On isomorphic realization of automata with α_0-products

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1. Notions and notations

In this section we give a brief summary of some basic concepts to be used in the sequel.

An *automaton* is a triplet $\mathbf{A}=(A, X, \delta)$ with finite *state set* A, finite *input set* X and *transition* $\delta: A\times X\to A$. The sets A and X are nonempty. The transition is also treated in the extended sense, i.e., as a mapping $A\times X^*\to A$, where X^* is the free monoid generated by X. Take a word $p\in X^*$. The *transition induced by* p is the state map $\delta_p: A\to A$ with $\delta_p(a)=\delta(a, p)$ $(a\in A)$. The collection of these transitions forms a monoid $S(\mathbf{A})$ under composition of mappings. We call $S(\mathbf{A})$ the *characteristic monoid* of $\mathbf{A}$.

The concepts as *subautomaton, homomorphism, congruence relation* and *isomorphism* are used with their usual meaning. Given an automaton $\mathbf{A}=(A, X, \delta)$ and a state $a\in A$, the *subautomaton generated by* a has state set $\{\delta(a, p)|p\in X^*\}$. An automaton (B, Y, δ') is an *X-subautomaton* of an automaton (A, X, δ) if $B\subseteq A$, $Y\subseteq X$ and δ' is the restriction of δ to $B\times Y$. The *factor automaton* of an automaton $\mathbf{A}$ with respect to a congruence relation θ of $\mathbf{A}$ is denoted $\mathbf{A}/\theta$. We write $\theta_1<\theta_2$ to mean that θ_1 is a refinement of θ_2 and $\theta_1\neq\theta_2$. An automaton is called *simple* if it has only the trivial congruence relations ω (identity relation) and ι (total relation). Thus *trivial* (i.e., one-state) automata are simple.

Let $\mathbf{A}_i=(A_i, X_i, \delta_i)$ $(i=1, ..., n,\ n\geqq 0)$ be automata. Take a finite nonempty set X and a family of *feedback functions* $\varphi_i: A_1\times...\times A_n\times X\to X_i$ $(i=1, ..., n)$. By the *product* $\mathbf{A}_1\times...\times\mathbf{A}_n[X, \varphi]$ we mean the automaton $(A_1\times...\times A_n, X, \delta)$, where

$$\delta((a_1, ..., a_n), x) = (\delta_1(a_1, x_1), ..., \delta_n(a_n, x_n))$$

with

$$x_i = \varphi_i(a_1, ..., a_n, x) \quad (i = 1, ..., n)$$

for all $(a_1, ..., a_n)\in A_1\times...\times A_n$ and $x\in X$. The integer n is referred to as the length of the product. If, for every i, φ_i is independent of the state variables $a_i, ..., a_n$, we speak about an α_0*-product*. In an α_0-product a feedback function φ_i is alternatively treated as a mapping $A_1\times...\times A_{i-1}\times X\to X_i$. Moreover, φ_i extends to a mapping $A_1\times...\times A_{i-1}\times X^*\to X_i^*$ in a natural way.

Let $\mathcal{K}$ be a (possibly empty) class of automata. We will use the following notations:

$\mathbf{P}_{\alpha_0}(\mathcal{K})$:=all α_0-products of automata from $\mathcal{K}$;
$\mathbf{P}_{1\alpha_0}(\mathcal{K})$:=all α_0-products with length at most 1 of automata from $\mathcal{K}$;
$\mathbf{S}(\mathcal{K})$:=all subautomata of automata from $\mathcal{K}$;
$\mathbf{H}(\mathcal{K})$:=all homomorphic images of automata from $\mathcal{K}$;
$\mathbf{I}(\mathcal{K})$:=all isomorphic images of automata from $\mathcal{K}$;
$\mathcal{K}^*$:=the collection of all automata $\mathbf{A}=(A, X, \delta)$ such that there is an automaton $\mathbf{B}=(A, Y, \delta')\in\mathcal{K}$ with the following properties: (i) $\mathbf{B}$ is an X-subautomaton of $\mathbf{A}$; (ii) for every sign $x\in X$ there is a word $p\in Y^*$ inducing the same transition as p, i.e., $\delta'_p=\delta_x$. (Note that we have $S(\mathbf{A})=S(\mathbf{B})$.)

We call a class $\mathcal{K}$ of automata an *α_0-variety* if it is closed under $\mathbf{H}$, $\mathbf{S}$ and $\mathbf{P}_{\alpha_0}$. An α_0-variety is never empty. An *α_0^*-variety* is an α_0-variety $\mathcal{K}$ with $\mathcal{K}^*\subseteq\mathcal{K}$. For later use we note that $\mathbf{HSP}_{\alpha_0}(\mathcal{K})$ ($\mathbf{HSP}_{\alpha_0}(\mathcal{K}^*)$) is the smallest α_0-variety (α_0^*-variety) containing a class $\mathcal{K}$. Similarly, $\mathbf{ISP}_{\alpha_0}(\mathcal{K})$ is the smallest class containing $\mathcal{K}$ and closed under $\mathbf{I}$, $\mathbf{S}$ and $\mathbf{P}_{\alpha_0}$. It is worth noting that $\mathbf{SP}_{1\alpha_0}(\mathcal{K})$ contains all X-subautomata of automata in $\mathcal{K}$.

A class $\mathcal{K}_0$ is said to be *isomorphically α_0-complete* for $\mathcal{K}$ if $\mathcal{K}\subseteq\mathbf{ISP}_{\alpha_0}(\mathcal{K}_0)$. The following statement is a direct consequence of results in [5] (see also [3], [4]):

Proposition 1.1. If $\mathcal{K}_0$ is isomorphically α_0-complete for $\mathcal{K}$ and $\mathbf{A}\in\mathcal{K}$ is a simple automaton then $\mathbf{A}\in\mathbf{ISP}_{1\alpha_0}(\mathcal{K}_0)$.

