n_j = 0 \end{cases}$$

and take a new operational symbol $f_i \notin F$ with arity $u = u_1 + \ldots + u_m$. Then construct the *DR* transducer $\mathfrak{B} = (F, A \cup \{b\}, F', P', a_0)$ where

(a) $F'=F\cup\{f_i|i\in|P|\}$;
(b) $b\notin A$ is a new state;
(c) P' is defined as follows: the rule

$$af(x_1,\ldots,x_m)\to f_i(b_{1_1}x_1,\ldots,b_{1_{u_1}}x_1,\ldots,b_{m_1}x_m,\ldots,b_{m_{u_m}}x_m) \quad (40)$$

is in P' iff the conditions

(i) the i-th production of P is of the form (39) and

(ii) $$b_{j_1},\ldots,b_{j_{u_j}}=\begin{cases}a_{j_1},\ldots,a_{j_{n_j}} & \text{if } n_j>0\\ b & \text{if } n_j=0\end{cases}\quad j\in[m]$$

hold, moreover the rule $bf(x_1,\ldots,x_m)\to f(bx_1,\ldots,bx_m)$ is in P' for each $m\geqq 0$, $f\in F_m$.

Next, introduce the H transducer $\mathfrak{C}=(F',c,G,P'',c)$ where the rule

$$cf_i(x_1,\ldots,x_u)\to\bar q(cx_1,\ldots,cx_{n_1},\ldots,cx_{u_1+\ldots+u_{m-1}+1},\ldots,cx_{u_1+\ldots+u_{m-1}+n_m}) \quad (41)$$

is in P'' iff the i-th production of P is (39), moreover, to make $\mathfrak{C}$ totally defined, let the rule $cf(x_1,\ldots,x_m)\to q$ be in P'' with an arbitrary $q\in T_G(\{c\}\times X_m)$ for each $m\geqq 0$, $f\in F_m$.

First note that $\mathfrak{B}$ is nondeleting since $u_j\geqq 1$, $(j\in[m])$, $\tau_{\mathfrak{B}(b)}|T_F=\iota_F$ and $\mathfrak{C}$ is linear. To prove $\tau_{\mathfrak{A}}=\tau_{\mathfrak{B}}\circ\tau_{\mathfrak{C}}$ it is enough to show that for each $a\in A$, $p\in T_F$ and $q\in T_G$ the equivalence

$$ap\overset{*}{\underset{\mathfrak{A}}{\Rightarrow}}q \quad (42)$$

if and only if

$$(\exists r\in T_{F'})(ap\overset{*}{\underset{\mathfrak{B}}{\Rightarrow}}r\wedge cr\overset{*}{\underset{\mathfrak{C}}{\Rightarrow}}q) \quad (43)$$

holds. We proceed by induction on $h(p)$.

If $h(p)=0$, that is $p=f\in F_0$, then $af\to q\in P$ iff there exists an $i\in[|P|]$ for which $af\to f_i\in P'$ and $cf_i\to q\in P''$. Now let $h(p)>0$, that is $p=f(p_1,\ldots,p_m)$, where $m>0$. Suppose that the production applied at the first step of (42) is (39). Then

$$a_{j_k}p_j\overset{*}{\underset{\mathfrak{A}}{\Rightarrow}}q_{j_k}\quad(j\in[m],\ k\in[n_j]) \quad (44)$$

under some $q_{j_k}\in T_G$ for which $q=\bar q(q_{1_1},\ldots,q_{1_{n_1}},\ldots,q_{m_1},\ldots,q_{m_{n_m}})$ holds. From here, by induction hypothesis we have

$$(\exists r'_{j_k}\in T_{F'})(a_{j_k}p_j\overset{*}{\underset{\mathfrak{B}}{\Rightarrow}}r'_{j_k}\wedge cr'_{j_k}\overset{*}{\underset{\mathfrak{C}}{\Rightarrow}}q_{j_k})\quad(j\in[m],\ k\in[n_j]) \quad (45)$$

