A note on the generalized ν_1-product

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A hierarchy of products was introduced in [1]. This hierarchy contains one kind of product, the ν_i-product, for every positive integer i, and the work [1] deals with the isomorphic completeness with respect to the ν_i-products. As regards another representations, the metric representation was studied in [6], [8]. The work [6] contains the characterization of the metrically complete systems with respect to the ν_i-products. In [8] it is shown that the ν_1-product is metrically equivalent to the general product. The works [2], [3], [4], [5] are devoted to the investigation of the homomorphic representation. In [3] and [4] some special compositions of the α_0-product and ν_i-products was studied and it is proved that these compositions are just as strong as the general product with respect to the homomorphic representation. The work [5] deals with the commutative automata. It is shown that there are finite homomorphically complete systems with respect to the ν_1-product for this class. In [2] the hierarchy of the ν_i-products was investigated. It is proved that this hierarchy is proper as regards the homomorphic representations. Finally, the work [7] compares the isomorphic and homomorphic representation powers of α_i-products and ν_j-products.

In this paper, connecting with the work [1], we give a sufficient condition for a system of automata to be isomorphically S-complete with respect to the generalized ν_1-product. This condition is a special case of condition (2) of Theorem 2 in [1], but the construction of the automata from these systems is simpler than the general construction given in [1]. Since our work is closely related to [1], we shall use its notions and notations.

Our result is the following statement.

Theorem. A system Σ of automata is isomorphically S-complete with respect to the generalized ν_1-product if Σ contains an automaton which has a state a and input word q such that the states a, aq, ..., aq^{s-1} are pairwise different and $aq^s=a$ for some integer $s>1$.

Proof. Let us assume that Σ satisfies the condition. Then without loss of generality we may suppose that Σ contains an automaton **A** which has a state a and input word q such that a, aq, ..., aq^{p-1} are pairwise different, $aq^p=a$, and p is a prime number. Let us denote by $0, 1, ..., p-1$ the states a, aq, ..., aq^{p-1}, respectively. Depending on p, we shall distinguish two cases.

Case 1. Let us suppose that $p=2$. By the proof of Theorem 2 in [1], it is enough to prove that for any $n\geqq 3$ the automaton $\mathbf{T}'_n$ can be simulated isomorphically by a generalized ν_1-product of automata from Σ, where $\mathbf{T}'_n=(\{t_1, t_2, t_3\}, \{0, \dots, n-1\}, \delta'_n)$ and

$$t_1(k) = k+1 \pmod n \quad (k = 0, \dots, n-1),$$

$$t_2(0) = 1,\ t_2(1) = 0,\ t_2(k) = k \ (k = 2, \dots, n-1),$$

$$t_3(0) = t_3(1) = 0,\ t_3(k) = k \ (k = 2, \dots, n-1).$$

Now let $n\geqq 3$ be an arbitrary fixed integer. Let us take an integer k for which $2^k+1\geqq n$ holds and denote by m the number 2^k+1. Form the generalized ν_1-product $\mathbf{A}^m(X, \varphi, \gamma)$ where

$$X = \{x_1, x_2, x_3\}\cup\{y_t\colon 0 \leqq t \leqq m-1\}$$

and the mappings γ and φ are defined in the following way:

$$\gamma(t) = \{t-1 \pmod m\} \quad (t = 0, \dots, m-1),$$

$$\varphi_t(0, x_1) = q,\ \varphi_t(1, x_1) = q^2 \quad (t = 0, \dots, m-1),$$

$$\varphi_t(0, x_2) = \varphi_t(1, x_2) = q^2 \ \text{ if } \ 0 \leqq t \leqq m-3,$$

$$\varphi_t(0, x_2) = \varphi_t(1, x_2) = q \ \text{ if } \ m-3 < t \leqq m-1,$$

$$\varphi_t(0, x_3) = q^2,\ \varphi_t(1, x_3) = q \ \text{ if } \ t \neq m-2,$$

$$\varphi_{m-2}(0, x_3) = q, \quad \varphi_{m-2}(1, x_3) = q^2,$$

$$\varphi_t(0, y_j) = q^2,\ \varphi_t(1, y_j) = \begin{cases} q & \text{if } t=j, \\ q^2 & \text{otherwise,} \end{cases} \quad (j = 0, \dots, m-1;\ t = 0, \dots, m-1)$$

