On metric equivalence of v_i-products

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In [7] it is shown that the v_3-product is metrically equivalent to the product. Here we strengthen this result by proving that already the v_1-product is metrically equivalent to the general product. It is also obtained that, if a class $\mathcal{K}$ of automata is not metrically complete for the product, then $\mathbf{HSP}_g(\mathcal{K})=\mathbf{HSP}_{v_1}(\mathcal{K})$.

In this paper by an automaton we mean a finite automaton. The only exceptions are varieties of automata; they may contain automata with infinite state-sets. For all notions and notations not defined here, see [1], [7], [8] and [9].

We start with

Lemma 1. *If a finite class $\mathcal{K}$ of automata is not metrically complete for the product, then every finitely generated automaton $\mathfrak{A}=(X, A, \delta)$ from $\mathbf{HSPP}_{v_1}(\mathcal{K})$ is in $\mathbf{HSP}_{v_1}(\mathcal{K})$.*

Proof. First let us note that the concept of the v_1-product can be generalized in a natural way to products with infinitely many factors, and every automaton in $\mathbf{PP}_{v_1}(\mathcal{K})$ is a v_1-product with possibly infinitely many factors. Thus, take a v_1-product

$$\mathfrak{B} = (X, B, \delta') = \prod (\mathfrak{A}_i \mid i\in I)[X, \varphi, \gamma]$$

with $|X|=m$ and $\mathfrak{A}_i=(X_i, A_i, \delta_i)\in\mathcal{K}$ $(i\in I)$. Let $\{a_1, \ldots, a_n\}$ be a generating set of $\mathfrak{A}$. Suppose that a subautomaton of $\mathfrak{B}$ can be mapped homomorphically onto $\mathfrak{A}$, and let $\mathbf{b}_i$ be a counter image of a_i $(i=1, \ldots, n)$ under this homomorphism. Denote by $\mathfrak{B}'=(X, B', \delta'')$ the subautomaton of $\mathfrak{B}$ generated by $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$. Moreover, set $u=\max\{|A_i|\ i\in I\}$ and $v=|\mathcal{K}|$. Let $k\geqq 0$ be a fixed integer such that, for arbitrary $\mathfrak{C}=(X_{\mathfrak{C}}, C, \delta_{\mathfrak{C}})\in\mathcal{K}$, $c\in C$, $p\in X_{\mathfrak{C}}^*$ with $|p|\geqq k$ and $x_1, x_2\in X_{\mathfrak{C}}$, $cpx_1=cpx_2$. (Since $\mathcal{K}$ is not metrically complete, there exists such a k.) We shall show the existence of a v_1-product $\bar{\mathfrak{B}}=(X, \bar{B}, \bar{\delta})$ of automata from $\{\mathfrak{A}_i|i\in I\}$ with a number of factors not exceeding vu^{nt}, where $t=\dfrac{m^{k+1}-1}{m-1}$ if $m>1$, and $t=k+1$ for $m=1$, such that a subautomaton $\bar{\mathfrak{B}}'=(X, \bar{B}', \bar{\delta}')$ of $\bar{\mathfrak{B}}$ is isomorphic to $\mathfrak{B}'$.

Define the binary relation ϱ on I in the following way: $i\equiv j(\varrho)$ $(i, j\in I)$ if and only if $\mathfrak{A}_i=\mathfrak{A}_j$ and $\delta_i(\mathrm{pr}_i(\mathbf{b}_r), \varphi_i(\mathbf{b}_r, p))=\delta_j(\mathrm{pr}_j(\mathbf{b}_r), \varphi_j(\mathbf{b}_r, p))$ hold for arbitrary r $(1\leqq r\leqq n)$ and $p\in X^*$ with $|p|\leqq k$. By the choice of k, $\delta_i(\mathrm{pr}_i(\mathbf{b}_r), \varphi_i(\mathbf{b}_r, q))=$ $=\delta_j(\mathrm{pr}_j(\mathbf{b}_r), \varphi_j(\mathbf{b}_r, q))$ is valid for any r $(1\leqq r\leqq n)$ and $q\in X^*$. Moreover, since

t is the number of words over X with length less than or equal to k, we have at most vu^{nt} ϱ-classes. From every ϱ-class take exactly one element, and let $\{i_1, \ldots, i_l\}$ be their set. Form the v_1-product

$$\bar{\mathfrak{B}} = (X, \bar{B}, \bar{\delta}) = \prod (\mathfrak{A}_{i_j} | j = 1, \ldots, l)[X, \varphi', \gamma']$$

in the following way:

(i) For every j $(1 \leqq j \leqq l$, $\gamma'(i_j) = \emptyset$ if $\gamma(i_j) = \emptyset$, and $\gamma'(i_j) = \{i_{j_1}\}$ $(j_1 \in \{1, \ldots, l\})$ if $\gamma(i_j) = \{j_2\}$ and $i_{j_1} \equiv j_2(\varrho)$.

