On homomorphic simulation of automata by α_0-products

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1. Introduction

The concept of the α_0-product of automata is equivalent to the cascade composition or loop-free product (see [1, 7]). In an α_0-product, the feedback functions admit only strict letter-to-letter replacement as opposed to the generalized α_0-product where input words may correspond to input letters. Thus the generalized α_0-product is closely related to the wreath product of transformation semigroups and/or monoids, see [1, 4]. The α_0-product and the above generalization are usually studied in conjunction with homomorphic realization or homomorphic simulation. The difference between the concepts of homomorphic realization and homomorphic simulation is similar to the difference between the α_0-product and the generalized α_0-product: for simulation the action of an input letter is related to the action of an input word rather than to the action of an input letter. It is a matter of fact that the homomorphic realization is equivalent to the homomorphic simulation with respect to the generalized α_0-product. In the present paper we study homomorphic simulations of automata by α_0-products. We give a sufficient condition on a class $\mathcal{K}$ of automata ensuring that an automaton be homomorphically simulated by a generalized α_0-product over $\mathcal{K}$ if and only if it is homomorphically simulated by an α_0-product of automata from $\mathcal{K}$. As an application it is shown that a class $\mathcal{K}$ is complete with respect to the homomorphic simulation by the generalized α_0-product if and only if it is complete with respect to the homomorphic simulation by the α_0-product, as far as nonempty words are considered.

2. Preliminaries

For a finite nonempty set X we let X^* denote the free monoid of all *words* over X and write X^+ for the free semigroup $X^*-\{\lambda\}$, where λ is the *empty word.* We set $X^\lambda=X\cup\{\lambda\}$. The *length* of a word $u\in X^*$ is denoted $|u|$. If $u=x_1\dots x_n$ with the x's in X, then for each $i\in[n]=\{1, \dots, n\}$ we define $u(i)=x_i$ and $u[i]=x_1\dots x_{i-1}$.

An *automaton* is a triple $\mathbf{A}=(A, X, \delta)$ with finite nonempty set A *(state set), X (input letters)* and *transition* δ: $A\times X\to A$ that extends to a mapping $A\times X^*\to A$ as usual. If $u\in X^*$ we write $u^{\mathbf{A}}$ for the transformation $A\to A$ given by $au^{\mathbf{A}}=\delta(a, u)$,

$a \in A$. The *characteristic monoid (semigroup)* $S_1(\mathbf{A})$, $(S(\mathbf{A}))$ of $\mathbf{A}$ consists of all the transformations $u^{\mathbf{A}}$ with $u \in X^*$ $(u \in X^+)$.

Let $\mathbf{A}=(A, X, \delta)$ be an automaton. We define $\mathbf{A}^*=(A, S_1(\mathbf{A}), \delta^*)$ and $\mathbf{A}^+=(A, S(\mathbf{A}), \delta^+)$ to be the automata with $\delta^*(a, s)=as$ and $\delta^+(a, t)=at$, for all $a \in A$, $s \in S_1(\mathbf{A})$ and $t \in S(\mathbf{A})$. Likewise we put $\mathbf{A}^\lambda=(A, \{u^{\mathbf{A}} | u \in X^\lambda\}, \delta^\lambda)$ with $\delta^\lambda(a, u^{\mathbf{A}})=au^{\mathbf{A}}$. The automata $\mathbf{A}^*$ and $\mathbf{A}^+$ thus correspond to the transformation monoid and the transformation semigroup of $\mathbf{A}$, see [3].

Given a family of automata $\mathbf{A}_i=(A_i, X_i, \delta_i)$ $(i \in [n],\ n \geqq 0)$ and a finite nonempty set X together with *feedback functions*

$$\varphi_i\colon A_1 \times \ldots \times A_{i-1} \times X \to X_i,$$

the α_0-*product* (cf. [8])

$$\mathbf{A}_1 \times \ldots \times \mathbf{A}_n(X, \varphi)$$

is defined to be the automaton $\mathbf{A}=(A, X, \delta)$ with

$$A = A_1 \times \ldots \times A_n$$

and

$$\delta((a_1, \ldots, a_n).\, x) = (\delta_1(a_1, x_1), \ldots, \delta_n(a_n, x_n)),$$

$$x_i = \varphi_i(a_1, \ldots, a_{i-1}, x) \quad (i \in [n]),$$

for all $(a_1, \ldots, a_n) \in A$ and $x \in X$. The α_0-product is equivalent to the cascade composition or the loop-free product (cf. [1, 7]).

We let $\mathbf{H}$, $\mathbf{S}$ and $\mathbf{P}_{\alpha_0}$ denote the operator corresponding to the formation of homomorphic images, subautomata and α_0-products, resp. Thus, if $\mathcal{K}$ is a class of automata, then $\mathbf{P}_{\alpha_0}(\mathcal{K})$ is the class of all α_0-products of automata from $\mathcal{K}$. Further, we let $\mathbf{P}_{1\alpha_0}(\mathcal{K})$ be the class

$$\{\mathbf{A}(X, \varphi) | \mathbf{A} \in \mathcal{K},\ \mathbf{A}(X, \varphi) \text{ is an } \alpha_0\text{-product}\}$$

and define $\mathcal{K}^* = \bigcup(\mathbf{P}_{1\alpha_0}(\mathbf{A}^*) | \mathbf{A} \in \mathcal{K})$, $\mathcal{K}^+ = \bigcup(\mathbf{P}_{1\alpha_0}(\mathbf{A}^+) | \mathbf{A} \in \mathcal{K})$ and $\mathcal{K}^\lambda = \bigcup(\mathbf{P}_{1\alpha_0}(\mathbf{A}^\lambda) | \mathbf{A} \in \mathcal{K})$.