Thus, any isomorphically α_0-complete class for $\mathcal{K}$ must "essentially" contain all simple automata in $\mathcal{K}$. The converse fails in general, yet it holds for some important classes: the class of all automata and the classes of permutation automata, monotone automata and definite automata are equally good examples (see [2], [3], [6], [7], [9]). Isomorphically α_0-complete classes for the class of all commutative automata essentially consist of automata very close to simple commutative automata (cf. [7]). In a sense there is a unique nontrivial simple nilpotent automaton. On the other hand no finite subclass of nilpotent automata is isomorphically α_0-complete for the class of all nilpotent automata. Thus, the class of nilpotent automata is a counterexample. Isomorphically α_0-complete classes for nilpotent automata are studied in [8].

Some more notation. The cardinality of a set A is denoted $|A|$. The symbol $\mathbf{E}$ denotes the automaton $(\{0, 1\}, \{x_0, x_1\}, \delta)$ with $\delta(0, x_0)=0$, $\delta(0, x_1)=\delta(1, x_0)=$ $=\delta(1, x_1)=1$. We call $\mathbf{E}$ the *elevator*.

The relation of the α_0-product to other product concepts is explained in [3]. The Krohn—Rhodes Decomposition Theorem gives a basis for studying α_0-products. For this, see [1], [3], [4].

2. Preliminary results

Let $\mathbf{A}=(A, X, \delta)$ be an automaton. As usual, we say that $\mathbf{A}$ is *strongly connected* if it is generated by any state $a\in A$. Further, $\mathbf{A}$ is called a *cone* if there is a state $a_0\in A$ with the following properties:

(i) $\delta(a_0, x)=a_0$, for all $x\in X$,
(ii) $A-\{a_0\}$ is nonempty and every state $a\in A-\{a_0\}$ generates $\mathbf{A}$.

Obviously, the state a_0 with the above properties is unique, whence it will be referred to as the *apex* of $\mathbf{A}$. The set $A-\{a_0\}$ constitutes the *base* of $\mathbf{A}$. It should be noted that every simple automaton is either a strongly connected automaton or a cone or an automaton $(\{a_1, a_2\}, X, \delta)$ with $\delta(a_i, x)=a_i$, $i=1, 2$, $x\in X$.

Theorem 2.1. Let $\mathcal{K}$ be a class of automata with $\mathbf{H}(\mathcal{K})\subseteq\mathcal{K}$, $\mathbf{S}(\mathcal{K})\subseteq\mathcal{K}$ and $\mathcal{K}^*\subseteq\mathcal{K}$. If $\mathbf{E}\in\mathcal{K}$ then for an arbitrary class $\mathcal{K}_0$, $\mathcal{K}\subseteq\mathbf{ISP}_{\alpha_0}(\mathcal{K}_0)$ if and only if every strongly connected automaton and every cone belonging to $\mathcal{K}$ is in $\mathbf{ISP}_{\alpha_0}(\mathcal{K}_0)$.

Proof. The necessity of the statement is trivial. For the sufficiency let $\mathbf{A}=(A, X, \delta)$ be an automaton in $\mathcal{K}$. We are going to apply induction on $|A|$ to show that $\mathbf{A}\in\mathbf{ISP}_{\alpha_0}(\mathcal{K}_0)$. Since $\mathcal{K}^*\subseteq\mathcal{K}$ and $\mathbf{ISP}_{\alpha_0}(\mathcal{K}_0)$ is closed under X-subautomata, it can be assumed that for every word $p\in X^*$ there is a sign $\bar{p}\in X$ inducing the same transition as p, i.e., $\delta(a, \bar{p})=\delta(a, p)$ for all $a\in A$.

If $|A|=1$ then $\mathbf{A}$ is strongly connected and $\mathbf{A}\in\mathbf{ISP}_{\alpha_0}(\mathcal{K}_0)$. Suppose that $|A|>1$. If $\mathbf{A}$ is strongly connected or a cone then $\mathbf{A}\in\mathbf{ISP}_{\alpha_0}(\mathcal{K}_0)$ by assumption. Otherwise two cases arise.

Case 1: $\mathbf{A}$ contains a nontrivial proper subautomaton $\mathbf{B}=(B, X, \delta)$ generated by a state $b_0\in B$. Let $\varrho\subseteq A\times A$ be the relation defined by $a\varrho b$ if and only if $a=b$ or $a, b\in B$. A straightforward computation proves that ϱ is a congruence relation of $\mathbf{A}$. For every state $b\in B$ fix an $x_b\in X$ with $\delta(b_0, x_b)=b$. Take the α_0-product

$$\mathbf{C} = (C, X, \delta') = \mathbf{A}/\varrho\times\mathbf{B}[X, \varphi],$$

where $\varphi_1(x)=x$,

$$\varphi_2(\{a\}, x) = \begin{cases} x_{b_0} & \text{if } \delta(a, x)\notin B, \\ x_b & \text{if } \delta(a, x) = b\in B \end{cases}$$

and $\varphi_2(B, x)=x$ for every $x\in X$ and $a\in A-B$. Set

$$C' = \{(\{a\}, b_0)\,|\,a\in A-B\}\cup\{(B, b)\,|\,b\in B\}.$$

It is immediately seen that $\mathbf{C}'=(C', X, \delta')$ is a subautomaton of $\mathbf{C}$ isomorphic to $\mathbf{A}$. Since both $\mathbf{A}/\varrho$ and $\mathbf{B}$ are in $\mathcal{K}$ and have fewer states than $\mathbf{A}$, we have $\mathbf{A}/\varrho$, $\mathbf{B}\in\mathbf{ISP}_{\alpha_0}(\mathcal{K}_0)$ from the induction hypothesis. The result follows by the fact that $\mathbf{ISP}_{\alpha_0}(\mathcal{K}_0)$ is closed under $\mathbf{I}$, $\mathbf{S}$ and $\mathbf{P}_{\alpha_0}$.

