On the hierarchy of ν_i-products of automata

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In order to decrease the feedback complexity of the Gluškov-type product of automata, a hierarchy of products was introduced by F. Gécseg in [6]. This hierarchy, referred to as the α_i-hierarchy, contains one product concept for each nonnegative integer i. The α_0-product is also known as the loop-free product, the series-parallel composition or the cascade composition [11, 1, 13]. Another hierarchy, the ν_i-hierarchy appears in [2], where i is any positive integer. Using the main result of [3] it has been shown in [5] that for homomorphic realization the α_i-hierarchy collapses at $i=2$. One of the aims of the present paper is to show that the ν_i-hierarchy is strict. For some classes of automata even the ν_1-product has a surprising power. This has been demonstrated in [2] for the first time and then in [7, 4]. In fact there are classes of automata for which the ν_1-product is much stronger than the α_0-product. In this paper we prove that the opposite can also be true for some classes.

An automaton is a system $\mathbf{A}=(A, X, \delta)$ with finite nonempty sets A and X, the state set and the input set, and transition $\delta: A\times X\to A$. The transition is also used in the extended sense, i.e. as a map $\delta: A\times X^*\to A$ where X^* is the free monoid of all words over X. Let $\mathbf{A}_j=(A_j, X_j, \delta_j)$ $(j=1, \ldots, n,\ n\geqq 0)$ be automata, and take a family of feedback functions

$$\varphi_j: A_1\times\ldots\times A_n\times X \to X_j$$

$(j=1, \ldots, n)$, where X is a new finite nonempty set of input letters. The Gluškov-type product (cf. [10]) of the automata $\mathbf{A}_j$ with respect to the feedback functions φ_j is defined to be the automaton

$$\mathbf{A}_1\times\ldots\times\mathbf{A}_n(X, \varphi)$$

with state set $A=A_1\times\ldots\times A_n$, input set X and transition δ given by

$$\mathrm{pr}_j(\delta(a, x)) = \delta_j(\mathrm{pr}_j(a), \varphi_j(a, x)),$$

for all $a\in A$, $x\in X$ and $1\leqq j\leqq n$. The Gluškov-type product is also called the general product, or g-product, for short. Let $i\geqq 1$ be any integer. Following [2], the above defined g-product is called a ν_i-product if for every integer $j=1, \ldots, n$ there is a set

$v(j) \subseteq \{1, \ldots, n\}$ with cardinality not exceeding i such that each feedback function

$$\varphi_j(a_1, \ldots, a_n, x)$$

is independent of any state variable a_k with $k \notin v(j)$. For the definition of the α_i-products see [6, 8].

Let $\mathcal{K}$ be a class of automata. We shall use the following notations:

$\mathbf{P}_g(\mathcal{K})$:=all g-products of automata from $\mathcal{K}$;
$\mathbf{P}_{\alpha_i}(\mathcal{K})$:=all α_i-products of automata from $\mathcal{K}$;
$\mathbf{P}_{v_i}(\mathcal{K})$:=all v_i-products of automata from $\mathcal{K}$;
$\mathbf{S}(\mathcal{K})$:=all subautomata of automata from $\mathcal{K}$;
$\mathbf{H}(\mathcal{K})$:=all homomorphic images of automata from $\mathcal{K}$.

In the sequel we shall also make use of a few simple facts.

Lemma 1. For every class $\mathcal{K}$, $\mathbf{HSP}_{\alpha_0}(\mathcal{K})$ is the smallest class containing $\mathcal{K}$ and closed under the operators $\mathbf{H}$, $\mathbf{S}$ and $\mathbf{P}_{\alpha_0}$.

The proof of Lemma 1 can be found in [8]. We note that a similar statement is true for the g-product.

Lemma 2. Let $\mathbf{A}=\mathbf{A}_1\times\ldots\times\mathbf{A}_n(X, \varphi)$ be a v_i-product of automata $\mathbf{A}_j=(A_j, X_j, \delta_j)$. Let π be a permutation of the set $\{1, \ldots, n\}$. There exists a v_i-product $\mathbf{A}'=\mathbf{A}_{\pi(1)}\times\ldots\times\mathbf{A}_{\pi(n)}(X, \varphi')$ which is isomorphic to $\mathbf{A}$, an isomorphism $\mathbf{A}\to\mathbf{A}'$ is the map $(a_1, \ldots, a_n)\mapsto(a_{\pi(1)}, \ldots, a_{\pi(n)})$ $((a_1, \ldots, a_n)\in A_1\times\ldots\times A_n)$.