(ii) For every j $(1 \leqq j \leqq l)$ and $x \in X$, $\varphi'_{i_j}(x) = \varphi_{i_j}(x)$ if $\gamma'(i_j) = \emptyset$.

(iii) For every j $(1 \leqq j \leqq l)$, if $\gamma'(i_j) = \{i_{j_1}\}$, then $\varphi'_{i_j}(a, x) = \varphi_{i_j}(a, x)$ $(a \in A_{i_{j_1}}$, $x \in X)$.

Moreover, let $\bar{\mathbf{b}}_i$ $(i = 1, \ldots, n)$ be those states of $\bar{\mathfrak{B}}$ which, for every $j (= 1, \ldots, l)$, satisfy the equality $\mathrm{pr}_{i_j}(\bar{\mathbf{b}}_i) = \mathrm{pr}_{i_j}(\mathbf{b}_i)$. Denote by $\bar{\mathfrak{B}}' = (X, \bar{B}', \bar{\delta}')$ the subautomaton of $\bar{\mathfrak{B}}$ generated by $\{\bar{\mathbf{b}}_1, \ldots, \bar{\mathbf{b}}_n\}$. Moreover, consider the mapping $\psi\colon B' \to \bar{B}'$ given by $\psi(\mathbf{b}_i p) = \bar{\mathbf{b}}_i p$ $(p \in X^*, i = 1, \ldots, n)$. Clearly, ψ is an isomorphism of $\mathfrak{B}'$ onto $\bar{\mathfrak{B}}'$. □

Lemma 2. If a finite class $\mathcal{K}$ of automata is not metrically complete for the product, then the equality $\mathbf{HSPP}_g(\mathcal{K}) = \mathbf{HSPP}_{v_1}(\mathcal{K})$ holds.

Proof. Obviously, $\mathbf{HSPP}_{v_1}(\mathcal{K}) \subseteq \mathbf{HSPP}_g(\mathcal{K})$. Thus, it is enough to show $\mathbf{HSPP}_g(\mathcal{K}) \subseteq \mathbf{HSPP}_{v_1}(\mathcal{K})$. This latter inclusion holds if and only if $\mathbf{HSPP}_g(\mathcal{K}) \cap \cap \mathcal{K}_X \subseteq \mathbf{HSPP}_{v_1}(\mathcal{K}) \cap \mathcal{K}_x$ for all input alphabet X, where $\mathcal{K}_X$ is the similarity class of all automata with input alphabet X. Since automata identities have at most two variables, $\mathbf{HSPP}_g(\mathcal{K}) \cap \mathcal{K}_X = \mathbf{HSP}(\{\mathfrak{A}_2\})$, where $\mathfrak{A}_2$ is a free automaton of the variety $\mathbf{HSPP}_g(\mathcal{K}) \cap \mathcal{K}_X$ generated by two elements. Let $\mathfrak{A}_1$ be a free automaton in $\mathbf{HSPP}_g(\mathcal{K}) \cap \mathcal{K}_X$ generated by a single element. One can show that every finitely generated automaton in $\mathbf{HSPP}_g(\mathcal{K}) \cap \mathcal{K}_X$ is in $\mathbf{HSP}_g(\mathcal{K})$, $\mathfrak{A}_2$ can be represented homomorphically by a quasi-direct square of $\mathfrak{A}_1$, or by a quasi-direct product of $\mathfrak{A}_1$ by a two-state discrete automaton with a single input signal depending on the forms of the p-identities holding in $\mathfrak{A}_2$ (see the Theorem in [3] and the proof of Theorem 2.1 from [5]). Since every finitely generated automaton from $\mathbf{HSPP}_{\alpha_0}(\mathcal{K})$ is in $\mathbf{HSP}_{\alpha_0}(\mathcal{K})$, by the Theorem of [3] and Proposition 12 from [4], if a two-state discrete automaton is in $\mathbf{HSPP}_g(\mathcal{K})$ then it is in $\mathbf{HSQ}(\mathcal{K})$, where $\mathbf{Q}$ is the quasi-direct product operator. Therefore, to prove $\mathbf{HSPP}_g(\mathcal{K}) \subseteq \mathbf{HSPP}_{v_1}(\mathcal{K})$ it is sufficient to show that $\mathfrak{A}_1 \in \mathbf{HSPP}_{v_1}(\mathcal{K})$ for an arbitrary input alphabet X. By the proof of Theorem 2 of [7], we may suppose that there is a largest positive integer t such that for an automaton $\mathfrak{C} = (X, C, \delta_C)$ in $\mathcal{K}$, a state $c \in C$ and a word $r \in X^*$ with $|r| = t - 1$ the state cr is ambiguous.