Case 2: There are distinct states $a_1, a_2\in A$ with $\delta(a_i, x)=a_i$, $i=1, 2$, $x\in X$. Define $\varrho\subseteq A\times A$ by $a\varrho b$ if and only if $a=b$ or $a, b\in\{a_1, a_2\}$. Again, ϱ is a congruence relation of $\mathbf{A}$. Let

$$\mathbf{C} = (C, X, \delta') = \mathbf{A}/\varrho\times\mathbf{E}[X, \varphi]$$

be the α_0-product with $\varphi_1(x)=x$,

$$\varphi_2(\{a\}, x) = \begin{cases} x_1 & \text{if } \delta(a, x) = a_2, \\ x_0 & \text{otherwise} \end{cases}$$

and $\varphi_2(\{a_1, a_2\}, x)=x_0$, where $x\in X$ and $a\in A-\{a_1, a_2\}$. It follows that $\mathbf{C}'=(C', X, \delta')$ with

$$C' = \{(\{a\}, 0)\,|\,a\in A-\{a_1, a_2\}\}\cup\{(\{a_1, a_2\}, 0), (\{a_1, a_2\}, 1)\}$$

is a subautomaton of **C** isomorphic to **A**. Since $\mathscr{K}$ is closed under homomorphic images and $\mathbf{A}/\varrho$ has fewer states than **A** we have $\mathbf{A}/\varrho\in\mathbf{ISP}_{\alpha_0}(\mathscr{K}_0)$ from the induction hypothesis. On the other hand, $\mathbf{E}\in\mathscr{K}$ and **E** is a cone. Thus $\mathbf{E}\in\mathbf{ISP}_{\alpha_0}(\mathscr{K}_0)$ and we conclude $\mathbf{A}\in\mathbf{ISP}_{\alpha_0}(\mathscr{K}_0)$.

Remark. Let $\mathscr{K}$ be a class as in Theorem 2.1, i.e. $\mathscr{K}^*\subseteq\mathscr{K}$, $\mathbf{H}(\mathscr{K})\subseteq\mathscr{K}$ and $\mathbf{S}(\mathscr{K})\subseteq\mathscr{K}$. Assuming $\mathbf{E}\notin\mathscr{K}$ it follows that $\mathscr{K}$ consists of permutation automata. (See the last section for the definition of permutation automata.) Every permutation automaton is the disjoint sum of strongly connected permutation automata. Now obviously, if $\mathscr{K}$ contains a nontrivial strongly connected automaton then $\mathscr{K}\subseteq\mathbf{ISP}_{\alpha_0}(\mathscr{K}_0)$ for a class $\mathscr{K}_0$ if and only if $\mathbf{A}\in\mathbf{ISP}_{\alpha_0}(\mathscr{K}_0)$ for every strongly connected permutation automaton $\mathbf{A}\in\mathscr{K}$. (Or even, the same holds if α_0-product is replaced by the so-called quasi-direct product.) If in addition $\mathscr{K}$ is closed under X-subautomata then, as we shall see later, $\mathscr{K}\subseteq\mathbf{ISP}_{\alpha_0}(\mathscr{K}_0)$ if and only if every simple strongly connected permutation automaton in $\mathscr{K}$ is already contained by $\mathbf{ISP}_{1\alpha_0}(\mathscr{K}_0)$. Suppose now that every strongly connected automaton in $\mathscr{K}$ is trivial. Then, if $\mathscr{K}$ contains a nontrivial automaton, we have $\mathscr{K}\subseteq\mathbf{ISP}_{\alpha_0}(\mathscr{K}_0)$ if and only if $(\{0,1\},\{x\},\delta)\in\mathbf{ISP}_{1\alpha_0}(\mathscr{K}_0)$ with $\delta(0,x)=0$ and $\delta(1,x)=1$. Further, $\mathscr{K}\subseteq\mathbf{ISP}_{\alpha_0}(\mathscr{K}_0)$ holds for every $\mathscr{K}_0$ if $\mathscr{K}$ consists of trivial automata.

The following two lemmas establish some simple facts about homomorphic realization of cones and strongly connected automata in the presence of **E**.

Lemma 2.2. Let $\mathbf{A}=(A,X,\delta)$ be a cone in $\mathbf{HSP}_{\alpha_0}(\mathscr{K}\cup\{\mathbf{E}\})$. There exist an automaton $\mathbf{D}\in\mathbf{P}_{\alpha_0}(\mathscr{K})$ and an α_0-product $\mathbf{D}\times\mathbf{E}[X,\varphi]$ containing a subautomaton that can be mapped homomorphically onto **A**.

Proof. Let $\mathbf{B}=(B,X,\delta')=\mathbf{B}_1\times\ldots\times\mathbf{B}_n[X,\psi]$ be an α_0-product with $\mathbf{B}_t\in\mathscr{K}\cup\cup\{\mathbf{E}\}$, $t=1,\ldots,n$. Let $\mathbf{C}=(C,X,\delta')$ be a subautomaton of **B** and $h\colon C\to A$ a homomorphism of **C** onto **A**. We may assume **C** to be in a sense minimal: no proper subautomaton of **C** is mapped homomorphically onto **A**.