moreover, by the construction of $\mathfrak{B}$ and $\mathfrak{C}$, (40) and (41) are in P' and P'', respectively. Letting

$$r_{j_1},\ldots,r_{j_{u_j}}=\begin{cases}r'_{j_1},\ldots,r'_{j_{n_j}} & \text{if } n_j>0\\ p_j & \text{if } n_j=0\end{cases}\quad j\in[m],$$

and taking into consideration that $\tau_{\mathfrak{B}(b)}|T_F=\iota_F$ we get that

$$b_{j_k}p_j\overset{*}{\underset{\mathfrak{B}}{\Rightarrow}}r_{j_k},\quad(j\in[m],\ k\in[u_j])\wedge cr_{j_k}\overset{*}{\underset{\mathfrak{C}}{\Rightarrow}}q_{j_k}\quad(j\in[m],\ k\in[n_j]) \quad (46)$$

from where (43) follows with

$$r = f_i(r_{1_1}, \ldots, r_{1_{u_1}}, \ldots, r_{m_1}, \ldots, r_{m_{u_m}}). \tag{47}$$

Conversely, suppose that r in (43) is of the form of (47). Then the productions used in the first step of the derivations of (43) are (40) and (41), respectively. Therefore (39) is in P. Moreover, (46) implies (44), by induction hypothesis, hence we have (42). □

We note that if $\mathfrak{A}$ is linear in the above lemma then so is $\mathfrak{B}$. Hence we also obtain:

Consequence 9.

$$\mathcal{LDR} \subseteq \mathcal{LNDR} \circ \mathcal{LH}. \tag{48}$$

Applying our last two results we have two further interesting identities.

Theorem 10. For each $n \geqq 2$

$$\mathcal{DR}^n = \mathcal{NDR} \circ \mathcal{LH} \tag{49}$$

$$\mathcal{LDR}^n = \mathcal{LNDR} \circ \mathcal{LH} \tag{50}$$

Proof. $\mathcal{DR}^n \overset{(34)}{=} \mathcal{LNDR} \circ \mathcal{DR} \overset{(38)}{\subseteq} \mathcal{LNDR} \circ \mathcal{NDR} \circ \mathcal{LH} \overset{(25)}{=} \mathcal{NDR} \circ \mathcal{LH}$ and in the same way we get (50). □

By the above results we can easily verify the equality

$$\mathcal{DR}^2 = \mathcal{NH} \circ \mathcal{LNDR} \circ \mathcal{LH}, \tag{51}$$

namely, we have $\mathcal{DR}^2 \overset{(49)}{=} \mathcal{NDR} \circ \mathcal{LH} \subseteq \mathcal{DR} \circ \mathcal{LH} \overset{(28)}{=} \mathcal{NH} \circ \mathcal{LDR} \circ \mathcal{LH} \overset{(48)}{\subseteq} \mathcal{NH} \circ \mathcal{LNDR} \circ \mathcal{LH} \circ \mathcal{LH} \overset{(17)}{=} \mathcal{NH} \circ \mathcal{LNDR} \circ \mathcal{LH} \subseteq \mathcal{DR}^3 \overset{(36)}{=} \mathcal{DR}^2$.

The equalities (49), (51), (32) are able to produce the class $\mathcal{DR}^2$ as a composition of two or three simpler classes of tree transformations. Using them we obtain some additional presentations for the class $\mathcal{DR}^2$ summarized by the following lemma.

Lemma 11.