If $\mathbf{O}$ is one of the operators $\mathbf{S}$ and $\mathbf{P}_{\alpha_0}$, then by $\mathbf{O}^*(\mathcal{K})$ $(\mathbf{O}^+(\mathcal{K}), \mathbf{O}^\lambda(\mathcal{K}))$ we denote the class $\mathbf{O}(\mathcal{K}^*)$ $(\mathbf{O}(\mathcal{K}^+), \mathbf{O}(\mathcal{K}^\lambda))$. We have $\mathbf{P}^*_{\alpha_0}(\mathcal{K})=\mathbf{P}_{\alpha_0}(\{\mathbf{A}^* | \mathbf{A} \in \mathcal{K}\})$, $\mathbf{P}^+_{\alpha_0}(\mathcal{K})=\mathbf{P}_{\alpha_0}(\{\mathbf{A}^+ | \mathbf{A} \in \mathcal{K}\})$ and $\mathbf{P}^\lambda_{\alpha_0}(\mathcal{K})=\mathbf{P}_{\alpha_0}(\{\mathbf{A}^\lambda | \mathbf{A} \in \mathcal{K}\})$. Moreover, $\mathbf{A} \in \mathbf{HS}^*(\{\mathbf{B}\})$ for automata $\mathbf{A}=(A, X, \delta)$ and $\mathbf{B}=(B, Y, \delta')$ if and only if, there exist a set $B' \subseteq B$, an onto mapping $h\colon B' \to A$ and a mapping $\varphi\colon X \to Y^*$ such that $\delta(h(b), x) = h(\delta'(b, \varphi(x)))$ for all $b \in B'$ and $x \in X$. It is understood that $\delta'(b, \varphi(x)) \in B'$. Similar fact is true for the combined operators $\mathbf{HS}^+$ and $\mathbf{HS}^\lambda$. In [6] the relation $\mathbf{A} \in \mathbf{HS}^*(\{\mathbf{B}\})$ is expressed by saying that $\mathbf{A}$ is *homomorphically simulated* by $\mathbf{B}$. We also note that the operators $\mathbf{HS}^*$ and $\mathbf{HS}^+$ correspond to the covering relation (or division) of transformation monoids and/or transformation semigroups, see [4].

In the sequel we shall also make use of another view of the operators $\mathbf{P}^*_{\alpha_0}$, $\mathbf{P}^+_{\alpha_0}$ and $\mathbf{P}_{\alpha_0}$. Define the concept of the α_0^*-*product* (α_0^+-*product*, α_0^λ-*product*) in exact analogue with the α_0-product except for the fact that each feedback function φ assumes values in X_i^* (X_i^+, X_i^λ). In this setting $\mathbf{P}^*_{\alpha_0}$ $(\mathbf{P}^+_{\alpha_0}, \mathbf{P}^\lambda_{\alpha_0})$ becomes the operator of forming α_0^*-products (α_0^+-products, α_0^λ-products). It is apparent that the generalized α_0-products, i.e. the α_0^*-product and the α_0^+-product, are closely related to the wreath product of transformation semigroups, cf. [4].

The above defined operators and the combined ones, e.g. $\mathbf{HS^*P}_{\alpha_0}$, satisfy a number of simple closure properties that we shall use implicitly. In this paper the emphasis will be on the combinations $\mathbf{HS^*P}^*_{\alpha_0}$ vs $\mathbf{HS^*P}_{\alpha_0}$, and also on $\mathbf{HS^+P}^+_{\alpha_0}$ and $\mathbf{HS^+P}_{\alpha_0}$.

Also the operators $\mathbf{HSP}^*_{\alpha_0}$ and $\mathbf{HSP}^+_{\alpha_0}$ could be of interest. These are however discarded due to the following simple fact, see also [7].

Proposition 2.1. For every class $\mathscr{K}$ and modifier $m\in\{*, +, \lambda\}$ it holds that $\mathbf{S}^m\mathbf{P}_{\alpha_0}(\mathscr{K})\subseteqq\mathbf{S}^m\mathbf{P}^m_{\alpha_0}(\mathscr{K})=\mathbf{SP}^m_{\alpha_0}(\mathscr{K})$.

The inclusion $\mathbf{HS^*P}_{\alpha_0}(\mathscr{K})\subseteqq\mathbf{HS^*P}^+_{\alpha_0}(\mathscr{K})$, just as $\mathbf{HS^+P}_{\alpha_0}(\mathscr{K})\subseteqq\mathbf{HS^+P}^+_{\alpha_0}(\mathscr{K})$, cannot usually be turned to equality. E.g. if $\mathscr{K}$ consists of a single counter with prime length $p>1$, i.e. $\mathscr{K}=\{\mathbf{C}\}$ with $\mathbf{C}=([p], \{x\}, \delta)$, $\delta(i, x)\equiv i+1 \bmod p$, then $\mathbf{HS^*P}_{\alpha_0}(\mathscr{K})=\mathbf{HS^+P}_{\alpha_0}(\mathscr{K})$ consist of commutative automata with very simple structure. On the other hand, $\mathbf{HS^+P}^*_{\alpha_0}(\mathscr{K})=\mathbf{HS^+P}^+_{\alpha_0}(\mathscr{K})$ is the class of all automata that could be called p-automata: i.e. permutation automata whose characteristic monoid is a p-group. The latter observation follows from the Krohn—Rhodes Decomposition Theorem, see below. In the next section there is given a sufficient condition ensuring $\mathbf{HS^*P}^*_{\alpha_0}(\mathscr{K})=\mathbf{HS^*P}_{\alpha_0}(\mathscr{K})$. In fact the condition will quarantee that $\mathbf{HS^*P}^*_{\alpha_0}(\mathscr{K})=\mathbf{HS^*P}_{\alpha_0}(\mathscr{K})=\mathbf{HS^+P}^+_{\alpha_0}(\mathscr{K})=\mathbf{HS^+P}_{\alpha_0}(\mathscr{K})$.

