On the worst-case performance of the *NkF* bin-packing heuristic*

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Introduction

In bin packing, we are given a list

$$L = (s_1, s_2, \ldots, s_n)$$

of items (elements) with a weight function on items and a sequence of unit-capacity bins $B_1, B_2, \ldots$. In this paper, we assume that the item weights are real numbers in the range (0, 1] and that the list is given by the weights. The problem is to find a packing of the items in the bins such that the sum of the items in each bin is not greater than 1, and the number of bins used is minimized.

This problem is NP-hard [GJ] and therefore heuristic algorithms which give "good" solutions in an acceptable computing time are investigated [J], [JDUGG]. We are interested in the worst-case behaviour of the Next-k Fit (*NkF*) algorithm. For this, an upper and a lower bound were given in Johnson's paper. We shall improve both bounds.

Preliminary definitions and notations

For a list L, let $\mathrm{OPT}(L)$ be the number of bins in optimal packing. For a given heuristic algorithm A, let $A(L)$ be the number of bins used by A to pack L. Let $R_A^N = \max\left\{\frac{A(L)}{\mathrm{OPT}(L)} \mid L \text{ is a list with } \mathrm{OPT}(L)=N\right\}$.

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The *asymptotic worst-case ratio* of A is then defined as

$$R_A = \limsup_{N\to\infty} R_A^N.$$

Let

$$s(L) = \sum_{i=1}^{n} s_i$$

and let $s(B_i)$ denote the sum of the weights of the items in B_i.

We investigate the NkF algorithm, which is defined as follows: we always use k bins at the same time. If the next element, a_j, is coming, we place it into the first of the k used bins which has enough room for it. If no such bin has enough room, we close the first (oldest) of these k bins, open a new one, and put a_j into this bin (this will now be the k-th or youngest bin).

Johnson has proved for the asymptotic worst-case ratio of NkF that

$$1.7+\frac{3}{10k} \leqq R_{NkF} \leqq 2.$$

In this paper we prove that

$$R_{N2F} = 2$$

and that for $k \geqq 3$

$$1.7+\frac{3}{10(k-1)} \leqq R_{NkF} \leqq 1.75+\frac{7}{4(2k+3)}.$$

However, the exact worst-case ratio is not known for $k \geqq 3$.

Results

First we give an upper bound on R_{NkF} for $k \geqq 3$. Let L be an arbitrary list and let us pack the elements of L by means of NkF. Let $B_1, B_2, \ldots, B_r$ denote the sequence of bins used and let m be a fixed nonnegative integer. For any positive integer $i \leqq r+1-m$ the sequence of the bins $B_i, B_{i+1}, \ldots, B_{i+m-1}$ is called a *parcel consisting of m bins* if the following conditions hold

(a) $s(B_t) > 1/2 \quad (t = i, \ldots, i+m-1)$,

(b) $i+m-1 = r$ or $i+m-1 < r \,\&\, s(B_{i+m}) \leqq 1/2$.

We classify the bins of a parcel consisting of m bins with respect to their contents as follows:

(A) $\Sigma s_t \geqq 2/3 \,\&\, (\exists t)(s_t > 1/2)$,

(B) $\Sigma s_t \geqq 2/3 \,\&\, (\forall t)(s_t \leqq 1/2)$,

(C) $\Sigma s_t < 2/3 \,\&\, (\exists t)(s_t > 1/2)$,

(D) $\Sigma s_t < 2/3 \,\&\, (\forall t)(s_t \leqq 1/2)$, where t runs through the set of indices of the items contained in the considered bin. Obviously, we obtain a partition of the bins $B_i, \dots, B_{i+m-1}$. We shall use the terminology *X-bin* for a bin which is contained in the class determined by the property X, where $X \in \{A, B, C, D\}$. It may be observed that any D-bin contains at least two items; moreover, it contains an item with $s_t \leqq 1/3$.

For the D-bins, the following statement holds.