(a) For any $\mathcal{X} \in \{\mathcal{NDR}, \mathcal{DR}\}$ and $\mathcal{Y} \in \{\mathcal{LH}, \mathcal{H}, \mathcal{LDR}, \mathcal{DR}\}$

$$\mathcal{X} \circ \mathcal{Y} = \mathcal{DR}^2 \tag{52}$$

(b) For any $\mathcal{X} \in \{\mathcal{NH}, \mathcal{NDR}\}$, $\mathcal{Y} \in \{\mathcal{LNDR}, \mathcal{NDR}, \mathcal{LDR}, \mathcal{DR}\}$ and $\mathcal{Z} \in \{\mathcal{LH}, \mathcal{H}, \mathcal{LNDR}, \mathcal{DR}\}$

$$\mathcal{X} \circ \mathcal{Y} \circ \mathcal{Z} = \mathcal{DR}^2 \tag{53}$$

(c) $\mathcal{LDR} \circ \mathcal{DR} = \mathcal{DR}^2$ (54)

(d) For arbitrary $\mathcal{X} \in \{\mathcal{LNDR}, \mathcal{LDR}\}$ and $\mathcal{Y} \in \{\mathcal{LH}, \mathcal{LDR}\}$

$$\mathcal{X} \circ \mathcal{Y} = \mathcal{LDR}^2 \tag{55}$$

Proof. We prove the case (a) only, since (b), (c) and (d) can be verified in the same way applying (51), (32) and (50), respectively.

$$\mathcal{DR}^2 \overset{(49)}{=} \mathcal{NDR} \circ \mathcal{LH} \subseteq \mathcal{X} \circ \mathcal{Y} \subseteq \mathcal{DR}^2. \quad \square$$

3. On mixed composition of different subclasses

We now investigate the set of all classes of tree transformations being a composition of finitely many ones introduced in Section 1. To be more precise we need some further notions and notations. Let $S=\{\mathcal{DR}, \mathcal{NDR}, \mathcal{LDR}, \mathcal{LNDR}, \mathcal{H}, \mathcal{NH}, \mathcal{LH}\}$ and denote by $[S]$ the set of all classes of tree transformations generated by S with composition, that is

$$[S] = \{\mathcal{K}_1\circ\ldots\circ\mathcal{K}_n | n \geqq 1, \quad \mathcal{K}_i\in S\}.$$

One of the most important questions concerning $[S]$ is that whether $[S]$ is infinite. We know, by consequence 7, that $\mathcal{C}\subseteq\mathcal{DR}^2$ for any $\mathcal{C}\in[S]$, however, in spite of this, $[S]$ may be infinite. In this paper we do not answer this question, instead, we present a theorem which, we hope, gives a deep insight into the structure of $[S]$.

First define the classes $\mathcal{C}_k$ for each $k\geqq 0$ as follows

(a) $\mathcal{C}_0 = \mathcal{LNDR}$

(b) $\mathcal{C}_{k+1} = \begin{cases} \mathcal{C}_k\circ\mathcal{NH} & \text{if} \quad k = 2m \\ \mathcal{C}_k\circ\mathcal{LNDR} & \text{if} \quad k = 2m+1. \end{cases} \quad (m \geqq 0)$

Moreover, we shall use Table 1 in the following sense. Each row and each column of the table is marked by a class of tree transformations. Their composition, in row-column order, is written in the corresponding square of the table. To get the depicted form of this composition, the equalities and inclusions the serial numbers of which appear in the lower part of the square can be used. If no serial number is indicated, then the form of the corresponding composition is meant by definition. For example,

$$\mathcal{LDR}^2\circ\mathcal{NH} \overset{(55)}{=} \mathcal{LNDR}\circ\mathcal{LH}\circ\mathcal{NH} \overset{(31)}{=} \mathcal{LNDR}\circ\mathcal{H}.$$

Now we can prove our last theorem.

Theorem 12. There are two finite subsets S_1 and S_2 of $[S]$ such that for any element $\mathcal{C}$ of $[S]$ one of the following conditions holds

(a) $\mathcal{C}\in S_1$,
(b) there exist a $\mathcal{C}'\in S_2$ and a $k\geqq 0$ such that $\mathcal{C}=\mathcal{C}'\circ\mathcal{C}_k$,
(c) $\mathcal{C}=\mathcal{C}_k$ for some $k\geqq 0$.