Take the mappings:

$$\mu\colon \begin{array}{l} 0 \to (0, 0, \dots, 1), \\ \vdots \\ m-1 \to (1, 0, \dots, 0), \end{array}$$

$$\tau\colon \begin{array}{l} t_1 \to x_1^{m-2}, \\ t_2 \to x_2 y_1 \dots y_{m-3} y_{m-1} y_{m-2} \dots y_1 y_{m-1}, \\ t_3 \to y_0 y_1 \dots y_{m-3} x_3. \end{array}$$

Now we show that $\mathbf{T}'_m$ can be simulated isomorphically by $\mathbf{A}^m(X, \varphi, \gamma)$ under μ and τ. Indeed, the validity of the equations $\mu(\delta'_m(j, t_l))=\delta_{\mathbf{A}^m}(\mu(j), \tau(t_l))$ $(l=2, 3; j=0, \dots, m-1)$ follows from the definitions. To prove the validity of the equations $\mu(\delta'_m(j, t_1))=\delta_{\mathbf{A}^m}(\mu(j), \tau(t_1))$ $(j=0, \dots, m-1)$ let us observe the following connection. If

$$(u_0, \dots, u_{m-1})\in\{0, 1\}^m \ \text{ and}$$

$$\delta_{\mathbf{A}^m}((u_0, \dots, u_{m-1}), x_1) = (v_0, \dots, v_{m-1})$$

then

$$v_t = \delta_A(u_t, \varphi_t(u_{t-1 \pmod m}, x_1)) =$$
$$= u_t q^{u_{t-1 \pmod m}+1 \pmod 2} = u_t + u_{t-1 \pmod m} + 1 \pmod 2$$

holds for any $0 \leqq t \leqq m-1$. Now let us denote by $(v_0^{(s)}, \ldots, v_{m-1}^{(s)})$ the state $\delta_{A^m}((u_0, \ldots, u_{m-1}), x_1^s)$. Then using the above connection, by induction on s, it can be proved that

$$v_t^{(s)} = 1 + \sum_{j=0}^{s} \binom{s}{j} u_{t-j \pmod m} \pmod 2 \quad (t = 0, \ldots, m-1).$$

On the other hand, it is known that $\binom{p^k}{j} \equiv 0 \pmod p$ $(j=1, \ldots, p^k-1)$ holds for any prime $p>1$ and integer $k \geqq 1$. Using this, by induction on j, one can show that

$$\binom{p^k-1}{j}(-1)^j \equiv 1 \pmod p \quad (j = 0, \ldots, p^k-1).$$

From this, by $p=2$, we obtain

$$\binom{2^k-1}{j} \equiv 1 \pmod 2 \quad (j = 0, \ldots, 2^k-1).$$

Now let $0 \leqq i \leqq m-1$ be an arbitrary integer and let us denote by $(c_0, \ldots, c_{m-1})$ the state $\mu(i)$. Then

$$c_t = \begin{cases} 1 & \text{if } t = m-i-1, \\ 0 & \text{otherwise,} \end{cases} \quad (t = 0, \ldots, m-1).$$

Let $(c_0', \ldots, c_{m-1}')$ denote the state $\delta_{A^m}((c_0, \ldots, c_{m-1}), x_1^{m-2})$. Then, by the above equality for $v_t^{(s)}$, we obtain that

$$c_t' = 1 + \sum_{j=0}^{2^k-1} \binom{2^k-1}{j} c_{t-j \pmod m} \pmod 2 \quad (t = 0, \ldots, m-1).$$