Assume that the identity $zp = zq$ $(p, q \in X^*)$ does not hold in $\mathfrak{A}_1$, where z is a variable, $p = x_1 \ldots x_k x_{k+1} \ldots x_m$, $q = x_1 \ldots x_k y_{k+1} \ldots y_n$, and $x_{k+1} \neq y_{k+1}$ if $m, n > k$. If $m, n \leqq t$, or $m < t$ and $n \geqq t$, then by the proof of Theorem 2 in [7], $zp = zq$ is not satisfied by $\mathbf{P}_{v_1}(\mathcal{K})$. Thus, we may assume that $m, n \geqq t$.

Since $\mathfrak{A}_1 \in \mathbf{HSPP}_g(\mathcal{K})$, there are an automaton $\mathfrak{A} = (\bar{X}, A, \delta)$ in $\mathcal{K}$, a state $a_0 \in A$ and two words $p' = x'_1 \ldots x'_l x'_{l+1} \ldots x'_m$, $q' = x'_1 \ldots x'_l y'_{l+1} \ldots y'_n$ in $\bar{X}^*$ such that $a_0 p' \neq a_0 q'$, $l \geqq k$ and $x'_{l+1} \neq y'_{l+1}$ if $m, n > l$. We shall suppose that there are no words $\bar{p} = \bar{x}_1 \ldots \bar{x}_r \bar{x}_{r+1} \ldots \bar{x}_m$ and $\bar{q} = \bar{x}_1 \ldots \bar{x}_r \bar{y}_{r+1} \ldots \bar{y}_n$ in $\bar{X}^*$ with $a_0 \bar{p} \neq a_0 \bar{q}$ and $r > l$.

Let $a_i = a_0 x_1' \dots x_i'$ $(i=1, \dots, m)$ and

$$b_i = \begin{cases} a_0 x_1' \dots x_i' & \text{if} \quad 1 \leqq i \leqq l, \\ a_0 x_1' \dots x_l' y_{l+1}' \dots y_i' & \text{if} \quad l < i \leqq n. \end{cases}$$

In the sequel we can confine ourselves to the case $m, n > l$. Assume to the contrary, say $m = l$. Consider the ν_1-product $\mathfrak{B} = (X, B, \delta') = \mathfrak{A}[X, \varphi, \gamma]$ with $\gamma(1) = \{1\}$,

$$\varphi(b_i, x_{i+1}) = x_{i+1}' \quad (i = 0, \dots, \min\{m-1, u\}),$$

$$\varphi(b_i, y_{i+1}) = y_{i+1}' \quad (i = l, \dots, \min\{n-1, u\})$$

where u is the largest index for which the states $b_0, \dots, b_n$ are pairwise distinct, and φ is given arbitrarily in all other cases. Observe that if $b_i = b_j$ for $0 \leqq i < j \leqq n$ then $\delta(b_r, x') = \delta(b_r, y')$ for arbitrary $r \geqq i$ and $x', y' \in X$; otherwise $\mathscr{K}$ would be metrically complete for the product. (This observation will be used silently throughout the paper.) By the construction of $\mathfrak{B}$, $a_0 p_{\mathfrak{B}} = a_0 p_{\mathfrak{A}}'$ and $a_0 q_{\mathfrak{B}} = a_0 q_{\mathfrak{A}}'$. Therefore, $a_0 p_{\mathfrak{B}} \neq a_0 q_{\mathfrak{B}}$.

We say that a_m and b_n *induce disjoint cycles,* if the subautomata generated by a_m and b_n are disjoint. Otherwise they induce the same cycle.

Let us distinguish the following cases.

