On a problem of Ádám concerning precodes assigned to finite Moore automata

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To investigate the structure of finite Moore automata, the concepts of code, precode and complexity are introduced by Ádám [1] and investigated in [1—8]. Main motivation is the following.

Basic Problem [1]. For arbitrary finite X, let a constructive description of all reduced finite Moore automata, whose input set equals to X, be given.

Relating to this problem Ádám raised four open problems, one of which is the following.

Problem 3 [1]. Consider all pairs (D, D') of precodes with finite complexity such that $D \prec D'$ holds. Either determine the maximal value of $\Omega(D')-\Omega(D)$ (as a function of the cardinality of input set) or prove that the set of these differences is unbounded.

In autonomous case, this problem is solved in [8]. The answer is that the difference is unbounded. However, we show in [8] that the quotient $\Omega(D')/\Omega(D)$

$$(D \prec D',\ \Omega(D) \neq 0,\ \Omega(D') < \infty)$$

is bounded by 2. In this note, it is shown that, in multiple-input case, not only the difference but also the quotient is unbounded.

For the background and fundamental facts concerning codes, precodes and complexity, see [1] and [2].

1.

$\mathbf{N}$ and $\mathbf{N}_0$ mean the sets of positive integers and of nonnegative integers, respectively. For $t, k \in \mathbf{N}_0$, we denote $[t:k]=\langle i \in \mathbf{N}_0 | t \leqq i \leqq k \rangle$. For $n, m \in \mathbf{N}$, we write $X_{(n)}=\langle x_1, \ldots, x_n \rangle$ and $Y_{(m)}=\langle y_1, \ldots, y_m \rangle$. A *partial automaton* is a 5-tuple $A=([1:v], X_{(n)}, Y_{(m)}, \delta, \lambda)$ where:

(1) v, n and m are positive integers. $[1:v]$, $X_{(n)}$ and $Y_{(m)}$ are called the state set, the input set and the output set of A, respectively.

(2) δ is a partial mapping of $[1:v]\times X_{(n)}$ into $[1:v]$ called a state transition function (δ is extended as usual to a partial mapping of $[1:v]\times(X_{(n)})^*$ into $[1:v]$).
(3) λ is a mapping of $[1:v]$ *onto* $Y_{(m)}$ called an output function.
(4) For any $a\in[1:v]$ there exists a $p\in X^*$ such that $\delta(1,p)=a$.

If δ is defined for any element of $[1:v]\times X_{(n)}$, then A is said to be an *(initially connected finite) Moore automaton.*

Let $A=([1:v], X_{(n)}, Y_{(m)}, \delta, \lambda)$ be a Moore automaton. If $\lambda(\delta(a,p))\neq\lambda(\delta(b,p))$ holds for $a, b\in[1:v]$ and $p\in X^*$, then we say that p distinguishes between a and b. $\omega(a,b)$ is the minimal length of p which distinguishes between a and b. If there is no word which distinguishes between a and b, then we denote $\omega(a,b)=\infty$. Especially, $a=b$ implies $\omega(a,b)=\infty$. The *complexity* $\Omega(A)$ of A is defined by

$$\Omega(A) = \min\langle\omega(a,b)|a, b\in[1:v], a\neq b\rangle.$$

If $v=1$ then $\Omega(A)=0$.

The notions of codes and precodes were introduced in [1] as tools to describe Moore automata constructively. The following definition is from [6, 7]. It is of course essentially equivalent to Ádám's definition in [1].