Proof. Define S_1 and S_2 by

$$S_1 = S\cup\{\mathcal{DR}^2, \mathcal{LDR}\circ\mathcal{NDR}, \mathcal{LDR}^2, \mathcal{LDR}\circ\mathcal{NH}, \mathcal{H}\circ\mathcal{NDR}, \mathcal{LDR}^2\circ\mathcal{NDR}, \mathcal{LNDR}\circ\mathcal{H}\}$$

and

$$S_2 = \{\mathcal{H}, \mathcal{NH}, \mathcal{LH}, \mathcal{LDR}\circ\mathcal{NH}, \mathcal{LNDR}\circ\mathcal{H}\}.$$

For any $\mathcal{C}\in[S]$ there exists a minimal number $n\geqq 1$ such that $\mathcal{C}=\mathcal{K}_1\circ\ldots\circ\mathcal{K}_n$ for some $\mathcal{K}_i\in S$. We prove the theorem by induction on this number n.

If $\mathcal{C}=\mathcal{K}_1$ for some $\mathcal{K}_1\in S$ then, by $S\subseteq S_1$, case (a) holds.

Now let $\mathcal{C}=\mathcal{K}_1\circ\ldots\circ\mathcal{K}_{n+1}$ under a minimal $n\geqq 1$ and some $\mathcal{K}_i\in S$. Then, since our theorem is supposed to hold for $\mathcal{K}_1\circ\ldots\circ\mathcal{K}_n$, three main cases are possible.

Table 1.