Some more terminology. By a semigroup we always mean a finite semigroup. We put $S|T$, i.e., S divides T, for semigroups S and T, if and only if S is a homomorphic image of a subsemigroup of T. If S is a monoid (group), it is equivalent to saying that S is a homomorphic image of a submonoid (subgroup) of T, see [1, 4]. (When talking about a submonoid M of a semigroup S which is a monoid, M is not required to contain the identity of S.) The following fact is known, see [4] and also [7] for the group case.

Lemma 2.2. Let $\mathbf{A}=(A, X, \delta)$ be an automaton and M a submonoid of $S(\mathbf{A})$ or $S_1(\mathbf{A})$. There exists a nonempty set $B\subseteqq A$ with the following properties:

(i) The elements of M map B into itself.

(ii) The restriction of the identity of M to B is the identical mapping $B\to B$.

(iii) If m_1 and m_2 are distinct elements of M then $m_1(b)\neq m_2(b)$ for at least one $b\in B$.

To end this section we mention one more useful fact whose proof is omitted. For similar, in fact stronger statements, see [4]. A trivial automaton is an automaton with a single state.

Lemma 2.3. If $S_1(\mathbf{A})|S_1(\mathbf{B})$ for two automata $\mathbf{A}, \mathbf{B}$ and either $\mathbf{B}$ is nontrivial or $\mathbf{A}$ is trivial, then $\mathbf{A}\in\mathbf{HSP}^*_{\alpha_0}(\{\mathbf{B}\})$.

3. The results

We start with an auxiliary definition. Let $\mathbf{A}=\mathbf{A}_1\times\ldots\times\mathbf{A}_n(X, \varphi)$ $(n\geqq 1)$ be an α_0^+-product with components $\mathbf{A}_i=(A_i, X_i, \delta_i)$ and let $\mathbf{B}=(B, X, \delta)$ be a subautomaton of $\mathbf{A}$. For an integer $i\in[n]$ the *useful* states of $\mathbf{A}_i$ (with respect to $\mathbf{B}$) are those states in A_i which occur in the place of the i-th component of the elements of B.

Lemma 3.1. Let **A** and **B** be automata as above. Suppose that for each $x \in X$ an integer $k_x \geqq 1$ is given with

$$|\varphi_i(a_1, \ldots, a_{i-1}, x)| = k_x,$$

for all $i \in [n]$ and $(a_1, \ldots, a_{i-1}) \in A_1 \times \ldots \times A_{i-1}$. Assume further that for each i and x there is a word $p_i^x \in X_i^+$ with $|p_i^x| = k_x$ and $\delta_i(a_i, p_i^x) = a_i$ whenever $a_i \in A_i$ is useful. Then **B** is isomorphic to an automaton in $\mathbf{S}^+(\{\mathbf{A}'\})$ for an α_0-product $\mathbf{A}' = \mathbf{A}_1 \times \ldots \times \mathbf{A}_n(Y, \psi)$.

Proof. For every $x \in X$ let Y_x be a new set of $m_x = nk_x$ input letters, say

$$Y_x = \{y_j^x \mid j \in [m_x]\}.$$

Set $Y = \bigcup(Y_x \mid x \in X)$. To define the feedback function ψ_i, $i \in [n]$, let $a_1 \in A_1, \ldots, a_{i-1} \in A_{i-1}$ be fixed states and $y_j^x \in Y_x$. Let $t \in [k_x]$ be that integer with $j \equiv t \bmod k_x$. If there is an $s \in [i-1]$ such that a_s is not of the form $\delta_s(b_s, p_s^x[t])$ for some useful state $b_s \in A_s$, then $\psi_i(a_1, \ldots, a_{i-1}, y_j^x)$ is any letter in X_i. Otherwise there are uniquely determined useful states $b_1 \in A_1, \ldots, b_{i-1} \in A_{i-1}$ with $\delta_s(b_s, p_x^s[t]) = a_s$, $s \in [i-1]$. If $j \leqq (i-1)k_x$ or $j > ik_x$ then we define

$$\psi_i(a_1, \ldots, a_{i-1}, y_j^x) = p_i^x(t).$$

Finally, if $(i-1)k_x < j \leqq ik_x$, we put

$$\psi_i(a_1, \ldots, a_{i-1}, y_j^x) = q(t),$$

where

$$q = \varphi_i(b_1, \ldots, b_{i-1}, x).$$

This ends the definition of the α_0-product $\mathbf{A}'$.