Lemma 1. There are at most two D-bins among any $k+1$ successive bins of any parcel consisting of $m \geqq k+1$ bins.

Proof. Let $B_1^*, B_2^*, \dots, B_{k+1}^*$ denote the considered bins. Let $1 \leqq i < j \leqq k+1$ and let us suppose that B_i^* is the D-bin with smallest index and that B_j^* is the D-bin with second smallest index. If $j=k+1$, then the statement obviously holds. Now let us assume that $j<k+1$. After the packing of L the empty room in B_i^* is greater than 1/3. Accordingly, the empty room in it is greater than 1/3 when the first item is packed in B_j^*. Therefore $1/3 < s_1$ holds for this item. By our assumption, B_j^* is a D-bin; thus, $s_1 \leqq 1/2$ and during the further packing at least one item with weight less than 1/3 will be packed in B_j^*. Let us investigate the circumstance of the packing of the first such item. It should be observed that the bin B_i^* contains enough empty room for this item. Therefore, the packing of this item in B_j^* implies that at this time the bin B_i^* is already closed. This results that, up to the closing of B_i^* the content of B_j^* is not greater than 1/2. But then, the weight of the first packed item in B_{j+u}^* is greater than 1/2 if $u \in \{1, \dots, k+1-j\}$. This means that $B_{j+1}^*, \dots, B_{k+1}^*$ are of types A or C, which yields the validity of our statement.

Lemma 2. For any $k+1$ successive bins of any parcel consisting of $m \geqq k+1$ bins if there exists a C-bin among the considered bins and if there exists a D-bin among the bins succeeding the C-bin, then the bins succeeding the D-bin are of types A or C.

Proof. In the proof of Lemma 1 we only made use of the fact that B_i^* has empty room greater than 1/3 and this property holds for any C-bin, too; thus, by repeating the proof of Lemma 1, we obtain the validity of Lemma 2.

Any $k+2$ successive bins of a parcel consisting of $m \geqq k+2$ bins is called a *block*. We classify the blocks as follows:

(1) it contains at most one D-bin,

(2) it contains two D-bins or it contains three D-bins and at least one B-bin,

(3) it contains three D-bins and at least one A-bin; moreover, the remaining $k-2$ bins are of types A or C,

(4) it contains three D-bins and $k-1$ C-bins.

From Lemma 1 it follows that any block contains at most three D-bins, and so the above classification induces a partition of the blocks. We shall use the terminology *j-block* or block of type j if it has the j-th property for some $j \in \{1, \dots, 4\}$.

Now let us consider an arbitrary block of a parcel consisting of $m \geqq k+2$ bins. Let s denote the sum of the weights of the items contained in the bins of the block and let q' and q be the numbers of its A-bins and C-bins, respectively.

The following statement then holds.

Lemma 3. For any $r \in \{1, \ldots, 4\}$ if a block is of type r, then the r-th assertion holds for it among the following ones:

(1) $s \geqq (k+2)\frac{2}{3}-(q+1)\frac{1}{6}$,

(2) $s \geqq (k+2)\frac{2}{3}-(q+2)\frac{1}{6}$,

(3) $s \geqq (k+2)\frac{2}{3}-(q+3)\frac{1}{6} \,\&\, q+q' = k-1 \,\&\, q' > 0$,

(4) $s \geqq (k+2)\frac{2}{3}-(q+3)\frac{1}{6} \,\&\, q = k-1$.

Proof. In the cases $r=1$, $r=3$ and $r=4$ the statement follows from the definitions. If $r=2$ and the block contains only two D-bins, then the assertion is again obvious.

Now let us suppose that the considered block contains three D-bins and at least one B-bin. Let $B_1^*, \ldots, B_{k+2}^*$ denote the bins of the block. Since it contains three D-bins, by using Lemma 1 twice, we obtain that B_1^* and B_{k+2}^* are D-bins. Let us assume that B_j^* is the intermediate D-bin for some $2 \leqq j \leqq k+1$. By our assumption, the block contains a B-bin. Let B_l^* denote this bin, where $2 \leqq l \leqq k+1$ and $l \neq j$. We distinguish the following two cases.