Case 1. The states a_m and b_n induce disjoint cycles. By our assumptions on p' and q', $\{a_{l+1}, \dots, a_m\} \cap \{b_{l+1}, \dots, b_m\} = \emptyset$. Let u_1 $(0 \leqq u_1 < m)$ be the largest index such that the elements $a_0, a_1, \dots, a_{u_1}$ are pairwise distinct. The number u_2 $(0 \leqq u_2 < n)$ has the same meaning for $b_0, b_1, \dots, b_{u_2}$.

Take the ν_1-product

$$\mathfrak{B} = (X, B, \delta') = \mathfrak{A}[X, \varphi, \gamma]$$

where

$$\gamma(1) = \{1\},$$

$$\varphi(a_{l-k+i}, x_{i+1}) = x_{l-k+i+1}' \quad (0 \leqq i \leqq u_1 + k - l),$$

$$\varphi(b_{l+i}, y_{k+i+1}) = y_{l+i+1}' \quad (0 \leqq i \leqq u_2 - l),$$

and in all other cases φ is given arbitrarily. Take $\mathbf{b} = (a_{l-k})$. Then $\mathbf{b}p = (a_{l-k} x_{l-k+1}' \dots x_m' \bar{x}^{l-k})$ and $\mathbf{b}q = (a_{l-k} x_{l-k+1}' \dots x_l' y_{l+1}' \dots y_n' \bar{x}^{l-k})$, where $\bar{x} \in \bar{X}$ is arbitrary. (Remember that $m, n \geqq t$.) Therefore, $\mathbf{b}p \neq \mathbf{b}q$.

Case 2. The states a_m and b_n induce the same cycle, i.e., in the intersection of the subautomata generated by a_m and b_n there is a cycle C of length w. We distinguish some subcases.

Case 2.1. $m \not\equiv n \pmod{w}$. Then $w > 1$. Take an arbitrary ν_1-product $\mathfrak{B} = (X, A, \delta')$ of $\mathfrak{A}$ with a single factor. In $\mathfrak{B}$, for any $c \in C$, we have $cp \neq cq$.

Case 2.2. $m \equiv n \pmod{w}$. Some further subcases are needed.

Case 2.2.1. $a_m, b_n \in C$ or $m = n$.

If $\{a_{l+1}, \dots, a_m\} \cap \{b_{l+1}, \dots, b_n\} = \emptyset$, then let u_1 $(0 \leqq u_1 < m)$ be the largest index such that the elements $a_0, a_1, \dots, a_{u_1}$ are pairwise distinct. The number u_2 $(0 \leqq u_2 < n)$ has the same meaning for $b_0, b_1, \dots, b_{u_2}$.

Take the ν_1-product

$$\mathfrak{B} = (X, B, \delta') = \underbrace{(\mathfrak{A} \times \dots \times \mathfrak{A})}_{l-k+1 \text{ times}} [X, \varphi, \gamma]$$

where

$$\gamma(1) = \{1\}; \quad \gamma(i) = \{i-1\} \quad (i = 2, \dots, l-k+1),$$

$$\varphi_1(a_{l-k+i}, x_{i+1}) = x'_{l-k+i+1} \quad (0 \leqq i \leqq u_1+k-l),$$

$$\varphi_1(b_{l+i}, y_{k+i+1}) = y'_{l+i+1} \quad (0 \leqq i \leqq u_2-l),$$

and for every $j(=2, \dots, l-k+1)$,

$$\varphi_j(a_{l-k-(j-2)+i}, x_{i+1}) = x'_{l-k-(j-2)+i} \quad (0 \leqq i \leqq u_1-l+k+j-2),$$

$$\varphi_j(b_{l-(j-2)+i}, y_{k+i+1}) = \begin{cases} x'_{l-(j-2)+i} & \text{if } \ 0 \leqq i \leqq j-2 \\ y'_{l-(j-2)+i} & \text{if } \ j-2 < i \leqq u_2-l+j-2, \end{cases}$$

and in all other cases φ is given arbitrarily in accordance with the definition of the ν_1-product.

Take $\mathbf{b}=(a_{l-k}, a_{l-k-1}, \dots, a_0)$. Then $\mathbf{b}p=(a_{l-k}p_0, a_{l-k-1}p_1, \dots, a_0 p_{l-k})$ and $\mathbf{b}q=(a_{l-k}q_0, a_{l-k-1}q_1, \dots, a_0 q_{l-k})$ where, for every $j(=0, \dots, l-k)$, $p_j=x'_{l-k-j+1}\dots x'_m \bar{x}^{l-k-j}$ and $q_j=x'_{l-k-j+1}\dots x'_l y'_{l+1}\dots y'_n \bar{x}^{l-k-j}$, and $\bar{x}\in \bar{X}$ is arbitrary. Thus $p_{l-k}=p'$ and $q_{l-k}=q'$, implying $\mathbf{b}p\neq\mathbf{b}q$.