Denote by a_0 the apex and by A_0 the base of **A**. Set $C_0=h^{-1}(A_0)$, $C_1=h^{-1}(\{a_0\})$. Clearly then $\mathbf{C}_1=(C,X,\delta')$ is a subautomaton of **C**, and **C** is generated by any state $a\in C_0$.

Let $1\leqq i_1<\ldots<i_r\leqq n$ be all the indices $t=1,\ldots,n$ with $\mathbf{B}_t\in\mathscr{K}$. If $(a_1,\ldots,a_n),(b_1,\ldots,b_n)\in C_0$, we have $a_t=b_t$ whenever $t\notin\{i_1,\ldots,i_r\}$ for otherwise **C** would not be generated by every state in C_0. Let $j_1,\ldots,j_s\in\{1,\ldots,n\}-\{i_1,\ldots,i_r\}$ be those indices t such that for any $(a_1,\ldots,a_n)\in C_0$, $a_t=0$ if and only if $t\in\{j_1,\ldots,j_s\}$. For every $a=(a_1,\ldots,a_r)\in B_{i_1}\times\ldots\times B_{i_r}$ put $\bar a=(\bar a_1,\ldots,\bar a_n)\in B$ with $\bar a_{i_1}=a_1,\ldots,\bar a_{i_r}=a_r$, $\bar a_{j_1}=\ldots=\bar a_{j_s}=0$ and $\bar a_t=1$ otherwise.

To end the proof we give an α_0-product $\mathbf{B}'=\mathbf{B}_{i_1}\times\ldots\times\mathbf{B}_{i_r}\times\mathbf{E}[X,\psi']$ and a subautomaton $\mathbf{C}'=(C',X,\delta'')$ of $\mathbf{B}'$ such that **A** is a homomorphic image of $\mathbf{C}'$. For every $a\in B_{i_1}\times\ldots\times B_{i_r}$, $i=0,1$, $x\in X$ and $j=1,\ldots,r$, define

$$\psi'_j(a,i,x)=\psi_{i_j}(\bar a,x),$$

$$\psi'_{r+1}(a,i,x)=\begin{cases}x_1 & \text{if } \delta'(\bar a,x)\in C_1,\\ x_0 & \text{otherwise.}\end{cases}$$

Let $\mathbf{C}'$ be the subautomaton generated by the set

$$C_0' = \{(a, 0) \mid a \in B_{i_1} \times \ldots \times B_{i_r},\ \bar{a} \in C_0\}.$$

Set $C_1' = C' - C_0'$. It is clear from the construction that states in C_1' have 1 as their last components. Therefore, C_1' is the state set of a subautomaton of $\mathbf{C}'$. Moreover, for every $(a, 0), (b, 0) \in C_0'$ and $x \in X$ we have $\delta''((a, 0), x) = (b, 0)$ if and only if $\delta'(\bar{a}, x) = \bar{b}$, while $\delta''((a, 0), x) \in C_1'$ if and only if $\delta'(\bar{a}, x) \in C_1$. It follows that $\mathbf{A}$ is a homomorphic image of $\mathbf{C}'$, a homomorphism being the map that takes each state in C_1' to a_0 and each state $(a, 0) \in C_0'$ to $h(\bar{a})$.

If $\mathbf{A}$ were strongly connected we would not need the last factor of the α_0-product $\mathbf{B}'$ either. This gives the following:

Lemma 2.3. Every strongly connected automaton in $\mathbf{HSP}_{\alpha_0}(\mathscr{K} \cup \{\mathbf{E}\})$ is contained in $\mathbf{HSP}_{\alpha_0}(\mathscr{K})$.

Let $\mathbf{A} = (A, X, \delta)$ be a cone with apex a_0 and base A_0. Suppose that the relation $\varrho \subseteq A \times A$ defined by $a \varrho b$ if and only if $a = b = a_0$ or $a, b \in A_0$ is a congruence relation of $\mathbf{A}$, which is to say that for every $x \in X$ either $\delta(A_0, x) \subseteq A_0$ or $\delta(A_0, x) = \{a_0\}$. Set $X_0 = \{x \in X \mid \delta(A_0, x) \subseteq A_0\}$. Assuming $X_0 \neq \emptyset$, the automaton $\mathbf{A}_0 = (A_0, X_0, \delta)$ is a strongly connected X-subautomaton of $\mathbf{A}$, which is guaranteed if $|A_0| > 1$. By definition, we call $\mathbf{A}$ a 0-simple cone if and only if $X_0 \neq \emptyset$ and $\mathbf{A}_0$ is simple. Thus, $\mathbf{E}$ is both a simple cone and a 0-simple cone. Given a strongly connected automaton $\mathbf{A}_0 = (A_0, X_0, \delta_0)$, there is a natural way to imbed $\mathbf{A}_0$ into a 0-simple cone $\mathbf{A}_0^c$: define $\mathbf{A}_0^c = (A \cup \{a_0\}, X_0 \cup \{x_0\}, \delta)$ where $a_0 \notin A_0$, $x_0 \notin X_0$, $\delta(a, x_0) = a_0$ for every $a \in A_0 \cup \{a_0\}$ and $\delta(a_0, x) = a_0$, $\delta(a, x) = \delta_0(a, x)$ if $a \in A_0$, $x \in X_0$. Obviously, $\mathbf{A}_0^c$ is 0-simple if and only if $\mathbf{A}_0$ is simple.

If $\mathbf{A}$ is a simple cone (i.e., a simple automaton that is a cone) then $\mathbf{A} \in \mathbf{ISP}_{\alpha_0}(\mathscr{K})$ for a class $\mathscr{K}$ if and only if $\mathbf{A} \in \mathbf{ISP}_{1\alpha_0}(\mathscr{K})$. In the next statement we investigate what can be said about $\mathscr{K}$ if $\mathbf{ISP}_{\alpha_0}(\mathscr{K})$ contains a 0-simple cone.