Let $n\in\mathbf{N}$. A 6-tuple $D=(r, s, \beta, \gamma, \varphi, \mu)$ is said to be an *n-input precode* if the following eight postulates are fulfilled:

(A) r, s are nonnegative integers.
(B) β and φ are mappings of $[2:r+s+1]$ into $[1:r+1]$.
γ is a mapping of $[2:r+s+1]$ into $[1:n]$.
μ is a mapping of $[1:r+1]$ into $\mathbf{N}$.
(C) $\beta(a)<a$ for any $a\in[2:r+1]$.
(D) For $a, b\in[2:r+1]$, if $a<b$ then $(\beta(a), \gamma(a))<(\beta(b), \gamma(b))$ in the lexicographic order.
(E) For $a\in[r+2:r+s+1]$, $(\beta(a), \gamma(a))$ is the lexicographically smallest element in $([1:r+1]\times[1:n])-\langle(\beta(b), \gamma(b))|b\in[2:a-1]\rangle$.
(F) For $a\in[2:r+1]$, $\varphi(a)=a$.
(G) For $a\in[r+2:r+s+1]$, $\varphi(a)=1$ or $(\beta(\varphi(a)), \gamma(\varphi(a)))<(\beta(a), \gamma(a))$ in the lexicographic order.
(H) $\mu(a)\in\langle 1\rangle\cup\langle\mu(b)+1|b\in[1:a-1]\rangle$.

We denote $\mu(D)=\max\langle\mu(a)|a\in[1:r+1]\rangle$. If $m=\mu(D)$ then D is said to be an m-output precode.

It can be easily be seen that $r+s\leqq n(r+1)$ i.e., $s\leqq nr+n-r$. If $s=nr+n-r$, then the precode is said to be a *code*.

Let $D=(r, s, \beta, \gamma, \varphi, \mu)$ and $D'=(r', s', \beta', \gamma', \varphi', \mu')$ be n-input precodes. If $r+s\leqq r'+s'$ and β', γ', φ', μ' are extensions of β, γ, φ, μ then we denote $D\leqq D'$. We denote $D<D'$ if $D\leqq D'$ and $r+s<r'+s'$. If $D<D'$ and $r'+s'=r+s+1$ then we write $D\prec D'$.

It can easily be seen that, for any precode D, there exists a code C such that $D\leqq C$.

Let $D=(r, s, \beta, \gamma, \varphi, \mu)$ be an n-input m-output precode. Define a partial mapping δ_D of $[1:r+1]\times X_{(n)}$ into $[1:r+1]$ by

$$\delta_D(\beta(a), x_{\gamma(a)}) = \varphi(a) \quad \text{for any} \quad a\in[2:r+s+1].$$

Define a mapping λ_D of $[1:r+1]$ onto $Y_{(m)}$ by

$$\lambda_D(a) = y_{\mu(a)} \quad \text{for any} \quad a \in [1:r+1].$$

Then it is easy to verify that $\Psi(D)=([1:r+1], X_{(n)}, Y_{(m)}, \delta_D, \lambda_D)$ is a partial automaton. $\Psi(D)$ is an automaton iff D is a code.

The *complexity* $\Omega(D)$ of a precode D is defined by

$$\Omega(D) = \min \langle \Omega(\Psi(C)) | C \text{ is a code such that } D \leqq C \rangle.$$

2.

Let n, w, t be positive integers such that $n \geqq 2$ and $w \geqq 2$. Define an n-input precode $D=(r, s, \beta, \gamma, \varphi, \mu)$ as follows:

(1) $r = 4t+4w-2$ and $s = nr+n-r-1$.

(2) $(\beta(2b), \gamma(2b), \varphi(2b)) = (b, 1, 2b)$ and $(\beta(2b+1), \gamma(2b+1), \varphi(2b+1)) = (b, n, 2b+1)$ for any $b \in [1:2t+2w-1]$.

(3) $\mu(a) = a$ for any $a \in [1:3t+3w-1]$.

$\mu(a) = a-w-t$ for any $a \in [3t+3w:4t+4w-1]$.

(4) For each $a \in [r+2:r+s+1]$, the a-th row is determined as follows:

(a) $\beta(a)$, $\gamma(a)$ are determined uniquely by Postulate (E).

(b) If $\beta(a) \in [2t+2w:3t+3w-2] \cup [3t+3w:4t+4w-2]$ and $\gamma(a) = 1$ then $\varphi(a) = a+1$.