Case 1. Let us suppose that $l<j$. Then $l \leqq k$, and so, at the time of opening of B_l^*, the bin B_1^* is open. On the other hand, B_1^* is a D-bin, and so, after the packing of all elements of L, the bin B_1^* contains empty room with weight $\frac{1}{3}+\Delta$, where $\Delta>0$. But then B_1^* contains empty room with weight at least $\frac{1}{3}+\Delta$ when the first item is packed in the bin B_l^*. Therefore, $\frac{1}{3}+\Delta<s_1$ holds for this item. Moreover, since B_l^* is a B-bin, $s_1 \leqq 1/2$. We now distinguish two subcases.

If at the time of the packing of the second item of B_l^*, the bin B_1^* is open, then for the weight s_2 of this item $\frac{1}{3}+\Delta<s_2 \leqq 1/2$ again holds. But then

$$s(B_1^*)+s(B_l^*) \geqq \frac{2}{3}-\Delta+2\left(\frac{1}{3}+\Delta\right) \geqq 2\cdot\frac{2}{3},$$

and so

$$s = s(B_1^*)+s(B_l^*)+\sum_{t\neq 1, t\neq l} s(B_t^*) \geqq 2\cdot\frac{2}{3}+k\cdot\frac{2}{3}-(q+2)\frac{1}{6},$$

which yields the validity of our statement.

If at the time of the packing of the second item of B_l^* the bin B_1^* is closed, then up to the closing of B_1^* the content of B_l^* is not greater than $\frac{1}{2}$. This results that the weight of the first packed item in B^*_{l+u} is greater than 1/2 if $u \in \{1, ..., k+1-l\}$. This means that the bins $B^*_{l+1}, ..., B^*_{k+1}$ are of types A or C, which contradicts our assumption on B^*_j. Therefore, this case is impossible.

Case 2. Let us suppose that $j<l$. Then $2 \leqq j < l \leqq k+1$, and so, at the opening of B^*_l the bin B^*_j is open. Next, in the same way as in Case 1 we obtain that $\frac{1}{3}+\Delta < s_1 \leqq 1/2$ holds for the first packed item in B^*_l.

If at the time of the packing of the second item of B^*_l the bin B^*_j is open, then, similarly as in Case 1, we obtain the validity of (2).

If at the considered time B^*_j is closed, then up to the closing of B^*_j the content of B^*_l is not greater than 1/2. This yields that the weight of the first packed item in B^*_{l+u} is greater than 1/2 if $u \in \{1, ..., k+2-l\}$. But then, $B^*_{l+1}, ..., B^*_{k+2}$ are of types A or C, which contradicts our assumption. Therefore this case is impossible, which completes the proof of Lemma 3.

Lemma 4. For any parcel consisting of m bins, the following assertions hold:

(I) there exists at most one D-bin among the last $z=\min\{k, m\}$ bins of the parcel;

(II) if $m \geqq k+2$, then the type of the block consisting of the last $k+2$ bins of the parcel is less than 4;

(III) if the first block among two successive blocks of the parcel is of type 4, then the type of the second block is 1 or 2, and in the last case the block contains at least one A-bin.

Proof. For assertions (I) and (II), we have to distinguish two subcases according to the definition of the parcel.

Case I/a. Let us suppose that the last z bins of the considered parcel are the last z bins of the packing of L. Then, these bins are all open at the packing of the very last item of L. Let $B^*_1, ..., B^*_z$ denote the considered bins and let us assume that B^*_i and B^*_j are D-bins, where $1 \leqq i < j \leqq z$. Then B^*_i has empty room with weight $\frac{1}{3}+\Delta$, where $\Delta > 0$. Therefore, $\frac{1}{3}+\Delta < s_1 \leqq 1/2$ holds for the first packed item (s_1) in B^*_j, and so $s_2 < 1/3$ holds for the weight s_2 of the second item of B^*_j. At the time of the packing of this item, the bin B^*_i is open and has empty room with weight $\frac{1}{3}+\Delta$; thus the NkF algorithm places this item in B^*_i, which is a contradiction. Therefore, there exists at most one D-bin among the considered z bins.