Lemma 2.4. If a 0-simple cone $\mathbf{A} = \mathbf{A}_0^c$ is in $\mathbf{ISP}_{\alpha_0}(\mathscr{K})$ then either $\mathbf{A} \in \mathbf{ISP}_{1\alpha_0}(\mathscr{K})$ or $\mathbf{E} \in \mathbf{ISP}_{1\alpha_0}(\mathscr{K})$ and there is an automaton $\mathbf{D} \in \mathscr{K}$ such that $\mathbf{A}$ is isomorphic to a subautomaton of an α_0-product of $\mathbf{E}$ with $\mathbf{D}$.

Proof. Let $\mathbf{A}_0 = (A_0, X_0, \delta_0)$ and $\mathbf{A} = (A, X, \delta)$ so that $A = A_0 \cup \{a_0\}$, $X = X_0 \cup \{x_0\}$ where $a_0 \notin A_0$, $x_0 \notin X_0$, $\delta(a, x_0) = a_0$ $(a \in A)$, $\delta(a_0, x) = a_0$ and $\delta(a, x) = \delta_0(a, x)$ $(a \in A_0, x \in X_0)$. Since $\mathbf{A} \in \mathbf{ISP}_{\alpha_0}(\mathscr{K})$ there exist an α_0-product $\mathbf{B} = (B, X, \delta') = \mathbf{B}_1 \times \ldots \times \mathbf{B}_n[X, \varphi]$ $(\mathbf{B}_t \in \mathscr{K},\ t = 1, \ldots, n)$ and a subautomaton $\mathbf{C} = (C, X, \delta')$ of $\mathbf{B}$ such that $\mathbf{A}$ is isomorphic to $\mathbf{C}$ under a mapping $h\colon A \to C$. We may assume that n is minimal, i.e., whenever an α_0-product of automata from $\mathscr{K}$ contains a subautomaton isomorphic to $\mathbf{A}$, the length of that product is at least n.

Suppose that $\mathbf{A} \notin \mathbf{ISP}_{1\alpha_0}(\mathscr{K})$. We then have $n > 1$. Let $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ be arbitrary states in C. For every $t = 1, \ldots, n$, put $a \theta_t b$ if and only if $a_1 = b_1, \ldots, a_t = b_t$. Further, let $a \varrho b$ if and only if $a = b = h(a_0)$ or $a, b \in h(A_0)$. Each of these relations is a congruence relation of $\mathbf{C}$, and since n is minimal, $\theta_1 > \ldots > \theta_n (= \omega)$ and $\theta_1 \neq \iota$. Since $\mathbf{A}$ is 0-simple this leaves $n = 2$, $\theta_1 = \varrho$ and $\theta_2 = \omega$. It then follows that $\mathbf{E}$ is isomorphic to a subautomaton of an α_0-product of $\mathbf{B}_1$ with a single factor and $\mathbf{A}$ is isomorphic to a subautomaton of an α_0-product of $\mathbf{E}$ with $\mathbf{B}_2$.

Let $\mathbf{A}_0^c=(A_0\cup\{a_0\}, X_0\cup\{x_0\}, \delta)$ be a 0-simple cone with $\mathbf{A}_0=(A_0, X_0, \delta_0)$, and take an arbitrary automaton $\mathbf{B}=(B, Y, \delta')$. It is not difficult to give a necessary and sufficient condition ensuring that $\mathbf{A}_0^c$ is isomorphic to an α_0-product of $\mathbf{E}$ with $\mathbf{B}$. Clearly this can happen if and only if there are a pair of functions $h\colon A_0\to B$, $\varphi\colon X_0\to Y$, a state $b_0\in B$ and two not necessarily distinct signs $y_0, y_1\in Y$ such that:

(i) h is injective;

(ii) for every $a_1, a_2\in A_0$ and $x\in X_0$ we have $\delta_0(a_1, x)=a_2$ if and only if $\delta'(h(a_1), \varphi(x))=h(a_2)$;

(iii) $\delta'(h(A_0), y_0)=\{b_0\}$, $\delta'(b_0, y_1)=b_0$.

If also $b_0\notin h(A_0)$ and $y_0=y_1$ then $\mathbf{A}_0^c$ is isomorphic to an α_0-product of $\mathbf{B}$ with a single factor.

3. The main result

An automaton $\mathbf{A}=(A, X, \delta)$ is called *permutation automaton* if δ_x is a permutation of the state set for every $x\in X$. This is equivalent to saying that δ_p is a permutation for every $p\in X^*$ or that $S(\mathbf{A})$ is a group. Let $\mathcal{K}_p$ denote the class of all permutation automata. It is known that $\mathcal{K}_p$ is an α_0^*-variety, see [1]. Moreover, from the Krohn—Rhodes Decomposition Theorem we have $\mathcal{K}_p=\mathbf{HSP}_{\alpha_0}(\{\mathbf{A}(G)|G$ is a simple group$\})$ where the group-like automaton $\mathbf{A}(G)$ on a (finite) group G is defined to be the automaton (G, G, δ) with $\delta(g, h)=gh$, $g, h\in G$.

Another class of automata we shall be dealing with is the class $\mathcal{K}_m$ of all monotone automata. By definition, an automaton $\mathbf{A}=(A, X, \delta)$ is *monotone* if $\delta(a, pq)=a$ implies $\delta(a, p)=a$, for all $a\in A$ and $p, q\in X^*$. This is equivalent to requiring the existence of an ordering $\leqq$ on A such that $a\leqq\delta(a, p)$ for all $a\in A$ and $p\in X^*$ (or $a\leqq\delta(a, x)$ for all $a\in A$ and $x\in X$). The class $\mathcal{K}_m$ is known to be an α_0^*-variety. Further, it is the α_0-variety generated by $\mathbf{E}$, i.e. $\mathcal{K}_m=\mathbf{HSP}_{\alpha_0}(\{\mathbf{E}\})$ (see [1], [10], [11]).