If $(\beta(a), \gamma(a)) = (3t+3w-1, 1)$ then $\varphi(a) = 2t+2w$.

If $(\beta(a), \gamma(a)) = (4t+4w-1, 1)$ then $\varphi(a) = 3t+3w$.

(c) If $\beta(a) \in [2t+2w:3t+2w-1]$ and $\gamma(a) = n$ then $\varphi(a) = 3t+3w-1$.

If $\beta(a) \in [3t+3w:4t+3w-2]$ and $\gamma(a) = n$ then $\varphi(a) = 4t+4w-1$.

(d) Otherwise, $\varphi(a)=1$.

It is easy to verify that D satisfies Postulates (A)—(H).

The state transition function and the output function of the partial automaton $\Psi(D)=([1:4t+4w-1], X_{(n)}, Y_{(m)}, \delta_D, \lambda_D)$ is shown in the following table:

a	$\delta_D(a, x_1)$	$\delta_D(a, x_j)$ $(j\in[2:\ n-1])$	$\delta_D(a, x_n)$	$\lambda_D(a)$
1	2	1	3	1
2	4	1	5	2
3	6	1	7	3
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$2t+2w-2$	$4t+4w-4$	1	$4t+4w-3$	$2t+2w-2$
$2t+2w-1$	$4t+4w-2$	1	$4t+4w-1$	$2t+2w-1$
$2t+2w$	$2t+2w+1$	1	$3t+3w-1$	$2t+2w$
$2t+2w+1$	$2t+2w+2$	1	$3t+3w-1$	$2t+2w+1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$3t+2w-2$	$3t+2w-1$	1	$3t+3w-1$	$3t+2w-2$
$3t+2w-1$	$3t+2w$	1	$3t+3w-1$	$3t+2w-1$
$3t+2w$	$3t+2w+1$	1	1	$3t+2w$
$3t+2w+1$	$3t+2w+2$	1	1	$3t+2w+1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$3t+2w-2$	$3t+3w-1$	1	1	$3t+3w-2$
$3t+3w-1$	$2t+2w$	1	1	$3t+3w-1$
$3t+3w$	$3t+3w+1$	1	$4t+4w-1$	$2t+2w$
$3t+3w+1$	$3t+3w+2$	1	$4t+4w-1$	$2t+2w+1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$4t+3w-2$	$4t+3w-1$	1	$4t+4w-1$	$3t+2w-2$
$4t+3w-1$	$4t+3w$	1	1	$3t+2w-1$
$4t+3w$	$4t+3w+1$	1	1	$3t+2w$
$4t+3w+1$	$4t+3w+2$	1	1	$3t+2w+1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$4t+4w-2$	$4t+4w-1$	1	1	$3t+3w-2$
$4t+4w-1$	$3t+3w$	1	—	$3t+3w-1$

Let $D'=(r, s+1, \beta, \gamma, \varphi, \mu)$ be a precode such that $D\prec D'$. Then D' is a code, i.e., $\Psi(D')$ is a Moore automaton. We have $(\beta(r+s+2), \gamma(r+s+2))=(4t+4w-1, n)$ and D' is determined only by the value $\varphi(r+s+2)$. It can easily be seen that arbitrary choice of $\varphi(r+s+2)\in[1:4w+4t-1]$ makes D' to satisfy the postulates for codes. We shall show that $\varphi(r+s+2)\neq 1$ implies $\Omega(D')=w$, and $\varphi(r+s+2)=1$ implies $\Omega(D')=t+w$.

Case 1: $\varphi(r+s+2)\neq 1$, i.e., $\delta_{D'}(4t+4w-1, x_n)\neq 1$.