If $t = m-i-2 \pmod m$ then from the definition of c_t it follows that $c_{t-j \pmod m} = 0$ $(j=0, \ldots, 2^k-1)$, and so, $c'_{m-i-2 \pmod m} = 1$. If $t \neq m-i-2 \pmod m$ then among the elements $c_{t-j \pmod m}$ $(j=0, \ldots, 2^k-1)$ one and only one is different from 0, and so, $c_t' = 1 + \binom{2^k-1}{j} \pmod 2$ for some $0 \leqq j \leqq 2^k-1$. Since $\binom{2^k-1}{j} \equiv 1 \pmod 2$ this implies the equality $c_t' = 0$. Summarizing, we obtained that

$$c_t' = \begin{cases} 1 & \text{if } t = m-i-2 \pmod m, \\ 0 & \text{otherwise,} \end{cases} \quad (t = 0, \ldots, m-1).$$

Now let us observe that $(c_0', \ldots, c_{m-1}') = \mu(i+1 \pmod m)$ and so,

$$\mu(\delta_m'(i, t_1)) = \mu(i+1 \pmod m) = (c_0', \ldots, c_{m-1}') = \delta_{A^m}((c_0, \ldots, c_{m-1}), x_1^{m-2}) =$$
$$= \delta_{A^m}((\mu(i), \tau(t_1))$$

which completes the proof of the Case 1.

Case 2. Let us suppose that $p>2$ and let $n \geqq 3$ be an arbitrary fixed integer again. Let k be an integer such that $p^k+1 \geqq 2n$ and, let $s=p^k+1$ and $m=s/2$. Form the generalized v_1-product $\mathbf{A}^s(X, \varphi, \gamma)$ where

$$X = \{x_1, ..., x_s\} \cup \{y_r\colon 0 \leqq r \leqq s-2\} \cup \{v_r, z_r\colon 0 \leqq r \leqq s-4\} \cup \{w_r\colon 0 \leqq r \leqq s-1\}$$

and the mappings γ and φ are defined in the following way: for any $t \in \{0, ..., s-1\}$, $j \in \{0, ..., p-1\}$, $r \in \{0, ..., s-1\}$

$$\gamma(t) = \{t-1 \pmod s\},$$

$$\varphi_t(j, x_1) = q^{p-1-j} \quad \text{if} \quad 0 \leqq j < p-1, \quad \varphi_t(p-1, x_1) = q^p,$$

$$\varphi_t(j, x_2) = q^{p-1} \quad \text{if} \quad t \in \{s-3, s-2, s-1\},\ \varphi_t(j, x_2) = q^p \quad \text{if} \quad 0 \leqq t < s-3,$$

$$\varphi_{s-3}(0, x_3) = q^2, \ \varphi_t(j, x_3) = q^p \quad \text{otherwise},$$

$$\varphi_{s-2}(0, x_4) = q, \quad \varphi_t(j, x_4) = q^p \quad \text{otherwise},$$

$$\varphi_t(p-1, x_5) = q \quad \text{if} \quad t \neq 0 \quad \text{and} \quad \varphi_t(j, x_5) = q^p \quad \text{otherwise},$$

$$\varphi_t(p-1, x_6) = q \quad \text{if} \quad t \in \{s-2, s-1\} \quad \text{and} \quad \varphi_t(j, x_6) = q^p \quad \text{otherwise},$$

$$\varphi_t(0, x_7) = q^{p-1} \quad \text{if} \quad t = s-3 \quad \text{and} \quad \varphi_t(j, x_7) = q^p \quad \text{otherwise},$$

$$\varphi_t(p-2, x_8) = \begin{cases} q & \text{if} \quad t = s-1, \\ q^2 & \text{if} \quad t \neq s-1, \end{cases} \quad \text{and} \quad \varphi_t(j, x_8) = q^p \quad \text{otherwise},$$

$$\varphi_t(j, y_r) = \begin{cases} q^{p-1} & \text{if} \quad t = r \quad \text{and} \quad j = p-1, \\ q^p & \text{otherwise}, \end{cases} \quad (r = 0, ..., s-2),$$

$$\varphi_t(j, v_r) = \begin{cases} q^2 & \text{if} \quad t = r \quad \text{and} \quad j = p-2, \\ q^p & \text{otherwise}, \end{cases} \quad (r = 0, ..., s-4),$$

$$\varphi_t(j, z_r) = \begin{cases} q & \text{if} \quad t = r \quad \text{and} \quad j = p-1, \\ q^p & \text{otherwise}, \end{cases} \quad (r = 0, ..., s-4),$$

$$\varphi_t(j, w_r) = \begin{cases} q^{p-2} & \text{if} \quad t = r \quad \text{and} \quad j = p-2, \\ q^p & \text{otherwise}, \end{cases} \quad (r = 0, ..., s-1).$$