Let $x \in X$ be any letter and define

$$u^x = u_n^x \ldots u_1^x,\ u_j^x = y_{(j-k)k_x+1}^x \ldots y_{jk_x}^x,$$

$(j \in [n])$. Denote by δ' the transition of $\mathbf{A}'$. To see that **B** is in $\mathbf{S}^+(\{\mathbf{A}'\})$, it suffices to show that for any $b = (b_1, \ldots, b_n) \in B$ and $x \in X$ we have $\delta'(b, u^x) = \delta(b, x)$. This is however obvious, for if $\delta(b, x) = c = (c_1, \ldots, c_n)$, then for each $i \in [n]$ we can compute as follows:

$$\begin{aligned}
\delta'((b_1, \ldots, b_{i-1}, b_i, c_{i+1}, \ldots, c_n), u_i^x) &= \\
= (\delta_1(b_1, p_1^x), \ldots, \delta_{i-1}(b_{i-1}, p_{i-1}^x)),& \\
\delta_i(b_i, \varphi_i(b_1, \ldots, b_{i-1}, x)),& \\
\delta_{i+1}(c_{i+1}, p_{i+1}^x), \ldots, \delta_n(c_n, p_n^x)) &= \\
= (b_1, \ldots, b_{i-1}, c_i, c_{i+1}, \ldots, c_n).&
\end{aligned}$$

A straightforward induction argument completes the proof.

Recall that a permutation automaton $\mathbf{A} = (A, X, \delta)$ is an automaton such that δ_x is a permutation of the state set for each $x \in X$. Equivalently, **A** is a permutation automaton if and only if $S_1(\mathbf{A})$ is a group. Note that $S_1(\mathbf{A}) = S(\mathbf{A})$ for a permutation automaton.

Remark. If the automata $\mathbf{A}_i$ of the previous lemma were permutation automata, then a much simpler argument could be applied. In fact we could define

$$Y_x = \{y_j^x | j \in [k_x]\}, \quad Y = \bigcup (Y_x | x \in X),$$

and then

$$\psi_i(a_1, \ldots, a_{i-1}, y_j^x) = (\varphi_i(b_1, \ldots, b_{i-1}, x))(j),$$

where the states b_s, $s \in [n]$, are successively determined by the condition

$$\delta_s(b_s, (\varphi_s(b_1, \ldots, b_{s-1}, x))[j]) = a_s.$$

For a more general form of the following definition see [4]. Let M be a monoid and $\mathbf{A}=(A, X, \delta)$ an automaton. We write $M \| S(\mathbf{A})$ $(M \| S_1(\mathbf{A}))$ if and only if there exists a submonoid M' of $S(\mathbf{A})$ $(S_1(\mathbf{A}))$ which can be mapped homomorphically onto M and such that $M' \subseteq \{u^{\mathbf{A}} | u \in X^*, \ |u|=n\}$ for an integer $n>0$ $(n \geqq 0)$. Notice that $M \| S(\mathbf{A})$ if and only if $M \| S_1(\mathbf{A})$, for if $M \| S_1(\mathbf{A})$ with $n=0$ then M' is trivial and so is M.

Theorem 3.2. Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two classes of automata. Assume that $\mathcal{K}_1$ contains an automaton $\mathbf{A}_0$ such that $S_1(\mathbf{A}_0)$ is a nontrivial monoid. Assume further that for every $\mathbf{A} \in \mathcal{K}_1$ there is $\mathbf{B} \in \mathcal{K}_2$ with $S_1(\mathbf{A}) \| S(\mathbf{B})$. Then $\mathbf{HS^*P}^*_{\alpha_0}(\mathcal{K}_1) \subseteq \subseteq \mathbf{HS^+P}^{\,!}_{\alpha_0}(\mathcal{K}_2)$.

Proof. First note that $\mathbf{HS^*P}^*_{\alpha_0}(\mathcal{K}_1) = \mathbf{HS^*P}^*_{\alpha_0}(\mathcal{K}_1 - \mathcal{K}_0)$, where $\mathcal{K}_0$ is the class of all automata with trivial characteristic monoid. (The class $\mathcal{K}_0$ can also be called the class of discrete automata, for an automaton belongs to $\mathcal{K}_0$ if and only if each input letter induces the identical state transformation.) Thus it suffices to prove that $\mathbf{HS^*P}^*_{\alpha_0}(\mathcal{K}_1 - \mathcal{K}_0) \subseteq \mathbf{HS^+P}_{\alpha_0}(\mathcal{K}_2)$; or even, by Proposition 2.1, it is enough to show the inclusion $\mathbf{P}^*_{\alpha_0}(\mathcal{K}_1 - \mathcal{K}_0) \subseteq \mathbf{HS^+P}_{\alpha_0}(\mathcal{K}_2)$.

Let $\mathbf{A} = \mathbf{A}_1 \times \ldots \times \mathbf{A}_n(X, \varphi)$ be any α_0^*-product with components $\mathbf{A}_i \in \mathcal{K}_1 - \mathcal{K}_0$. If $n=0$ then $\mathbf{A}$ is trivial, so that $\mathbf{A} \in \mathbf{HS^+P}_{\alpha_0}(\mathcal{K}_2)$. Assume $n>0$. For every $i \in [n]$ there are an automaton $\mathbf{B}_i = (B_i, X_i, \delta^i) \in \mathcal{K}_2$, a submonoid M_i of $S(\mathbf{B}_i)$ and an integer $k_i > 0$ such that $M_i \subseteq \{u^{\mathbf{B}_i} | u \in X_i^+, |u|=k_i\}$ and $S_1(\mathbf{A}_i)$ is a homomorphic image of M_i. Let k be the l.c.m. of the numbers k_i. If $u_0^{\mathbf{B}_i}$ is the identity of M_i and $|u_0|=k_i$, then for any $u^{\mathbf{B}_i} \in M_i$ with $|u|=k_i$ we have $u^{\mathbf{B}_i} = w^{\mathbf{B}_i}$ where $w = u u_0^{k/k_i - 1}$. It follows that $M_i \subseteq \{u^{\mathbf{B}_i} | u \in X_i^+, \ |u|=k\}$.