Lemma 3. Let $\mathbf{A}=\mathbf{A}_1\times\ldots\times\mathbf{A}_n(X, \varphi)$ be a v_i-product with $n\geqq 1$ and components $\mathbf{A}_j=(A_j, X_j, \delta_j)$. Let $\mathbf{B}=(B, X, \delta)$ be a subautomaton of $\mathbf{A}$, $j_0\in\{1, \ldots, n\}$ a fixed integer and $a\in A_{j_0}$. If $\mathrm{pr}_{j_0}(b)=a$ for all $b\in B$ then there is a v_i-product $\mathbf{A}'=\mathbf{A}_1\times\ldots\times\mathbf{A}_{j_0-1}\times\mathbf{A}_{j_0+1}\times\ldots\times\mathbf{A}_n(X, \varphi')$ such that $\mathbf{A}'$ contains a subautomaton $\mathbf{B}'$ isomorphic to $\mathbf{B}$, an isomorphism $\mathbf{B}\to\mathbf{B}'$ is the map $(a_1, \ldots, a_{j_0-1}, a, a_{j_0+1}, \ldots, a_n)\mapsto(a_1, \ldots, a_{j_0-1}, a_{j_0+1}, \ldots, a_n)$.

We are now ready to state our main result.

Theorem. There exists a class $\mathcal{K}$ of automata such that $\mathbf{HSP}_{v_i}(\mathcal{K})\subset\mathbf{HSP}_{v_{i+1}}(\mathcal{K})\subset\mathbf{HSP}_{\alpha_0}(\mathcal{K})$ holds for all $i\geqq 1$.

Proof. Let p be a prime number. We define an automaton $\mathbf{D}_p=(D_p, \{x, y\}, \delta)$ as follows:

$$D_p = \{0, \ldots, p\},$$

$$\delta(j, x) = \begin{cases} j+1 \bmod p & \text{if } j < p, \\ p & \text{if } j = p, \end{cases}$$

$$\delta(j, y) = p, \quad j\in D_p.$$

Let $\mathcal{K}=\{D_p|p$ is a prime$\}$. We set out to prove the following properties of $\mathcal{K}$.

(1) $\mathbf{HSP}_g(\mathcal{K})\subseteq\mathbf{HSP}_{\alpha_0}(\mathcal{K})$,

(2) $\mathbf{HSP}_{v_i}(\mathcal{K})\subset\mathbf{HSP}_{v_{i+1}}(\mathcal{K})$ for all $i\geqq 1$.

Supposing (1) and (2) have been shown, the proof is easily completed. Since $\mathbf{HSP}_{v_{i+1}}(\mathscr{K}) \subseteq \mathbf{HSP}_g(\mathscr{K})$ holds obviously, from (1) we have $\mathbf{HSP}_{v_{i+1}}(\mathscr{K}) \subseteq \subseteq \mathbf{HSP}_{\alpha_0}(\mathscr{K})$, which in turn implies $\mathbf{HSP}_{v_i}(\mathscr{K}) \subset \mathbf{HSP}_{\alpha_0}(\mathscr{K})$ by (2). Thus $\mathbf{HSP}_{v_i}(\mathscr{K}) \subset \mathbf{HSP}_{\alpha_0}(\mathscr{K})$ for all $i \geqq 1$.

Proof of (1). For every prime number p, define $\mathbf{C}_p = (C_p, \{x\}, \delta)$ by

$$C_p = \{0, \dots, p-1\},$$

$$\delta(j, x) = j+1 \bmod p, j \in C_p.$$

Moreover, let $\mathbf{E} = (E, \{x, y\}, \delta)$ with $E = \{0, 1\}$, $\delta(0, x) = 0$, $\delta(0, y) = \delta(1, x) = = \delta(1, y) = 1$. Thus $\mathbf{C}_p$ is the counter with length p and $\mathbf{E}$ is the elevator. Set

$$\mathscr{K}' = \{\mathbf{C}_p | \, p \quad \text{is a prime}\} \cup \{\mathbf{E}\}.$$

From the proof of the main result of [5] we have $\mathbf{HSP}_g(\mathscr{K}) = \mathbf{HSP}_{\alpha_0}(\mathscr{K}')$. To end the proof, by Lemma 1, it suffices to show that $\mathscr{K}' \subseteq \mathbf{HSP}_{\alpha_0}(\mathscr{K})$. That is however obvious for we have $\mathbf{C}_p \in \mathbf{S}(\{\mathbf{D}_p\})$ and $\mathbf{E} \in \mathbf{H}(\{\mathbf{D}_p\})$, each prime number p.