	$\mathcal{DR}$	$\mathcal{NDR}$	$\mathcal{LDR}$	$\mathcal{LNDR}$	$\mathcal{H}$	$\mathcal{NH}$	$\mathcal{LH}$
$\mathcal{DR}$	$\mathcal{DR}^2$	$\mathcal{DR}$ (18)	$\mathcal{DR}^2$ (52)	$\mathcal{DR}$ (19)	$\mathcal{DR}^2$ (52)	$\mathcal{DR}$ (20)	$\mathcal{DR}^2$ (52)
$\mathcal{NDR}$	$\mathcal{DR}^2$ (52)	$\mathcal{NDR}$ (21)	$\mathcal{DR}^2$ (52)	$\mathcal{NDR}$ (22)	$\mathcal{DR}^2$ (52)	$\mathcal{NDR}$ (23)	$\mathcal{DR}^2$ (52)
$\mathcal{LDR}$	$\mathcal{DR}^2$ (54)	$\mathcal{LDR} \circ \mathcal{NDR}$	$\mathcal{LDR}^2$	$\mathcal{LDR}$ (24)	$\mathcal{LNDR} \circ \mathcal{H}$ (48) (15)	$\mathcal{LDR} \circ \mathcal{NH}$	$\mathcal{LDR}^2$ (55)
$\mathcal{LNDR}$	$\mathcal{DR}^2$ (32)	$\mathcal{NDR}$ (25)	$\mathcal{LDR}^2$ (33)	$\mathcal{LNDR}$ (36)	$\mathcal{LNDR} \circ \mathcal{H}$	$\mathcal{C}_1$	$\mathcal{LDR}^2$ (55)
$\mathcal{H}$	$\mathcal{DR}$ (6)	$\mathcal{H} \circ \mathcal{NDR}$	$\mathcal{DR}$ (27)	$\mathcal{H} \circ \mathcal{C}_0$	$\mathcal{H}$ (11)	$\mathcal{H}$ (12)	$\mathcal{H}$ (13)
$\mathcal{NH}$	$\mathcal{DR}$ (8)	$\mathcal{NDR}$ (10)	$\mathcal{DR}$ (28)	$\mathcal{NH} \circ \mathcal{C}_0$	$\mathcal{H}$ (14)	$\mathcal{NH}$ (16)	$\mathcal{H}$ (29)
$\mathcal{LH}$	$\mathcal{DR}$ (7)	$\mathcal{H} \circ \mathcal{NDR}$ (10) (31)	$\mathcal{LDR}$ (9)	$\mathcal{LH} \circ \mathcal{C}_0$	$\mathcal{H}$ (15)	$\mathcal{H}$ (31)	$\mathcal{LH}$ (17)
$\mathcal{DR}^2$	$\mathcal{DR}^2$ (36)	$\mathcal{DR}^2$ (18)	$\mathcal{DR}^2$ (52) (36)	$\mathcal{DR}^2$ (19)	$\mathcal{DR}^2$ (52) (36)	$\mathcal{DR}^2$ (20)	$\mathcal{DR}^2$ (52) (36)
$\mathcal{LDR} \circ \mathcal{NDR}$	$\mathcal{DR}^2$ (52) (54) (36)	$\mathcal{LDR} \circ \mathcal{NDR}$ (21)	$\mathcal{DR}^2$ (52) (54) (36)	$\mathcal{LDR} \circ \mathcal{NDR}$ (22)	$\mathcal{DR}^2$ (52) (54) (36)	$\mathcal{LDR} \circ \mathcal{NDR}$ (23)	$\mathcal{DR}^2$ (52) (54) (36)
$\mathcal{LDR}^2$	$\mathcal{DR}^2$ (54) (36)	$\mathcal{LDR}^2 \circ \mathcal{NDR}$	$\mathcal{LDR}^2$ (37)	$\mathcal{LDR}^2$ (24)	$\mathcal{LNDR} \circ \mathcal{H}$ (55) (15)	$\mathcal{LNDR} \circ \mathcal{H}$ (55) (31)	$\mathcal{LDR}^2$ (55) (37)
$\mathcal{LDR} \circ \mathcal{NH}$	$\mathcal{DR}^2$ (8) (54)	$\mathcal{LDR} \circ \mathcal{NDR}$ (10)	$\mathcal{DR}^2$ (28) (54)	$\mathcal{LDR} \circ \mathcal{NH} \circ \mathcal{C}_0$	$\mathcal{LNDR} \circ \mathcal{H}$ (14)	$\mathcal{LDR} \circ \mathcal{NH}$ (16)	$\mathcal{LNDR} \circ \mathcal{H}$ (29) (48) (15)
$\mathcal{H} \circ \mathcal{NDR}$	$\mathcal{DR}^2$ (52) (6)	$\mathcal{H} \circ \mathcal{NDR}$ (21)	$\mathcal{DR}^2$ (52) (6)	$\mathcal{H} \circ \mathcal{NDR}$ (22)	$\mathcal{DR}^2$ (52) (6)	$\mathcal{H} \circ \mathcal{NDR}$ (23)	$\mathcal{DR}^2$ (52) (6)
$\mathcal{LDR}^2 \circ \mathcal{NDR}$	$\mathcal{DR}^2$ (52) (54) (36)	$\mathcal{LDR}^2 \circ \mathcal{NDR}$ (21)	$\mathcal{DR}^2$ (52) (54) (36)	$\mathcal{LDR}^2 \circ \mathcal{NDR}$ (22)	$\mathcal{DR}^2$ (52) (54) (36)	$\mathcal{LDR}^2 \circ \mathcal{NDR}$ (23)	$\mathcal{DR}^2$ (52) (54) (36)
$\mathcal{LNDR} \circ \mathcal{H}$	$\mathcal{DR}^2$ (6) (32)	$\mathcal{LDR}^2 \circ \mathcal{NDR}$ (31) (55) (10)	$\mathcal{DR}^2$ (27) (32)	$\mathcal{LNDR} \circ \mathcal{H} \circ \mathcal{C}_0$	$\mathcal{LNDR} \circ \mathcal{H}$ (11)	$\mathcal{LNDR} \circ \mathcal{H}$ (12)	$\mathcal{LNDR} \circ \mathcal{H}$ (13)