Case I/b. Let us suppose that the considered $B^*_1, ..., B^*_z$ bins are not the last z bins of the packing of L and that $s(B^*_{z+1}) \leqq 1/2$ holds for the following bin B^*_{z+1} of the packing. Now let us assume that B^*_i and B^*_j are D-bins, where $1 \leqq i < j \leqq z$. Then B^*_i has empty room with weight $\frac{1}{3}+\Delta$, where $\Delta > 0$. Therefore, $\frac{1}{3}+\Delta < s_1 \leqq 1/2$ holds for the first packed item (s_1) in B^*_j, and so $s_2 < 1/3$ holds for the weight s_2 of the second item of B^*_j. Thus, at the time of the packing of this item the bin B^*_i

is closed. This yields that, up to the closing of B_i^* the content of B_j^* is not greater than 1/2. But then, the weight of the first packed item in B_{j+u}^* is greater than 1/2 if $u\in\{1, ..., z-j+1\}$. This contradicts our assumption on B_{z+1}^*. Therefore, there is at most one D-bin among the considered z bins.

Case II/a. Let us assume that the bins of the considered block are the last $k+2$ bins of the packing of L and that the block is of type 4. Let us $B_1^*, ..., B_{k+2}^*$ denote the considered bins. Then, by using Lemma 1 twice, we obtain that B_1^* and B_{k+2}^* are D-bins. Now let us suppose that B_j^* is the intermediate D-bin for some $2\leqq j\leqq k+1$. If $j>2$, then B_2^* is a C-bin, since the block contains only D-bins and C-bins. But then, by Lemma 2, we obtain that the bins $B_{j+1}^*, ..., B_{k+2}^*$ are not of type D, which is a contradiction. Thus, $j=2$ and $B_3^*, ..., B_{k+1}^*$ are of type C. Since the considered $k+2$ bins are the last $k+2$ bins of the packing, the bins $B_3^*, ..., B_{k+2}^*$ are all open when the second item is placed in B_{k+2}^*. On the other hand, B_3^* is a C-bin, and so it has empty room with weight $\frac{1}{3}+\Delta$, where $\Delta>0$. Thus, $\frac{1}{3}+\Delta<s_1\leqq\leqq 1/2$ holds for the weight s_1 of B_{k+2}^* and $s_2<1/3$ holds for the weight s_2 of the second item of B_{k+2}^*. But, at the time of the packing of this item, B_3^* is open and it has empty room with weight $\frac{1}{3}+\Delta$; thus the NkF algorithm places this item in B_3^*, which is a contradiction. Therefore, the type of the considered block is less than 4.

Case II/b. Let us suppose that the considered m bins are not the last m bins of the packing and that $s(B')\leqq 1/2$ holds for the bin B' immediately succeeding the last bin of the parcel. Moreover, let us assume that the block is of type 4. Let $B_1^*, ..., B_{k+2}^*$ denote the bins of the block, and let B_{k+3}^* denote the bin B'. Then, by our assumption, $s(B_{k+3}^*)\leqq 1/2$. Now, in the same ways as in Case II/a, we obtain that B_1^*, B_2^*, B_{k+2}^* are D-bins and $B_3^*, ..., B_{k+1}^*$ are C-bins. Since B_3^* is a C-bin, it has empty room with weight $\frac{1}{3}+\Delta$, where $\Delta>0$. Thus, $\frac{1}{3}+\Delta<s_1\leqq 1/2$ holds for the weight s_1 of the first packed item in B_{k+2}^*. On the other hand, B_{k+2}^* is a D-bin, and so $s_2<1/3$ holds for the weight s_2 of the second item of B_{k+2}^*. Thus, at the time of the packing of this item, the bin B_3^* is closed. Therefore, up to the closing of B_3^* the content of B_{k+2}^* is not greater than 1/2. But then, the weight of the first packed item in B_{k+3}^* is greater than 1/2, which contradicts our assumption. Therefore, the type of the considered block is less than 4.