Proof of (2). Let $i \geqq 1$ be any integer and $m = \prod (p_j | \, j = 1, \dots, i+1)$, where p_j is the j-th prime. Define $\mathbf{M} = (M, \{x, y\}, \delta)$ to be the automaton with

$$M = \{0, \dots, m\},$$

$$\delta(j, x) = \begin{cases} j+1 \bmod m & \text{if} \quad j < m, \\ m & \text{if} \quad j = m, \end{cases}$$

$$\delta(j, y) = \begin{cases} j+1 \bmod m & \text{if} \quad 0 < j < m, \\ m & \text{if} \quad j = 0 \quad \text{or} \quad j = m, \end{cases}$$

for all $j \in M$. We prove that $\mathbf{M} \notin \mathbf{HSP}_{v_i}(\mathscr{K})$ while $\mathbf{M} \in \mathbf{HSP}_{v_{i+1}}(\mathscr{K})$.

Assume that, on the contrary, $\mathbf{M} \in \mathbf{HSP}_{v_i}(\mathscr{K})$. Let

$$\mathbf{D}_{q_1} \times \dots \times \mathbf{D}_{q_n}(\{x, y\}, \varphi)$$

be a v_i-product of automata from $\mathscr{K}$ that contains a subautomaton $\mathbf{A} = (A, \{x, y\}, \delta)$ which is mapped onto $\mathbf{M}$ under a suitable homomorphism h. We may choose n to be the least (positive) integer with the above property, i.e. if a v_i-product of automata from $\mathscr{K}$ contains a subautomaton that can be mapped homomorphically onto $\mathbf{M}$ then the number of factors of that product is at least n. Also, the subautomaton $\mathbf{A}$ can be chosen such that none of its proper subautomata is mapped homomorphically onto $\mathbf{M}$.

Let us write A as the disjoint union $A = A_0 \cup A_1$ where $A_0 = h^{-1}(M - \{m\})$ and $A_1 = h^{-1}(\{m\})$. Let $a \in A_0$ be a state. Since a is a generator of A, if $\mathrm{pr}_j(a) = q_j$ for an integer $j = 1, \dots, n$, then $\mathrm{pr}_j(b) = q_j$ for all $b \in A$. By Lemma 3, there exists a v_i-product

$$\mathbf{D}_{q_1} \times \dots \times \mathbf{D}_{q_{j-1}} \times \mathbf{D}_{q_{j+1}} \times \dots \times \mathbf{D}_{q_n}(\{x, y\}, \varphi')$$

that contains a subautomaton isomorphic to $\mathbf{A}$. This contradicts the minimality of n. Thus $\mathrm{pr}_j(a) \neq q_j$ for all $a \in A_0$ and $j = 1, \dots, n$. Suppose now that there is an $a \in A_1$ such that for all $j = 1, \dots, n$ we have $\mathrm{pr}_j(a) \neq q_j$. Let $b \in A_0$ be a state and $u \in \{x, y\}^*$

a word with $\delta(b, u)=a$. Let $v=x^k$ where k denotes the length of u. We have $c=\delta(b, v)\in A_0$, henceforth $\mathrm{pr}_j(c)\neq q_j$ for all $j=1, \ldots, n$. The special structure of the automata $\mathbf{D}_{q_j}$ guarantees that $a=c$. This contradiction yields that for every $a\in A_1$ there is an integer $j(1\leqq j\leqq n)$ with $\mathrm{pr}_j(a)=q_j$.

Let $a_0=(a_{0,1}, \ldots, a_{0,n}), \ldots, a_{q-1}=(a_{q-1,1}, \ldots, a_{q-1,n})$ be all the states in A_0, so that $a_{t,j}\neq q_j$, $0\leqq t\leqq q-1$, $1\leqq j\leqq n$. By the minimality of **A** and the special structure of the automata $\mathbf{D}_{q_j}$ it follows that the letter x induces a cyclic permutation of the states a_t, say $\delta(a_t, x)=a_{t+1 \bmod q}$. Also q is the l.c.m. of the primes $q_1, \ldots, q_n$. Since h is a homomorphism of **A** onto **M**, we have $q\equiv 0 \bmod m$. Without loss of generality we may suppose $\delta(a_0, y)=a\in A_1$. Thus $\mathrm{pr}_j(a)=q_j$ for some j. By Lemma 2, we may take $j=1$. Since $\mathrm{pr}_1(a)=q_1$ we must have $\varphi_1(a_0, y)=y$. Let $\nu(1)=\{j_1, \ldots, j_k\}$, so that $k\leqq i$. Define $\bar{q}$ to be the l.c.m. of the primes on the list $q_{j_1}, \ldots, q_{j_k}$. Obviously then $q\equiv 0 \bmod \bar{q}$. Since m is the product of $i+1$ distinct primes and $\bar{q}$ is the product of at most i distinct primes, from $q\equiv 0 \bmod m$ and $q\equiv 0 \bmod \bar{q}$ we obtain $\bar{q}<q$. Let us now consider the state $a_{\bar{q}}=(a_{\bar{q},1}, \ldots, a_{\bar{q},n})$. For every $l=1, \ldots, k$ we have $\delta(a_{0,j_l}, x^{\bar{q}})=a_{\bar{q},j_l}\neq q_{j_l}$. Since $\bar{q}\equiv 0 \bmod q_{j_l}$ we see that $a_{\bar{q},j_l}=a_{0,j_l}$. Since we have a ν_i-product it follows that $\varphi_1(a_{\bar{q}}, y)=\varphi_1(a_0, y)=y$. We conclude $\delta(a_{\bar{q}}, y)\in A_1$. Since h is a homomorphism of **A** onto **M** we see that $\bar{q}\equiv 0 \bmod m$. This is however clearly impossible for m is the product of $i+1$ distinct primes and $\bar{q}$ is the product of at most i distinct primes.