Let $i \in [n]$ be a fixed integer. Since M_i is a submonoid of $S(\mathbf{B}_i)$, there is a (nonempty) set $B_i' \subseteq B_i$ as in Lemma 2.2. Define the automaton $\mathbf{B}_i' = (B_i', M_i, \delta_i)$ by $\delta_i(b, m) = m(b)$, for all $b \in B_i'$ and $m \in M$. M_i is isomorphic to $S_1(B_i')$ and every transformation in $S_1(B_i')$ is induced by a letter in M_i. Since $S_1(\mathbf{A}_i)$ is a homomorphic image of $S_1(B_i')$ and $S_1(A_i)$ is nontrivial, from Lemma 2.3 we obtain $\mathbf{A}_i \in \mathbf{HSP}^*_{\alpha_0}(\{B_i'\})$.

We have seen that $\mathbf{A}_i \in \mathbf{HSP}^*_{\alpha_0}(\{\mathbf{B}_i'\})$ for all i. Consequently also $\mathbf{A} \in \in \mathbf{HSP}^*_{\alpha_0}(\{\mathbf{B}_1', \ldots, \mathbf{B}_n'\})$, and since the members of each $S_1(\mathbf{B}_i')$ are induced by input letters, $\mathbf{A} \in \mathbf{HSP}_{\alpha_0}(\{\mathbf{B}_1', \ldots, \mathbf{B}_n'\})$. Let $\mathbf{B}' = \mathbf{B}_{i_1}' \times \ldots \times \mathbf{B}_{i_t}'(X, \varphi')$ be an α_0-product of the automata $\mathbf{B}_1', \ldots, \mathbf{B}_n'$ containing a subautomaton which can be mapped homomorphically onto $\mathbf{A}$. We define an α_0^+-product $\mathbf{B} = \mathbf{B}_{i_1} \times \ldots \times \mathbf{B}_{i_t}(X, \psi)$ as follows. For each $j \in [t]$, let $u_j \in X_{i_j}^+$ be a fixed word with $|u_j|=k$, and to each $(b_1, \ldots, b_{j-1}) \in \in B_{i_1}' \times \ldots \times B_{i_{j-1}}'$ and $x \in X$ let us correspond a word $u = u(b_1, \ldots, b_{j-1}, x) \in X_{i_j}^+$ with $|u|=k$ and $u^{\mathbf{B}_{i_j}} = \varphi_j'(b_1, \ldots, b_{j-1}, x)$. Then for all $j \in [t]$, $(b_1, \ldots, b_{j-1}) \in$

$\in B_{i_1}\times\ldots\times B_{i_{j-1}}$ and $x\in X$ let

$$\psi_j(b_1, \ldots, b_{j-1}, x) = \begin{cases} u(b_1, \ldots, b_{j-1}, x) & \text{if} \quad (b_1, \ldots, b_{j-1})\in B'_{i_1}\times\ldots B'_{i_{j-1}}, \\ u_j & \text{otherwise.}\end{cases}$$

It is easy to see that **B** contains an isomorphic copy of **B**′, in fact **B**′ is a subautomaton of **B**. The α_0^+-product **B** and the subautomaton **B**′ satisfies the assumptions of Lemma 3.1, therefore $\mathbf{B}'\in \mathbf{S}^+\mathbf{P}_{\alpha_0}(\{\mathbf{B}_1, \ldots, \mathbf{B}_n\})\subseteq \mathbf{S}^+\mathbf{P}_{\alpha_0}(\mathscr{K}_2)$. Since $\mathbf{A}\in\mathbf{HS}(\{\mathbf{B}'\})$ it follows that $\mathbf{A}\in\mathbf{HS}^+\mathbf{P}_{\alpha_0}(\mathscr{K}_2)$. The proof is complete.

Notice that also $\mathbf{HS}^*\mathbf{P}^*_{\alpha_0}(\mathscr{K}_1)\subseteq\mathbf{HS}^*\mathbf{P}_{\alpha_0}(\mathscr{K}_2)$ and $\mathbf{HS}^+\mathbf{P}^+_{\alpha_0}(\mathscr{K}_1)\subseteq\mathbf{HS}^+\mathbf{P}_{\alpha_0}(\mathscr{K}_2)$.

It should be noted that if $\mathscr{K}_1$ consists of discrete automata one of which is nontrivial, then $\mathbf{HS}^*\mathbf{P}^*_{\alpha_0}(\mathscr{K}_1)=\mathscr{K}_0$, the class of all discrete automata. Moreover, $\mathbf{HS}^*\mathbf{P}^*_{\alpha_0}(\mathscr{K}_1)\subseteq\mathbf{HS}^+\mathbf{P}_{\alpha_0}(\mathscr{K}_2)$ if and only if $\mathscr{K}_2$ contains an automaton **A** which is not definite, i.e., which has two distinct states a_1, a_2 and a nonempty input word u with $a_iu^{\mathbf{A}}=a_i$, $i=1, 2$.