Having defined the classes $\mathcal{K}_p$ and $\mathcal{K}_m$, put $\mathcal{K}_{pm}=\mathbf{HSP}_{\alpha_0}(\mathcal{K}_p\cup\mathcal{K}_m)=\mathbf{HSP}_{\alpha_0}(\mathcal{K}_p\cup\{\mathbf{E}\})=\mathbf{HSP}_{\alpha_0}(\{\mathbf{A}(G)|G$ is a simple group$\}\cup\{\mathbf{E}\})$. It follows from Stiffler's switching rules that $\mathbf{A}\in\mathcal{K}_{pm}$ if and only if there is an α_0-product $\mathbf{B}$ of a permutation automaton with a monotone automaton such that $\mathbf{A}\in\mathbf{HS}(\{\mathbf{B}\})$. For this and other characterizations of the class $\mathcal{K}_{pm}$, see [1] and [10]. It is immediate from our definition that $\mathcal{K}_{pm}$ is an α_0-variety. Or even, it is an α_0^*-variety.

Lemma 3.1. Let $\mathbf{A}$ be a strongly connected automaton. Then $\mathbf{A}\in\mathcal{K}_{pm}$ if and only if $\mathbf{A}\in\mathcal{K}_p$.

Proof. Use Lemma 2.3.

Corollary. If $\mathbf{A}=\mathbf{A}_0^c$ is a cone in $\mathcal{K}_{pm}$ then $\mathbf{A}_0$ a strongly connected permutation automaton.

Lemma 3.2. Let $\mathbf{A}=(A, X, \delta)\in\mathcal{K}_{pm}$ be a cone with apex a_0 and base A_0. If $\delta(a, p)=\delta(b, p)\in A_0$ holds for some $a, b\in A_0$ and $p\in X^*$ then $a=b$.

Proof. From Lemma 2.2 it follows that $\mathbf{A}$ is a homomorphic image of a subautomaton $\mathbf{C}=(C, X, \delta')$ of an α_0-product $\mathbf{B}\times\mathbf{E}[X, \varphi]$ where $\mathbf{B}$ is a permutation automaton, say $\mathbf{B}=(B, X_1, \delta_1)$. Denote by h an onto homomorphism $\mathbf{C}\to\mathbf{A}$. Set $C_0=h^{-1}(A_0)$. We may assume that every state in C_0 is a generator of $\mathbf{C}$. Each state in C_0 must have 0 as its second component since otherwise we would have $C\subseteq B\times\{1\}$, and this would yield that $\mathbf{C}$ and $\mathbf{A}$ are permutation automata.

Let $(a_1, 0), (b_1, 0)\in C_0$ with $h(a_1, 0)=a, h(b_1, 0)=b$. Take a word $q\in X^*$ with $\delta(a, pq)=a$. We have $\delta(a, (pq)^n)=\delta(b, (pq)^n)=a$, and hence $\delta'((a_1, 0), (pq)^n)$, $\delta'((b_1, 0), (pq)^n)\in C_0$, for all $n\geq 1$. Define $r=\varphi_1(pq)$. For every integer $n\geq 1$ we have $\delta'((a_1, 0), (pq)^n)=(\delta_1(a_1, r^n), 0)$ and $\delta'((b_1, 0), (pq)^n)=(\delta_1(b_1, r^n), 0)$, Since $\mathbf{B}$ is a permutation automaton, there is an $n\geq 1$ with $a_1=\delta_1(a_1, r^n)$ and $b_1=\delta_1(b_1, r^n)$. Thus we obtain $a=h(a_1, 0)=h(\delta'((a_1, 0), (pq)^n))=h(\delta'((b_1, 0), (pq)^n))=h(b_1, 0)=b$.

Theorem 3.3. Let $\mathcal{K}\subseteq\mathcal{K}_{pm}$ be a class containing $\mathbf{E}$, closed under X-subautomata and homomorphic images and such that $\mathcal{K}^*\subseteq\mathcal{K}$. A class $\mathcal{K}_0$ is isomorphically α_0-complete for $\mathcal{K}$ if and only if the following conditions hold:

(i) every simple cone and every simple strongly connected permutation automaton belonging to $\mathcal{K}$ is in $\mathbf{ISP}_{1\alpha_0}(\mathcal{K}_0)$,

(ii) for every 0-simple cone $\mathbf{A}_0^c\in\mathcal{K}$ there is a $\mathbf{B}\in\mathcal{K}_0$ such that $\mathbf{A}_0^c$ is isomorphic to a subautomaton of an α_0-product of $\mathbf{E}$ with $\mathbf{B}$.

Proof. The necessity of (i) comes from Proposition 1.1 while (ii) is necessary in virtue of Lemma 2.4.

For the converse recall that $\mathcal{K}$ satisfies the assumptions of Theorem 2.1. Therefore, by Theorem 2.1, it suffices to show that every strongly connected automaton and every cone belonging to $\mathcal{K}$ is contained by $\mathbf{ISP}_{\alpha_0}(\mathcal{K}_0)$.