Let $a, b\in[1:4t+4w-1]$ such that $a<b$. We have $\omega(a,b)\neq 0$ iff $\lambda_{D'}(a)=$ $=\lambda_{D'}(b)$ iff $a=2t+2w+i, b=3t+3w+i$ for some $i\in[0:t+w-1]$. Let $i\in[0:t+w-1]$. Since

$$\lambda_{D'}(2t+2w+i) = \lambda_{D'}(3t+3w+i),$$

$$\delta_{D'}(2t+2w+i, x_j) = \delta_{D'}(3t+3w+i, x_j) \quad \text{for any} \quad j\in[2:n-1],$$

we have

$$\omega(2t+2w+i,\, 3t+3w+i) =$$
$$= \min\langle \omega(\delta_{D'}(2t+2w+i, x_1), \delta_{D'}(3t+3w+i, x_1))+1,$$
$$\omega(\delta_{D'}(2t+2w+i, x_n), \delta_{D'}(3t+3w+i, x_n))+1\rangle.$$

Thus we have

$$\omega(3t+2w-1,\, 4t+3w-1) =$$
$$= \min\langle \omega(3t+2w, 4t+3w)+1, \omega(3t+3w-1, 1)+1\rangle = 1.$$
$$\omega(3t+3w-1,\, 4t+4w-1) =$$
$$= \min\langle \omega(2t+2w, 3t+3w)+1, \omega(1, \delta_{D'}(4t+4w-1, x_n))+1\rangle = 1.$$

For $i\in[0:t-2]$,

$$\omega(2t+2w+i,\, 3t+3w+i) =$$
$$= \min\langle \omega(2t+2w+i+1, 3t+3w+i+1)+1, \omega(3t+3w-1, 4t+4w-1)+1\rangle = 2.$$

For $i\in[0:w-2]$,

$$\omega(3t+2w+i,\, 4t+3w+i) = \omega(3t+2w+i+1,\, 4t+3w+i+1)+1.$$

Hence,

$$\omega(3t+3w-2,\, 4t+4w-2) = 2,$$
$$\omega(3t+3w-3,\, 4t+4w-3) = 3,$$
$$\dots$$
$$\omega(3t+2w,\, 4t+3w) = w.$$

Consequently, $\Omega(D')=\max\langle 0, 1, 2, \dots, w\rangle = w$.

Case 2: $\varphi(r+s+2)=1$, i.e., $\delta_{D'}(4t+4w-1, x_n)=1$.
Let $a, b\in[1:4t+4w-1]$ such that $a<b$. Just as in Case 1, we have

$$\omega(a, b) \neq 0 \quad \text{iff} \quad a = 2t+2w+i, \quad b = 3t+3w+i \quad \text{for some} \quad i\in[0:t+w-1].$$

$$\omega(3t+2w-1,\, 4t+3w-1) = \min\langle \omega(3t+2w, 4t+3w)+1, \omega(3t+2w-1, 1)+1\rangle = 1.$$

$$\omega(3t+2w+i,\, 4t+3w+i) = \omega(3t+2w+i+1,\, 4t+3w+i+1)+1 \text{ for any } i\in[0:w-2].$$

We have

$$\omega(3t+3w-1,\, 4t+4w-1) = \omega(2t+2w,\, 3t+3w)+1.$$

For $i\in[0:t-2]$,

$$\omega(2t+2w+i,\, 3t+3w+i) =$$
$$= \min\langle \omega(2t+2w+i+1, 3t+3w+i+1)+1, \omega(3t+3w-1, 4t+4w-1)+1\rangle =$$
$$= \min\langle \omega(2t+2w+i+1, 3t+3w+i+1)+1, \omega(2t+2w, 3t+3w)+2\rangle.$$

It follows that

$$\omega(3t+2w-1, 4t+3w-1) = 1,$$
$$\omega(3t+2w-2, 4t+3w-2) = 2,$$
$$\omega(3t+2w-3, 4t+3w-3) = 3,$$
$$\dots$$
$$\omega(2t+2w, 3t+3w) = t,$$
$$\omega(3t+3w-1, 4t+4w-1) = t+1,$$
$$\omega(3t+3w-2, 4t+4w-2) = t+2,$$
$$\dots$$
$$\omega(3t+2w, 4t+3w) = t+w.$$

Consequently, $\Omega(D')=\max\langle 0, 1, 2, \dots, t+w\rangle=t+w$.