Next we give a reformulation of Theorem 3.2 and discuss some consequences. For a monoid M, define $\mathrm{Aut}\,(M)=(M, M, \delta)$ with $\delta(m_1, m_2)=m_1m_2$. If $\mathscr{M}$ is a class of monoids, set $\mathrm{Aut}\,(\mathscr{M})=\{\mathrm{Aut}\,(M)|M\in\mathscr{M}\}$.

Corollary 3.3. Let $\mathscr{M}$ be a class of monoids and $\mathscr{K}$ a class of automata. Suppose that for each $M\in\mathscr{M}$ there is az automaton $\mathbf{A}\in\mathscr{K}$ with $M\|S(\mathbf{A})$. Then $\mathbf{HS}^*\mathbf{P}^*_{\alpha_0}(\mathrm{Aut}\,(\mathscr{M}))=\mathbf{HSP}_{\alpha_0}(\mathrm{Aut}\,(\mathscr{M}))\subseteq\mathbf{HS}^+\mathbf{P}_{\alpha_0}(\mathscr{K})$.

Corollary 3.4. Let $\mathscr{K}_1$ and $\mathscr{K}_2$ be two classes of automata such that for each $\mathbf{A}\in\mathscr{K}_1$ there is an automaton $\mathbf{B}\in\mathscr{K}_2$ with $S_1(\mathbf{A})|S_1(\mathbf{B})$. Suppose further that either $\mathscr{K}_1$ consists of trivial automata or $\mathscr{K}_2$ contains a nontrivial automaton. Then $\mathbf{HS}^*\mathbf{P}^*_{\alpha_0}(\mathscr{K}_1)\subseteq\mathbf{HS}^+\mathbf{P}^\lambda_{\alpha_0}(\mathscr{K}_2)$.

Proof. If $\mathscr{K}_1$ consists of discrete automata then the result is obvious. Otherwise there is an automaton $\mathbf{A}_0\in\mathscr{K}_1$ such that $S_1(\mathbf{A}_0)$ is nontrivial. If $S_1(\mathbf{A})|S_1(\mathbf{B})$ then $S_1(\mathbf{A})\|S(\mathbf{B}^\lambda)$. Thus the inclusion $\mathbf{HS}^*\mathbf{P}^*_{\alpha_0}(\mathscr{K}_1)\subseteq\mathbf{HS}^+\mathbf{P}^\lambda_{\alpha_0}(\mathscr{K}_2)$ is obtained by applying Theorem 3.2 for $\mathscr{K}_1$ and $\mathscr{K}_2^\lambda$.

Corollary 3.5. Let $\mathscr{K}$ be any class of automata. If for every $\mathbf{A}\in\mathscr{K}$ there exists $\mathbf{B}\in\mathscr{K}$ with $S_1(\mathbf{A})\|S(\mathbf{B})$ then $\mathbf{HS}^*\mathbf{P}^*_{\alpha_0}(\mathscr{K})=\mathbf{HS}^*\mathbf{P}_{\alpha_0}(\mathscr{K})=\mathbf{HS}^+\mathbf{P}^+_{\alpha_0}(\mathscr{K})=\mathbf{HS}^+\mathbf{P}_{\alpha_0}(\mathscr{K})$. Moreover, $\mathbf{HS}^*\mathbf{P}^*_{\alpha_0}(\mathscr{K})=\mathbf{HS}^+\mathbf{P}^\lambda_{\alpha_0}(\mathscr{K})$ holds universally.

The Krohn—Rhodes Decomposition Theorem (cf. [1, 4, 7]) is a basis for studying the α_0-product. Below we give one possible formalization in terms of the operators **H**, $\mathbf{S}^+$, $\mathbf{S}^*$, $\mathbf{P}^+_{\alpha_0}$ and $\mathbf{P}^*_{\alpha_0}$. Following [1], by U_3 we denote the three-element monoid with two right zeros. An irreducible semigroup is a semigroup S such that for every nonempty class $\mathscr{K}$, if $S|S_1(\mathbf{A})$ for some $\mathbf{A}\in\mathbf{HS}^*\mathbf{P}^*_{\alpha_0}(\mathscr{K})$ then there is an automaton $\mathbf{B}\in\mathscr{K}$ with $S|S_1(\mathbf{B})$. Equivalently this means that for every nonempty class $\mathscr{K}$, if $S|S(\mathbf{A})$ for an automaton $\mathbf{A}\in\mathbf{HS}^+\mathbf{P}^+_{\alpha_0}(\mathscr{K})$ or $\mathbf{A}\in\mathbf{HSP}_{\alpha_0}(\mathscr{K})$ then $S|S(\mathbf{B})$ for some $\mathbf{B}\in\mathscr{K}$. Notice that for a group G the conditions $G|S_1(\mathbf{A})$ and $G|S(\mathbf{A})$ are equivalent.

Theorem 3.6. *Krohn—Rhodes Decomposition Theorem.*

(1) For every group G let $\mathbf{A}_G$ be any automaton with $G|S(\mathbf{A}_G)$ and let $\mathbf{A}_0$ be an automaton with $U_3|S_1(\mathbf{A}_0)$ $(U_3|S(\mathbf{A}_0))$. Given an automaton **A**, define $\mathscr{K}=\{\mathbf{A}_G|G$ is a simple group with $G|S(\mathbf{A})\}$. Then $\mathbf{A}\in\mathbf{HS}^*\mathbf{P}^*_{\alpha_0}(\mathscr{K}\cup\{\mathbf{A}_0\})$

($\mathbf{A}\in\mathbf{HS^+P^+_{\alpha_0}}(\mathcal{K}\cup\{\mathbf{A}_0\})$). If $\mathbf{A}$ is a permutation automaton and $S_1(\mathbf{A})$ is nontrivial, then $\mathbf{A}\in\mathbf{HS^+P^*_{\alpha_0}}(\mathcal{K})$.