Let $\mathbf{A}=(A, X, \delta)\in\mathcal{K}$ be a cone with base A_0 and apex a_0. Since $\mathcal{K}^*\subseteq\mathcal{K}$ and $\mathbf{ISP}_{\alpha_0}(\mathcal{K}_0)$ is closed under X-subautomata, we may assume that for every $p\in X^*$ there is a $\bar{p}\in X$ inducing the same transition as p. If $\mathbf{A}$ is simple then $\mathbf{A}\in\mathbf{ISP}_{\alpha_0}(\mathcal{K}_0)$ by (i). If $\mathbf{A}$ is 0-simple then $\mathbf{A}$ is isomorphic to an α_0-product $\mathbf{A}_0^c[X, \varphi]$ with a single factor where $\mathbf{A}_0^c\in\mathcal{K}$ is a 0-simple cone. (Recall that $\mathcal{K}$ is closed under X-subautomata.) Therefore, we may assume that $\mathbf{A}$ is of the form $\mathbf{A}_0^c$. Now, by (ii), $\mathbf{A}$ is isomorphic to a subautomaton of an α_0-product of $\mathbf{E}$ with $\mathbf{B}$ where $\mathbf{B}\in\mathcal{K}_0$. Since $\mathbf{E}$ is a simple cone we have $\mathbf{E}\in\mathbf{ISP}_{1\alpha_0}(\mathcal{K}_0)$. It follows that $\mathbf{A}\in\mathbf{ISP}_{\alpha_0}(\mathcal{K}_0)$. Suppose that $\mathbf{A}$ is neither simple nor 0-simple. We proceed by induction on $|A|$. If $|A|=2$ our statement holds vacantly. Let $|A|>2$. There exists a congruence relation $\theta\neq\omega$ of $\mathbf{A}$ such that $a\theta b$ implies $a=b$ or $a, b\in A_0$, and such that A_0 contains at least two blocks of the partition induced by θ.

Let $C_0=\{a_0\}, C_1, \dots, C_n$ $(n\geq 2, |C_1|>1)$ be the blocks of θ. Since $\mathbf{A}$ is generated by any state in A_0, from Lemma 3.2 we have the following: for every $i, j\in\{1, \dots, n\}$ there exists a word $p\in X^*$ with $\delta(C_i, p)=C_j$. Consequently, for every $i\in\{1, \dots, n\}$ there is a pair of words (p_i, q_i) with $\delta(C_1, p_i)=C_i$, $\delta(C_i, q_i)=C_1$ and such that p_iq_i induces the identity map on C_1 while q_ip_i induces the identity map on C_i.

Set $X'=\{x\in X|\delta(C_1, x)\subseteq C_0\cup C_1\}$, $\mathbf{C}=(C_0\cup C_1, X', \delta')$, where $\delta'(c, x)=\delta(c, x)$ for all $c\in C_0\cup C_1$ and $x\in X'$. Obviously, both $\mathbf{A}/\theta$ and $\mathbf{C}$ are cones in $\mathcal{K}$. Fix a sign $x_0\in X'$ with $\delta'(C_1, x_0)=C_0$. Take the α_0-product

$$\mathbf{B}=(B, X, \delta'')=\mathbf{A}/\theta\times\mathbf{C}[X, \varphi]$$

where $\varphi_1(x)=x$ and

$$\varphi_2(C_i, x) = \begin{cases} x_0 & \text{if } \delta(C_i, x) = C_0 \\ \overline{p_i x q_j} & \text{if } \delta(C_i, x) = C_j \quad \text{and} \quad i, j \neq 0. \end{cases}$$

It is easy to check that $\mathbf{B}'=(B', X, \delta'')$ is a subautomaton of $\mathbf{B}$ where

$$B' = \{(C_0, a_0)\} \cup \{(C_i, a) \mid i = 1, \ldots, n,\ a \in C_1\}.$$

Further, the map $(C_0, a_0) \mapsto a_0$, $(C_i, a) \mapsto \delta(a, p_i)$ $(i=1, \ldots, n,\ a \in C_1)$ is an isomorphism of $\mathbf{B}'$ onto $\mathbf{A}$. Hence the result follows from the induction hypothesis.

Suppose now that $\mathbf{A}=(A, X, \delta) \in \mathscr{K}$ is a strongly connected automaton. From Lemma 3.1 we know that $\mathbf{A}$ is a permutation automaton. Just as before, we may assume that for every $p \in X^*$ there is a sign $\bar{p} \in X$ with $\delta_p = \delta_{\bar{p}}$. If $\mathbf{A}$ is simple then $\mathbf{A} \in \mathbf{ISP}_{1\alpha_0}(\mathscr{K}) \subseteq \mathbf{ISP}_{\alpha_0}(\mathscr{K})$. Otherwise let θ be a congruence relation of $\mathbf{A}$ different from ω and ι. Denote by $C_1, \ldots, C_n$ $(n \geqq 2,\ |C_1| > 1)$ the blocks of the partition induced by θ. Set $X' = \{x \in X \mid \delta(C_1, x) = C_1\}$. One shows that $\mathbf{A}$ is isomorphic to an α_0-product of $\mathbf{A}/\theta$ with $\mathbf{C}$, where $\mathbf{C}=(C_1, X', \delta')$, $\delta'(c, x)=\delta(c, x)$ $(c \in C_1, x \in X')$.

We note that a substantial part of the above proof as well as the proofs of Theorem 2.1 and Lemma 2.2 follow well-known ideas (see [1], [4], [5]).

Corollary. Let $\mathscr{K} \subseteq \mathscr{K}_p$ be closed under X-subautomata and homomorphic images and suppose that $\mathscr{K}^* \subseteq \mathscr{K}$. If $\mathscr{K}$ contains a nontrivial strongly connected automaton then a class $\mathscr{K}_0$ is isomorphically α_0-complete for $\mathscr{K}$ if and only if $\mathbf{A} \in \mathbf{ISP}_{1\alpha_0}(\mathscr{K})$ holds for every simple strongly connected automaton $\mathbf{A}$ in $\mathscr{K}$.