We have shown that $\varphi(r+s+2)\neq 1$ implies $\Omega(D')=w$ and $\varphi(r+s+2)=1$ implies $\Omega(D')=w+t$. It follows that $\Omega(D)=\min\langle w, w+t\rangle=w$. We have shown the following.

Theorem 1. For any $n, w, t\in \mathrm{N}$ with $n\geqq 2$ and $w\geqq 2$, there exist n-input precodes D and D' such that $D\prec D'$, $\Omega(D)=w$ and $\Omega(D')=t+w$. □

In autonomous case, Problem 3 of Ádám is solved in [8] as follows:

Proposition 1. The set

$\langle\Omega(D')-\Omega(D)|D$ and D' are 1-input precodes such that $D\prec D'$ and $\Omega(D')<\infty\rangle$

coincides with all nonnegative integers. □

In multiple-input case, we have the following similar result which is an immediate consequence of Theorem 1.

Corollary 1. For any $n\in\mathrm{N}$ with $n\geqq 2$, the set

$\langle\Omega(D')-\Omega(D)|D$ and D' are n-input precodes such that $D\prec D'$ and $\Omega(D')<\infty\rangle$

coincides with all nonnegative integers. □

Consider the quotient $\Omega(D')/\Omega(D)$ instead of the difference $\Omega(D')-\Omega(D)$. In autonomous case, we have the following result [8].

Proposition 2. The set

$\langle\Omega(D')/\Omega(D)|D$ and D' are 1-input precodes such that $D\prec D'$, $\Omega(D)\neq 0$ and $\Omega(D')<\infty\rangle$

coincides with all rational numbers between 1 and 2. □

Though the quotient is bounded in autonomous case, it is unbounded in multiple-input case. The following is also immediate from Theorem 1.

Corollary 2. For any $n\in N$ with $n\geqq 2$, the set

$$\langle \Omega(D')/\Omega(D) | D \text{ and } D' \text{ are } n\text{-input precodes such that } D\prec D',\ \Omega(D)\neq 0 \text{ and } \Omega(D')<\infty\rangle$$

coincides with all rational numbers not less than 1. □

Contrary to expectation, the solution of the problem does not contribute to our investigation, especially to the Basic Problem. If we wish to proceed further in this line, we should make refinements of the problem, e.g., not only n but also r, s and/or m should be taken into account.

3.

In this section, we consider a modification of our problem in the sense that, instead of the cardinality n of the input set, the cardinality m of the output set is taken into account. Analogous to Theorem 1, we have the following result:

Theorem 2. For any $m, w, t\in N$ with $m\geqq 2$ and $w\geqq 2$, there exist m-output precodes D and D' such that $D\prec D'$, $\Omega(D)=w$, $\Omega(D')=t+w$.

Proof. Define a $(2t+2w)$-input precode $D=(r, s, \beta, \gamma, \varphi, \mu)$ as follows:

(1) $r=2t+2w+m-2$ and $s=(2t+2w)r+(2t+2w)-r-1$.

(2) $(\beta(a), \gamma(a), \varphi(a))=(a-1, 1, a)$ for any $a\in[2:m-1]$.

(3) $(\beta(a), \gamma(a), \varphi(a))=(m-1, a-m+1, a)$ for any $a\in[m:2t+2w+m-1]$.

(4) $\mu(a)=a$ for any $a\in[1:m-1]$.

$\mu(a)=m$ for any $a\in[m:2t+2w+m-1]$.

(5) For each $a\in[r+2, r+s+1]$, the a-th row is determined as follows:

(a) $\beta(a)$, $\gamma(a)$ are determined uniquely by Postulate (E).