Case (a). There exists a $\mathscr{C}''\in S_1$ such that $\mathscr{K}_1\circ\ldots\circ\mathscr{K}_n=\mathscr{C}''$, thus $\mathscr{C}=\mathscr{C}''\circ\mathscr{K}_{n+1}$. Here $\mathscr{C}$ can be given in one of the following three forms, verifying our theorem:

(i) $\mathscr{C}=\mathscr{C}''\circ\mathscr{C}_0$ if $\mathscr{C}''\in S_2$ and $\mathscr{K}_{n+1}=\mathscr{L}\mathscr{N}\mathscr{D}\mathscr{R}$;

(ii) $\mathscr{C}=\mathscr{C}_1$ if $\mathscr{C}''=\mathscr{L}\mathscr{N}\mathscr{D}\mathscr{R}$ and $\mathscr{K}_{n+1}=\mathscr{N}\mathscr{H}$;

(iii) $\mathscr{C}\in S_1$ in any other cases, by Table 1.

Case (b). There exist $\mathscr{C}''\in S_2$ and $k\geqq 0$ for which

$$\mathscr{K}_1\circ\ldots\circ\mathscr{K}_n=\mathscr{C}''\circ\mathscr{C}_k,\quad \text{so}\quad \mathscr{C}=\mathscr{C}''\circ\mathscr{C}_k\circ\mathscr{K}_{n+1}.$$

Now seven subcases detailed from (i) to (vii) can be raised proving again our theorem.

(i) $\mathscr{C}=\mathscr{D}\mathscr{R}^2$ if $\mathscr{K}_{n+1}=\mathscr{D}\mathscr{R}$, by (32);

(ii) $\mathscr{C}=\mathscr{C}''\circ\mathscr{N}\mathscr{D}\mathscr{R}\in S_1$ if $\mathscr{K}_{n+1}=\mathscr{N}\mathscr{D}\mathscr{R}$, by (10), (25) and table 1;

(iii) $\mathscr{C}=\begin{cases}\mathscr{L}\mathscr{D}\mathscr{R}^2 & \text{if } \mathscr{K}_{n+1}=\mathscr{L}\mathscr{D}\mathscr{R},\ k=0 \text{ and } \mathscr{C}''=\mathscr{L}\mathscr{H}, \text{ by (33) and (9)}\\ \mathscr{D}\mathscr{R}^2 & \text{if } \mathscr{K}_{n+1}=\mathscr{L}\mathscr{D}\mathscr{R}, \text{ and } k\geqq 1 \text{ or } \mathscr{C}''\neq\mathscr{L}\mathscr{H} \text{ since in this case } \mathscr{D}\mathscr{R}^2\overset{(51)}{=}\mathscr{N}\mathscr{H}\circ\mathscr{L}\mathscr{N}\mathscr{D}\mathscr{R}\circ\mathscr{L}\mathscr{H}\subseteqq\\ & \subseteqq\mathscr{C}''\circ\mathscr{C}_k\circ\mathscr{K}_{n+1}\subseteqq\mathscr{D}\mathscr{R}^2;\end{cases}$

(iv) $\mathscr{C}=\begin{cases}\mathscr{C}''\circ\mathscr{C}_k & \text{if } \mathscr{K}_{n+1}=\mathscr{L}\mathscr{N}\mathscr{D}\mathscr{R} \text{ and } k=2m, \text{ by (26)}\\ \mathscr{C}''\circ\mathscr{C}_{k+1} & \text{if } \mathscr{K}_{n+1}=\mathscr{L}\mathscr{N}\mathscr{D}\mathscr{R} \text{ and } k=2m+1;\end{cases}\quad (m\geqq 0)$