If $\{a_{l+1}, \dots, a_m\}\cap\{b_{l+1}, \dots, b_n\}\neq\emptyset$, then let r $(l+1\leqq r\leqq m)$ be the least index for which there is a b_j with $a_r=b_j$. Moreover, let s $(l+1\leqq s\leqq n)$ be the least index such that $b_s=a_r$. Then $r\neq s$, since in the opposite case $\bar{p}=x'_1\dots x'_r x'_{r+1}\dots x'_m$ and $\bar{q}=x'_1\dots x'_r y'_{r+1}\dots y'_n$ would contradict the choice of p' and q'. Assume that $r<s$. Let u $(0\leqq u<m)$ be the largest index for which the states $a_0, \dots, a_u$ are pairwise distinct. Take the ν_1-product

$$\mathfrak{B} = (X, B, \delta') = \underbrace{(\mathfrak{A} \times \dots \times \mathfrak{A})}_{l-k+1 \text{ times}} [X, \varphi, \gamma]$$

where

$$\gamma(1) = \{1\}; \quad \gamma(i) = \{i-1\} \quad (2 \leqq i \leqq l-k+1),$$

$$\varphi_1(a_{l-k+i}, x_{i+1}) = x'_{l-k+i+1} \quad (0 \leqq i \leqq u+k-l),$$

$$\varphi_1(b_{l+i}, y_{k+i+1}) = y'_{l+i+1} \quad (0 \leqq i \leqq r-l),$$

and for every $j(=2, \dots, l-k+1)$,

$$\varphi_j(a_{l-k-(j-2)+i}, x_{i+1}) = x'_{l-k-(j-2)+i} \quad (0 \leqq i \leqq u-l+k+j-2),$$

$$\varphi_j(b_{l-(j-2)+i}, y_{k+i+1}) = \begin{cases} x'_{l-(j-2)+i} & \text{if } \ 0 \leqq i \leqq j-2, \\ y'_{l-(j-2)+i} & \text{if } \ j-2 < i \leqq r-l+j-2, \end{cases}$$

and in all other cases φ is given arbitrarily. Take the state

$$\mathbf{b} = (a_{l-k}, a_{l-k-1}, \dots, a_0)\in B.$$

Then $\mathbf{b}p=(b_1', \ldots, b_{l-k}', a_0p')$ and $\mathbf{b}q=(b_1'', \ldots, b_{l-k}'', a_0x_1'\ldots x_l'y_{l+1}'\ldots y_r'\bar{q})$ where $\bar{q}\in\bar{X}^*$ satisfies the equality $|\bar{q}|=n-r$. One can easily check that $a_0p'\neq a_0x_1'\ldots x_l'y_{l+1}'\ldots y_r'\bar{q}$. Indeed, in the opposite case let $\bar{q}'$ be the initial segment of $\bar{q}$ with length $m-r$ if $m\leqq n$, and otherwise let $\bar{q}'=\bar{q}\bar{\bar{q}}$, where $\bar{\bar{q}}\in\bar{X}^*$ is arbitrary with $|\bar{q}\bar{\bar{q}}|=m-r$. From our assumptions it follows that $a_0x_1'\ldots x_l'y_{l+1}'\ldots y_r'\bar{q}'\neq a_0q'$. Therefore, by $r>l$, the pair $x_1'\ldots x_l'y_{l+1}'\ldots y_r'\bar{q}'$, q' contradicts the choice of p' and q'.

Case 2.2.2. $m\neq n$ and at least one of a_m and b_n is not in C.

Case 2.2.2.1. None of a_m and b_n is in C and $m<n$. Then the states $b_0, \ldots, b_n$ are pairwise distinct. Take the ν_1-product

$$\mathfrak{B} = (X, B, \delta') = \mathfrak{A}[X, \varphi, \gamma]$$

where

$$\gamma(1) = \{1\},$$

$$\varphi(b_i, x_{i+1}) = \begin{cases} x_{i+1}' & \text{if } 0 \leqq i < l, \\ y_{i+1}' & \text{if } l \leqq i < m, \end{cases}$$

$$\varphi(b_i, y_{i+1}) = \begin{cases} x_{i+1}' & \text{if } k \leqq i < l, \\ y_{i+1}' & \text{if } l \leqq i < n, \end{cases}$$

and φ is given arbitrarily in all other cases. Taking $\mathbf{b}=(b_0)$ we obtain $\mathbf{b}p=(b_m)$ and $\mathbf{b}q=(b_n)$.