Let $\mathscr{G}$ be a nonempty class of (finite) simple groups closed under division. (Recall that G_1 divides G_2 for groups G_1 and G_2, written $G_1 | G_2$, if and only if G_1 is a homomorphic image of a subgroup of G_2.) Denote by $\mathscr{K}(\mathscr{G})$ the class $\mathbf{HSP}_{\alpha_0}(\{\mathbf{A}(G) \mid G \in \mathscr{G}\})$; $\mathscr{K}(\mathscr{G})$ is an α_0^*-variety contained in $\mathscr{K}_p$. It follows from the Krohn—Rhodes Decomposition Theorem that every α_0^*-variety of permutation automata is of the form $\mathscr{K}(\mathscr{G})$ except for the α_0^*-variety consisting of all automata (A, X, δ) such that δ_x is the identity map for each $x \in X$. Moreover, if $\mathscr{G}$ contains a nontrivial simple group then for every permutation automaton $\mathbf{A}$ we have $\mathbf{A} \in \mathscr{K}(\mathscr{G})$ if and only if $G | S(\mathbf{A})$ implies $G \in \mathscr{G}$ for simple groups G. Since $\mathscr{K}(\mathscr{G}) \subseteq \mathscr{K}_p$, also $\mathscr{K}_m(\mathscr{G}) = \mathbf{HSP}_{\alpha_0}(\mathscr{K}(\mathscr{G}) \cup \mathscr{K}_m) \subseteq \mathscr{K}_{pm}$. We obviously have

$$\mathscr{K}_m(\mathscr{G}) = \mathbf{HSP}_{\alpha_0}(\mathscr{K}(\mathscr{G}) \cup \{\mathbf{E}\}) = \mathbf{HSP}_{\alpha_0}(\{\mathbf{A}(G) \mid G \in \mathscr{G}\} \cup \{\mathbf{E}\}).$$

Thus, $\mathscr{K}_m(\mathscr{G})$ is an α_0-variety in $\mathscr{K}_{pm}$, or even, it is an α_0^*-variety.

Corollary. $\mathscr{K}_m(\mathscr{G}) \subseteq \mathbf{ISP}_{\alpha_0}(\mathscr{K})$ if and only if the following hold:

(i) for every simple cone $\mathbf{A} \in \mathscr{K}_m(\mathscr{G})$ we have $\mathbf{A} \in \mathbf{ISP}_{1\alpha_0}(\mathscr{K}_0)$,

(ii) for every 0-simple cone $\mathbf{A}_0^c \in \mathscr{K}_m(\mathscr{G})$ there is a $\mathbf{B} \in \mathscr{K}_0$ such that $\mathbf{A}_0^c$ is isomorphic to a subautomaton of an α_0-product of $\mathbf{E}$ with $\mathbf{B}$.

Proof. Use Theorem 3.3 and the following fact: every simple strongly connected (permutation) automaton in $\mathscr{K}_m(\mathscr{G})$ is isomorphic to an X-subautomaton of a 0-simple cone $\mathbf{A}_0^c$ in $\mathscr{K}_m(\mathscr{G})$.

Corollary [2]. A class $\mathscr{K}_0$ is isomorphically α_0-complete for $\mathscr{K}_m$ if and only if $\mathbf{E} \in \mathbf{ISP}_{1\alpha_0}(\mathscr{K}_0)$.

Proof. Let $\mathcal{G}$ be the class of trivial groups. We have $\mathcal{K}_m=\mathcal{K}_m(\mathcal{G})$. On the other hand, every cone in $\mathcal{K}_m$ is similar to **E**. More exactly, if $\mathbf{A}\in\mathcal{K}_m$ is a cone then **A** is isomorphic to an α_0-product in $\mathbf{P}_{1\alpha_0}(\{\mathbf{E}\})$.

An automaton $\mathbf{A}=(A, X, \delta)$ is called *commutative* if $\delta(a, xy)=\delta(a, yx)$ for all $a\in A$ and $x, y\in X$, i.e., if $S(\mathbf{A})$ is commutative. Denote by $\mathcal{K}$ the class of all commutative automata; $\mathcal{K}$ is closed under X-subautomata and homomorphic images. Moreover, $\mathcal{K}^*\subseteq\mathcal{K}$ and $\mathcal{K}\subseteq\mathcal{K}_{pm}$. For a prime $p>1$ let $\mathbf{C}_p$ be a fixed automaton of the form $\mathbf{A}(Z_p)^c$, where Z_p is the cyclic group of order p. Every simple commutative automaton is in the class $\mathbf{ISP}_{1\alpha_0}(\{\mathbf{C}_p|p>1 \text{ is a prime}\})$, and every 0-simple commutative cone is in $\mathbf{ISP}_{1\alpha_0}(\{\mathbf{C}_p^c|p>1 \text{ is a prime}\})$.

Corollary [7]. *A class $\mathcal{K}_0$ is isomorphically α_0-complete for the class of all commutative automata if and only if the following hold:*

(i) $\mathbf{E}\in\mathbf{HSP}_{1\alpha_0}(\mathcal{K}_0)$,

(ii) *for every prime $p>1$ there is an $\mathbf{A}\in\mathcal{K}_0$ such that $\mathbf{C}_p^c$ is isomorphic to a subautomaton of an α_0-product of* **E** *with* **A**.

Abstract

Every isomorphically α_0-complete class for a class $\mathcal{K}$ of automata must essentially contain all simple automata belonging to $\mathcal{K}$. In this paper we present some classes $\mathcal{K}$ for which also the converse is true, or isomorphically α_0-complete classes can be characterized by means of automata in $\mathcal{K}$ close to simple automata.

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(Received Nov. 21, 1986)