(b) If $\beta(a)\in[m:t+w+m-2]\cup[t+w+m:2t+2w+m-2]$ and $\gamma(a)=1$ then $\varphi(a)=a+1$.

If $(\beta(a), \gamma(a))=(t+w+m-1, 1)$ then $\varphi(a)=m$.

If $(\beta(a), \gamma(a))=(2t+2w+m-1, 1)$ then $\varphi(a)=t+w+m$.

(c) If $\beta(a)-(m-2)=\gamma(a)\in[2:t+w+1]$ then $\varphi(a)=\beta(a)$.

If $\beta(a)-(t+w+m-2)=\gamma(a)\in[2:t+w+1]$ then $\varphi(a)=\beta(a)$.

(d) If $\beta(a)\in[m:t+m-1]$ and $\gamma(a)=2t+2w$ then $\varphi(a)=t+w+m-1$.

If $\beta(a)\in[t+w+m:2t+w+m-2]$ and $\gamma(a)=2t+2w$ then $\varphi(a)=2t+2w+m-1$.

(e) Otherwise, $\varphi(a)=1$.

Let $D'=(r, s+1, \beta, \gamma, \varphi, \mu)$ be a precode such that $D \prec D'$. Let

$$a, b \in [1:2t+2w+m-1]$$

such that $a<b$. If $a \in [1:m-1]$ then $\lambda_{D'}(a) \neq \lambda_{D'}(b)$ and thus $\omega(a, b)=0$. If $a \in [m:2t+2w+m-1]$ then there exist $i, j \in [0: t+w-1]$ such that

$$a = m+i \quad \text{or} \quad a = t+w+m+i,$$

$$b = m+j \quad \text{or} \quad b = t+w+m+j.$$

If $i \neq j$ then $\lambda_{D'}(a)=m=\lambda_{D'}(b)$ and

$$\lambda_{D'}(\delta_{D'}(a, x_{i+2})) = \lambda_{D'}(a) = m \neq 1 = \lambda_{D'}(1) = \lambda_{D'}(\delta_{D'}(b, x_{i+2})).$$

Hence $\omega(a, b)=1$. Consequently $\omega(a, b) \geqq 2$ implies that $a=m+i$ and $b=t+ +w+m+i$.

Similarly as in Theorem 1, we have, for any $i \in [0:t+w-1]$,

$$\omega(m+i, t+w+m+i) =$$

$$= \min \langle \omega(\delta_{D'}(m+i, x_1), \delta_{D'}(t+w+m+i, x_1))+1,$$

$$\omega(\delta_{D'}(m+i, x_n), \delta_{D'}(t+w+m+i, x_n))+1 \rangle.$$

Since $\delta_{D'}(t+m-1, x_{2t+2w})=t+w+m-1$ and $\delta_{D'}(t+w+m-1, x_{2t+2w})=1$, we have $\omega(t+m-1, t+w+m-1)=1$. In a similar way as in Theorem 1, we can verify the following:

If $\varphi(r+s+2) \neq 1$ then $\Omega(D') = \omega(t+m, 2t+w+m) = w$.

If $\varphi(r+s+2) = 1$ then $\Omega(D') = \omega(t+m, 2t+w+m) = t+w$. □

The followings results are immediate from the above theorem.

Corollary 3. For any $m \in \mathrm{N}$ with $m \geqq 2$, the set

$\langle \Omega(D')-\Omega(D) | D$ and D' are m-output precodes such that $D \prec D'$ and $\Omega(D') < \infty \rangle$

coincides with all nonnegative integers. □

Corollary 4. For any $m \in \mathrm{N}$ with $m \geqq 2$, the set

$\langle \Omega(D')/\Omega(D) | D$ and D' are m-output precodes such that $D \prec D'$, $\Omega(D) \neq 0$ and $\Omega(D') < \infty \rangle$

coincides with all rational numbers not less than 1. □

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