Take the mappings:

$$\mu\colon \begin{array}{l} 0 \to (0, 0, 0, ..., 0, 0, p-1), \\ 1 \to (0, 0, 0, ..., p-1, 0, 0), \\ \vdots \\ m-1 \to (0, p-1, 0, ..., 0, 0, 0), \end{array}$$

$$\tau\colon \begin{array}{l} t_1 \to x_1^{2(p^k-1)}, \\ t_2 \to x_2 y_2 \dots y_{s-4} w_0 \dots w_{s-4} x_3 x_4 v_{s-4} \dots v_0 x_5 y_{s-2} x_6, \\ t_3 \to y_0 \dots y_{s-4} x_7 w_{s-2} w_{s-1} w_0 \dots w_{s-4} z_{s-4} \dots z_0 x_8. \end{array}$$

Now we shall show that the automaton $\mathbf{T}'_m$ can be simulated isomorphically by $\mathbf{A}^s(X, \varphi, \gamma)$ under μ and τ.

The validity of $\mu(\delta'_m(j, t_l))=\delta_{\mathbf{A}^s}(\mu(j), \tau(t_l))$ $(l=2, 3; j=0, \dots, m-1)$ can be checked by a simple computation. To prove the validity of $\mu(\delta'_m(j, t_1))= =\delta_{\mathbf{A}^s}(\mu(j), \tau(t_1))$ let $(u_0, \dots, u_{s-1})\in\{0, \dots, p-1\}^s$ be arbitrary and let us denote by $(u_0^{(r)}, \dots, u_{s-1}^{(r)})$ the state $\delta_{\mathbf{A}^s}((u_0, \dots, u_{s-1}), x_1^r)$ for arbitrary integer $r\geqq 1$. Then

$$u_t^{(1)} = \delta_{\mathbf{A}}(u_t, \varphi_t(u_{t-1(\text{mod } s)}, x_1)) = u_t q^{(p-1-u_{t-1(\text{mod } s)})(\text{mod } p)} = u_t - u_{t-1(\text{mod } s)} - 1 \,(\text{mod } p).$$

Using this, by induction on r, it can be proved that

$$u_t^{(r)} = -1+\sum_{j=0}^{r}\binom{r}{j}(-1)^j u_{t-j(\text{mod } s)} (\text{mod } p) \quad (t = 0, \dots, s-1)$$

Now let $i\in\{0, \dots, m-1\}$ be arbitrary and let us denote by $(c_0, \dots, c_{s-1})$, $(c'_0, \dots, c'_{s-1})$, $(\bar{c}_0, \dots, \bar{c}_{s-1})$ the states $\mu(i)$, $\delta_{\mathbf{A}^s}(\mu(i), x_1^{(p^k-1)})$, $\delta_{\mathbf{A}^s}(\mu(i), x_1^{2(p^k-1)})$, respectively. Then from the definition of μ,

$$c_t = \begin{cases} p-1 & \text{if } \ t = s-2i-1, \\ 0 & \text{otherwise.} \end{cases} \quad (t = 0, \dots, s-1).$$

Consider the state $(c'_0, \dots, c'_{s-1})$. By the above equality for $u_t^{(r)}$, we obtain that

$$c'_t = -1+\sum_{j=0}^{p^k-1}\binom{p^k-1}{j}(-1)^j c_{t-j(\text{mod } s)} (\text{mod } p) \quad (t = 0, \dots, s-1).$$