(2) A semigroup S is irreducible if and only if S is a simple group or $S|U_3$.

The monoids M with $M|U_3$, $M\neq U_3$, are the trivial monoid and the two-element monoid U_2 with a right zero. Let $\mathcal{G}$ be a nonempty class of simple groups closed under division, i.e. such that $G\in\mathcal{G}$ and $H|G$ implies $H\in\mathcal{G}$ for every simple group H. We define:

$$\mathcal{K}_3(\mathcal{G}) = \mathbf{HSP}_{\alpha_0}(\mathrm{Aut}(\mathcal{G}\cup\{U_3\})),$$

$$\mathcal{K}_2(\mathcal{G}) = \mathbf{HSP}_{\alpha_0}(\mathrm{Aut}(\mathcal{G}\cup\{U_2\})),$$

$$\mathcal{K}_0(\mathcal{G}) = \mathbf{HSP}_{\alpha_0}(\mathrm{Aut}(\mathcal{G})).$$

Note that $\mathcal{K}_3(\mathcal{G})=\mathbf{HS^*P^*_{\alpha_0}}(\mathrm{Aut}(\mathcal{G}\cup\{U_3\}))$ and similarly for $\mathcal{K}_2(\mathcal{G})$ and $\mathcal{K}_0(\mathcal{G})$. The avoid trivial situations, when writing $\mathcal{K}_0(\mathcal{G})$, we shall always assume that $\mathcal{G}$ contains a nontrivial group. As a direct consequence of the Krohn—Rhodes Decomposition Theorem we have:

Corollary 3.7.
(i) $\mathcal{K}_3(\mathcal{G})\subseteq\mathbf{HS^*P^*_{\alpha_0}}(\mathcal{K})$ ($\mathcal{K}_3(\mathcal{G})\subseteq\mathbf{HS^+P^+_{\alpha_0}}(\mathcal{K})$) if and only if the following hold:
(i_1) For every $G\in\mathcal{G}$ there is $\mathbf{A}\in\mathcal{K}$ with $G|S(\mathbf{A})$.
(i_2) There is an automaton $\mathbf{A}\in\mathcal{K}$ with $U_3|S_1(\mathbf{A})$ ($U_3|S(\mathbf{A})$).
(ii) $\mathcal{K}_2(\mathcal{G})\subseteq\mathbf{HS^*P^*_{\alpha_0}}(\mathcal{K})$ ($\mathcal{K}_2(\mathcal{G})\subseteq\mathbf{HS^+P^+_{\alpha_0}}(\mathcal{K})$) if and only if ($\mathrm{i}_1$) and ($\mathrm{ii}_1$) hold:
(ii_1) There is $\mathbf{A}\in\mathcal{K}$ with $U_2|S_1(\mathbf{A})$ ($U_2|S(\mathbf{A})$).
(iii) $\mathcal{K}_0(\mathcal{G})\subseteq\mathbf{HS^*P^*_{\alpha_0}}(\mathcal{K})$ ($\mathcal{K}_0(\mathcal{G})\subseteq\mathbf{HS^+P^+_{\alpha_0}}(\mathcal{K})$) if and only if ($\mathrm{i}_1$) holds.

We note that $U_2|S_1(\mathbf{A})$ for an automaton $\mathbf{A}$ if and only if $\mathbf{A}$ is not a permutation automaton. In order to establish similar results for the operators $\mathbf{HS^*P}_{\alpha_0}$ and $\mathbf{HS^+P}_{\alpha_0}$, we need the following facts. Proposition 3.8 derives from a strong result in [2], for a direct proof see also [6].

Proposition 3.8. Let G be any group and $\mathbf{A}$ an automaton. If $G|S(\mathbf{A})$ then $G'\|S(\mathbf{A})$, where G' denotes the commutator group of G.

Corollary 3.9. Let G be a nonabelian simple group and $\mathbf{A}$ an automaton. If $G|S(\mathbf{A})$ then $G\|S(\mathbf{A})$.

Proposition 3.10. If for $i=2, 3$ we have $U_i|S(\mathbf{A})$ then $U_i\|S(\mathbf{A})$.

Proposition 3.11. Let G be a nontrivial simple group. If $G\|S(\mathbf{A})$ for an automaton $\mathbf{A}\in\mathbf{HSP}\alpha_0(\mathcal{K})$, where $\mathcal{K}$ is any class of automata, then $G\|S(\mathbf{B})$ for some $\mathbf{B}\in\mathcal{K}$.

The proof of Proposition 3.10 is trivial. Proposition 3.11 is from [5]. In the rest of the paper $\mathcal{G}$ denotes a fixed class of simple groups closed under division. Recall that when dealing with $\mathcal{K}_0(\mathcal{G})$ it is assumed that $\mathcal{G}$ contains a nontrivial group.

Theorem 3.12. Let $\mathscr{K}$ be a class of automata.