We still have to show that $\mathbf{M}\in\mathbf{HSP}_{\nu_{i+1}}(\mathscr{K})$. For this define the g-product

$$\mathbf{A} = (A, X, \delta) = \mathbf{D}_{p_1}\times\ldots\times\mathbf{D}_{p_{i+1}}(\{x, y\}, \varphi)$$

by

$$\varphi_j(a_1, \ldots, a_{i+1}, x) = x,$$

$$\varphi_j(a_1, \ldots, a_{i+1}, y) = \begin{cases} y & \text{if } a_1 = \ldots = a_{i+1} = 0, \\ x & \text{otherwise.} \end{cases}$$

Since the number of factors is $i+1$, this g-product is also a ν_{i+1}-product. Define

$$A_0 = \{a\in A|\mathrm{pr}_j(a) \neq p_j \quad \text{for all} \quad j = 1, \ldots, i+1\},$$

$$A_1 = A-A_0.$$

For an $a=(a_1, \ldots, a_{i+1})\in A_0$ let $h(a)=t$ be that integer $0\leqq t<m$ with $t\equiv a_j \bmod p_j$, $j=1, \ldots, i+1$. If $a\in A_1$ put $h(a)=m$. The mapping h is easily seen to be a homomorphism of **A** onto **M**. □

Remark. It is said that an automaton $\mathbf{A}=(A, X, \delta)$ satisfies the Letičevskiĭ criterion if there exist a state $a\in A$, input letters $x_1, x_2\in X$ and words $u_1, u_2\in X^*$ with $\delta(a, x_1)\neq\delta(a, x_2)$ and $\delta(a, x_1u_1)=\delta(a, x_2u_2)=a$. If only $\delta(a, x_1)\neq\delta(a, x_2)$ and $\delta(a, x_1u)=a$ hold for some $a\in A$, $x_1, x_2\in X$ and $u\in X_1^*$, we say that **A** satisfies the semi-Letičevskiĭ criterion. The above definitions extend to classes of automata: a class $\mathscr{K}$ satisfies the Letičevskiĭ criterion or the semi-Letičevskiĭ criterion if one of its members satisfies it. By a classical result in [12], $\mathbf{HSP}_g(\mathscr{K})$ is the class of all automata if and only if $\mathscr{K}$ satisfies the Letičevskiĭ criterion. It has been shown in [3] that the same is true for the α_2-product. If $\mathscr{K}$ does not satisfy the semi-Letičevskiĭ criterion then, by the proof of the main result in [5], $\mathbf{HSP}_g(\mathscr{K})=\mathbf{HSP}_{\alpha_0}(\mathscr{K})$. Also $\mathbf{HSP}_g(\mathscr{K})=\mathbf{HSP}_{\nu_1}(\mathscr{K})$ in this case as shown in [9]. Suppose now that $\mathscr{K}$

satisfies the semi-Letičevskiĭ criterion but does not satisfy the Letičevskiĭ criterion. In [5] it is proved that for every such $\mathscr{K}$ we have $\mathbf{HSP}_g(\mathscr{K})=\mathbf{HSP}_{\alpha_1}(\mathscr{K})$. The ν_i-products behave quite differently. The class $\mathscr{K}$ given in the proof of our Theorem satisfies the semi-Letičevskiĭ criterion but does not satisfy the Letičevskiĭ criterion, moreover, there exists no integer $i \geqq 1$ with $\mathbf{HSP}_g(\mathscr{K})=\mathbf{HSP}_{\nu_i}(\mathscr{K})$.

Open problems. (1) Suppose that $\mathscr{K}$ satisfies the Letičevskiĭ criterion. Does there exist an integer $i \geqq 1$ with $\mathbf{HSP}_g(\mathscr{K})=\mathbf{HSP}_{\nu_i}(\mathscr{K})$? (2) Does there exist an integer $i \geqq 1$ such that $\mathbf{HSP}_g(\mathscr{K})=\mathbf{HSP}_{\nu_i}(\mathscr{K})$ whenever $\mathscr{K}$ satisfies the Letičevskiĭ criterion? What is the least such i, if it exists?

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