If $t=s-2i-2 \,(\text{mod } s)$ then from the definition of c_t it follows that $c_{t-j(\text{mod } s)}=0$ $(j=0, \dots, p^k-1)$, and so, $c'_{s-2i-2(\text{mod } s)}=p-1$. If $t\neq s-2i-2 \,(\text{mod } s)$ then among the elements $c_{t-j(\text{mod } s)}$ $(j=0, \dots, p^k-1)$ exactly one element is different from 0 and this element is equal to $p-1$, and so, $c'_t=-1+\binom{p^k-1}{j}(-1)^j(p-1)$ for some $0\leqq j\leqq p^k-1$. From this, by $\binom{p^k-1}{j}(-1)^j\equiv 1 \,(\text{mod } p)$ $(j=0, \dots, p^k-1)$, we obtain that $c'_t=p-2$. Therefore

$$c'_t = \begin{cases} p-1 & \text{if } \ t = s-2i-2, \\ p-2 & \text{otherwise,} \end{cases} \quad (t = 0, \dots, s-1).$$

Now consider the state $(\bar{c}_0, \dots, \bar{c}_{s-1})$.

$$\bar{c}_t = -1+\sum_{j=0}^{p^k-1}\binom{p^k-1}{j}(-1)^j c'_{t-j(\text{mod } s)} (\text{mod } p) \ (t = 0, \dots, s-1).$$

If $t=s-2(i+1)-1 \,(\text{mod } s)$, then $c'_{t-j(\text{mod } s)}=p-2$ $(j=0, \dots, p^k-1)$. On the other hand $\binom{p^k-1}{j}(-1)^j\equiv 1 \,(\text{mod } p)$, and so, we obtain that $\bar{c}_{s-2(i+1)-1(\text{mod } s)}=p-1$. If $t\neq s-2(i+1)-1 \,(\text{mod } s)$ then among the elements $c'_{t-j(\text{mod } s)}$ $(j=0, \dots, p^k-1)$ exactly one element is different from $p-2$ and this element is equal to $p-1$. From this, by $\binom{p^k-1}{j}(-1)^j \equiv 1 \pmod p$ $(j=0, \ldots, p^k-1)$, we get that $\bar{c}_t=0$. Therefore,

$$\bar{c}_t = \begin{cases} p-1 & \text{if } t = s-2(i+1)-1 \pmod s, \\ 0 & \text{otherwise,} \end{cases} \quad (t = 0, \ldots, s-1).$$

Observe that $(\bar{c}_0, \ldots, \bar{c}_{s-1}) = \mu(i+1 \pmod m)$, and so,

$$\mu(\delta'_m(i, t_1)) = \mu(i+1 \pmod m) = (\bar{c}_0, \ldots, \bar{c}_{s-1}) = \delta_{\mathbf{A}^s}(\mu(i), x_1^{2(p^k-1)}) = $$
$$= \delta_{\mathbf{A}^s}(\mu(i), \tau(t_1))$$

which completes the proof of Case 2. This ends the proof of our Theorem.

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References

[1] DÖMÖSI, P. and IMREH, B., On ν_i-products of automata, Acta Cybernet., 6 (1983), 149—162.
[2] DÖMÖSI, P. and ÉSIK Z., On the hierarchy of ν_i-products of automata, Acta Cybernet., 8 (1988), 253—258.
[3] ÉSIK, Z., Loop products and loop-free products, Acta Cybernet., 8 (1987), 45—48.
[4] ÉSIK, Z. and GÉCSEG, F., On α_0-products and α_2-products, Theoret. Comput. Sci., 48 (1986), 1—8.
[5] GÉCSEG, F., On ν_i-products of commutative automata, Acta Cybernet., 7 (1985), 55—59.
[6] GÉCSEG, F. Metric representations by ν_i-products, Acta Cybernet., 7 (1985), 203—209.
[7] GÉCSEG, F. and IMREH, B., A comparison of α_i-products and ν_i-products, Foundations of Control Engineering, 12 (1987), 3—9.
[8] GÉCSEG, F. and IMREH, B., On metric equivalence of ν_i-products, Acta Cybernet., 8 (1987), 129—134.

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