(i) $\mathscr{K}_3(\mathscr{G}) \subseteq \mathbf{HS}^*\mathbf{P}_{\alpha_0}(\mathscr{K})$ if and only if (i_1)—(i_3) hold:

(i_1) For every nonabelian $G \in \mathscr{G}$ there is $\mathbf{A} \in \mathscr{K}$ with $G|S(\mathbf{A})$.

(i_2) For every abelian $G \in \mathscr{G}$ there is $\mathbf{A} \in \mathscr{K}$ with $G \| S(\mathbf{A})$.

(i_3) There is an automaton $\mathbf{A} \in \mathscr{K}$ with $U_3|S(\mathbf{A})$.

(ii) $\mathscr{K}_2(\mathscr{G}) \subseteq \mathbf{HS}^*\mathbf{P}_{\alpha_0}(\mathscr{K})$ if and only if (i_1), (i_2) and (ii_1) hold:

(ii_1) There is an automaton $\mathbf{A} \in \mathscr{K}$ with $U_2|S(\mathbf{A})$.

(iii) $\mathscr{K}_0(\mathscr{G}) \subseteq \mathbf{HS}^*\mathbf{P}_{\alpha_0}(\mathscr{K})$ if and only if (i_1) and (i_2) hold.

Proof. We only prove the first statement. Assuming $\mathscr{K}_3(\mathscr{G}) \subseteq \mathbf{HS}^*\mathbf{P}_{\alpha_0}(\mathscr{K})$ also $\mathscr{K}_3(\mathscr{G}) \subseteq \mathbf{HS}^*\mathbf{P}^*_{\alpha_0}(\mathscr{K})$. Thus (i_1) follows from the Krohn—Rhodes Decomposition Theorem. Let G be a nontrivial abelian simple group in $\mathscr{G}$, say $G = Z_p$, the cyclic group of order p. Let H be any nonabelian p-group. We have $\mathrm{Aut}\,(H) \in \mathscr{K}_3(\mathscr{G})$ from the Krohn—Rhodes Decomposition Theorem. Thus also $\mathrm{Aut}\,(H) \in \mathbf{HS}^*\mathbf{P}_{\alpha_0}(\mathscr{K})$ and, henceforth, there is an automaton $\mathbf{B} \in \mathbf{P}_{\alpha_0}(\mathscr{K})$ with $H|S(\mathbf{B})$. But then $H' \| S(\mathbf{B})$ follows from Proposition 3.8. Since H' is a nontrivial p-group we have $Z_p | H'$. Since $H' \| S(\mathbf{B})$ also $Z_p \| S(\mathbf{B})$ and, by Proposition 3.11, $Z_p \| S(\mathbf{A})$ for some $\mathbf{A} \in \mathscr{K}$. Thus (i_2) is satisfied by $\mathscr{K}$. To see that (i_3) holds, let $\mathbf{A}_0 = (A_0, X, \delta)$ be an automaton in $\mathscr{K}_3(\mathscr{G})$ with $U_3|S(\mathbf{A}_0)$ and such that none of the transformations $x^{\mathbf{A}_0}$, $x \in X$, is the identical mapping $A_0 \to A_0$. Since $\mathbf{A}_0 \in \mathbf{HS}^*\mathbf{P}_{\alpha_0}(\mathscr{K})$, the above property yields $\mathbf{A}_0 \in \mathbf{HS}^+\mathbf{P}_{\alpha_0}(\mathscr{K}) \subseteq \mathbf{HS}^+\mathbf{P}^+_{\alpha_0}(\mathscr{K})$. The Krohn—Rhodes Decomposition Theorem implies $U_3|S(\mathbf{A})$ for some $\mathbf{A} \in \mathscr{K}$. This ends the proof of the necessity.

Conversely the assumptions (i_1)—(i_3), Corollary 3.9 and Proposition 3.10 imply that for every $G \in \mathscr{G}$ there is $\mathbf{A} \in \mathscr{K}$ with $G \| S(\mathbf{A})$ and similarly for U_3. Apply Corollary 3.3.

Corollary 3.13. For each $i = 0, 2, 3$; $\mathscr{K}_i(\mathscr{G}) \subseteq \mathbf{HS}^*\mathbf{P}_{\alpha_0}(\mathscr{K})$ if and only if $\mathscr{K}_i(\mathscr{G}) \subseteq \mathbf{HS}^+\mathbf{P}_{\alpha_0}(\mathscr{K})$.

Corollary 3.14. The following are equivalent for a class $\mathscr{K}$ of automata:

(i) $\mathbf{HS}^*\mathbf{P}_{\alpha_0}(\mathscr{K})$ is the class of all automata.

(ii) $\mathbf{HS}^+\mathbf{P}_{\alpha_0}(\mathscr{K})$ is the class of all automata.

(iii) $\mathbf{HS}^+\mathbf{P}^+_{\alpha_0}(\mathscr{K})$ is the class of all automata.

Completeness criteria for the operator $\mathbf{HSP}_{\alpha_0}$ are formulated in [6, 3].

Abstract

A sufficient condition is given on a class $\mathscr{K}$ of automata ensuring that an automaton be homomorphically simulated by a generalized α_0-product (loop-free product) over $\mathscr{K}$ if and only if it is homomorphically simulated by an α_0-product with components in $\mathscr{K}$. As an application it is proved that a class $\mathscr{K}$ of automata is complete with respect to the homomorphic simulation by generalized α_0-products if and only if it is complete with respect to the homomorphic simulation by α_0-products, as far as nonempty words are considered.

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