(v) $\mathscr{C}=\begin{cases}\mathscr{D}\mathscr{R}^2 & \text{if } \mathscr{K}_{n+1}=\mathscr{H},\ \mathscr{C}''\neq\mathscr{L}\mathscr{H} \text{ or } k\geqq 2 \text{ because}\\ & \mathscr{D}\mathscr{R}^2\overset{(51)}{=}\mathscr{N}\mathscr{H}\circ\mathscr{L}\mathscr{N}\mathscr{D}\mathscr{R}\circ\mathscr{L}\mathscr{H}\subseteqq\mathscr{C}''\circ\mathscr{C}_k\circ\mathscr{K}_{n+1}\subseteqq\mathscr{D}\mathscr{R}^2\\ \mathscr{L}\mathscr{N}\mathscr{D}\mathscr{R}\circ\mathscr{H} & \text{if } \mathscr{K}_{n+1}=\mathscr{H},\ \mathscr{C}''=\mathscr{L}\mathscr{H} \text{ and } k=0,1 \text{ since}\\ & \text{in both cases } \mathscr{L}\mathscr{N}\mathscr{D}\mathscr{R}\circ\mathscr{H}\subseteqq\mathscr{C}''\circ\mathscr{C}_k\circ\mathscr{K}_{n+1}\overset{(14)}{=}\\ & \mathscr{L}\mathscr{H}\circ\mathscr{L}\mathscr{N}\mathscr{D}\mathscr{R}\circ\mathscr{H}\subseteqq\mathscr{L}\mathscr{D}\mathscr{R}^2\circ\mathscr{H}\overset{(55)}{=}\mathscr{L}\mathscr{N}\mathscr{D}\mathscr{R}\circ\\ & \circ\mathscr{L}\mathscr{H}\circ\mathscr{H}\overset{(15)}{=}\mathscr{L}\mathscr{N}\mathscr{D}\mathscr{R}\circ\mathscr{H};\end{cases}$

(vi) $\mathscr{C}=\begin{cases}\mathscr{C}''\circ\mathscr{C}_{k+1} & \text{if } \mathscr{K}_{n+1}=\mathscr{N}\mathscr{H} \text{ and } k=2m\\ \mathscr{C}''\circ\mathscr{C}_k & \text{if } \mathscr{K}_{n+1}=\mathscr{N}\mathscr{H} \text{ and } k=2m+1, \text{ by (16)};\end{cases}\quad (m\geqq 0)$

(vii) $\mathscr{C}=\begin{cases}\mathscr{D}\mathscr{R}^2 & \text{if } \mathscr{K}_{n+1}=\mathscr{L}\mathscr{H},\ \mathscr{C}''\neq\mathscr{L}\mathscr{H} \text{ or } k\geqq 2, \text{ similarly as in (v)}\\ \mathscr{L}\mathscr{D}\mathscr{R}^2 & \text{if } \mathscr{K}_{n+1}=\mathscr{L}\mathscr{H},\ \mathscr{C}''=\mathscr{L}\mathscr{H} \text{ and } k=0, \text{ by (50) and (9)}\\ \mathscr{L}\mathscr{N}\mathscr{D}\mathscr{R}\circ\mathscr{H} & \text{if } \mathscr{K}_{n+1}=\mathscr{L}\mathscr{H},\ \mathscr{C}''=\mathscr{L}\mathscr{H} \text{ and } k=1, \text{ see as in (v).}\end{cases}$

Case (c). $\mathscr{K}_1\circ\ldots\circ\mathscr{K}_n=\mathscr{C}_k$ for some $k\geqq 0$, so $\mathscr{C}=\mathscr{C}_k\circ\mathscr{K}_{n+1}$. This case can be handled similarly to the case (b), the detailed proof is omitted. □

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References

[1] BAKER, B. S., Composition of top-down and bottom-up tree transductions, Inf. and Control, v. 41, 1979, pp. 186—213.
[2] ENGELFRIET, J., Bottom-up and top-down tree transformations — A comparison, Math. Syst. Theory, v. 9, 1975, pp. 198—231.
[3] GÉCSEG, F. and M. STEINBY, Tree automata, Akadémiai Kiadó, Budapest, 1984.
[4] ROUNDS, W. C., Mappings and grammars on trees, Math. Syst. Theory, v. 4, 1970, pp. 257—287.

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