Case 2.2.2.2. $a_m\notin C$, $b_n\in C$ and $n>m$; or $a_m\notin C$, $b_n\in C$ and $n<m$. The states $a_0, a_1, \ldots, a_m$ are pairwise distinct. Take the ν_1-product

$$\mathfrak{B} = (X, B, \delta') = \mathfrak{A}[X, \varphi, \gamma]$$

where

$$\gamma(1) = \{1\},$$

$$\varphi(a_i, x_{i+1}) = x_{i+1}' \quad (0 \leqq i < m),$$

$$\varphi(a_i, y_{i+1}) = x_{i+1}' \quad (k \leqq i < \min\{m, n\}),$$

and φ is given arbitrarily in the remaining cases. Let $\mathbf{b}=(a_0)$. If $n>m$, then $\mathbf{b}p=(a_m)$ and $\mathbf{b}q=(a_m\bar{x}^{n-m})$, where $\bar{x}\in\bar{X}$ is arbitrary. Obviously, $a_m\neq a_m\bar{x}^{n-m}$, since in the opposite case $a_m\in C$. If $n<m$, then $\mathbf{b}p=(a_m)$ and $\mathbf{b}q=(a_n)$. □

Remark. Let $\mathcal{K}$ be an arbitrary class of automata. In [3] it is shown that if an identity does not hold in an infinite product of automata from $\mathcal{K}$, then there is a finite product of automata from $\mathcal{K}$ which does not satisfy the given identity either. (See also [2], where this result is generalized to automata with infinite input alphabets.) Moreover, by Theorem 1 of [7], the ν_1-product is equivalent to the product as regards metric completeness. Therefore, if $\mathcal{K}$ is metrically complete for the product, then none of the nontrivial p-identities holds in $\mathbf{HSPP}_{\nu_1}(\mathcal{K})$. Thus, using Lemma 2, we obtain that $\mathbf{HSPP}_g(\mathcal{K})=\mathbf{HSPP}_{\nu_1}(\mathcal{K})$ for arbitrary class of automata. However, Lemma 2 will be sufficient to prove our main result.

By Lemmas 1 and 2, we obtain

Corollary 3. If a class $\mathcal{K}$ of automata is not metrically complete for the product, then $\mathbf{HSP}_g(\mathcal{K})=\mathbf{HSP}_{\nu_1}(\mathcal{K})$.

Proof. The inclusion $\mathbf{HSP}_{\nu_1}(\mathscr{K})\subseteq\mathbf{HSP}_g(\mathscr{K})$ is obvious. If $\mathfrak{A}\in\mathbf{HSP}_g(\mathscr{K})$, then there exists a finite subset $\bar{\mathscr{K}}$ such that $\mathfrak{A}\in\mathbf{HSP}_g(\bar{\mathscr{K}})$. Therefore, by Lemmas 1 and 2, $\mathfrak{A}\in\mathbf{HSP}_{\nu_1}(\bar{\mathscr{K}})$. □

Let us note that by the proof of the Theorem in [6], $\mathbf{HSP}_{\alpha_0}(\mathscr{K})=\mathbf{HSP}_g(\mathscr{K})$ if $\mathscr{K}$ is not metrically complete for the product. Thus, for such classes $\mathscr{K}$, the equality $\mathbf{HSP}_{\alpha_0}(\mathscr{K})=\mathbf{HSP}_{\nu_1}(\mathscr{K})$ holds, too.

Now we are ready to state and prove the main result of the paper.

Theorem 4. The ν_1-product is metrically equivalent to the general product.

Proof. Let $\mathscr{K}$ be an arbitrary class of automata. If $\mathscr{K}$ is metrically complete for the product, then by Theorem 1 in [7], $\mathscr{K}$ is metrically complete with respect to the ν_1-product. If $\mathscr{K}$ is not metrically complete, then $\mathbf{HSP}_g(\mathscr{K})=\mathbf{HSP}_{\nu_1}(\mathscr{K})$, as it is stated in Corollary 3. □

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