Case III. Let us suppose that the parcel contains two successive blocks and that the first of them is of type 4. Let $B_1^*, ..., B_{k+2}^*$ denote the bins of the first block and $B_{k+3}^*, ..., B_{2k+4}^*$ the bins of the second block. Then, in the same way as above, we obtain that B_1^*, B_2^*, B_{k+2}^* are D-bins and B_{k+1}^* is a C-bin. But then, by Lemma 2, the bins $B_{k+3}^*, ..., B_{2k+1}^*$ are of types A or C. On the other hand, $B_{k+4}^*, ..., B_{2k+4}^*$ are $k+1$ successive bins of the parcel, and so, by Lemma 1, there are at most two D-bins among these bins. Since B_{k+3}^* is of type A or C, we obtain that the second block contains at most two D-bins. Now let us investigate the bins $B_{k+3}^*, ..., B_{2k+1}^*$. Since $k\geqq 3$, the number of the investigated bins is at least 2. If there exists an A-bin among these bins, then assertion (III) obviously holds. In the opposite case, B_{k+3}^* and B_{k+4}^* are C-bins. On the other hand, the bins $B_{k+4}^*, ..., B_{2k+4}^*$ are $k+1$ successive bins of the parcel, and so, by Lemma 2, we obtain that there exists at most one D-bin among these bins, which results the validity of assertion (III).

This ends the proof of Lemma 4.

For any parcel consisting of m bins let s denote the sum of the weights of the items contained in the bins of the parcel and let q' and q denote the numbers of its A-bins and C-bins, respectively. Let $w=q+q'$. Then, the following statement holds.

Theorem 1. For any parcel consisting of m bins

$$s \geqq \frac{2}{3}m-\frac{1}{6}(w+1)-\frac{1}{3}\frac{m-1}{k+2}.$$

Proof. Depending on the value of m, we distinguish five cases.

1. $m=0$. In this case the statement obviously holds.

2. $1 \leqq m \leqq k$. Then, by assertion (I) of Lemma 4, we obtain that the parcel contains at most one D-bin, and so

$$s \geqq \frac{2}{3}m-\frac{1}{6}(q+1) \geqq \frac{2}{3}m-\frac{1}{6}(w+1)-\frac{1}{3}\frac{m-1}{k+2}.$$

3. $m=k+1$. Then, by Lemma 1, the parcel contains at most two D-bins, and so

$$s \geqq \frac{2}{3}m-\frac{1}{6}(q+2) \geqq \frac{2}{3}m-\frac{1}{6}(w+1)-\frac{1}{3}\frac{m-1}{k+2}.$$

4. $m=r(k+2)$ where r is a positive integer. Let us index the successive blocks with the numbers $1, \ldots, r$ according to their sequence, and let $I=\{1, \ldots, r\}$. Let $i \in I$ and let q_i' and q_i denote the numbers of A-bins and C-bins of the i-th block, respectively. For any index $j \in \{1, \ldots, 4\}$, let u_j denote the number of j-blocks and I_j the set of indices of these blocks. By assertion (II) of Lemma 4, the r-th block is not a 4-block, and so, there exists a further block for any 4-block from the considered blocks. On the other hand, by assertion (III) of Lemma 4, the block succeeding some 4-block of the parcel is of type 1 or 2. Using this observation, we classify the 4-blocks into the following two classes.

The first class contains all 4-blocks for which the following block is of type 1. Let u_{41} denote the number of these 4-blocks and I_{41} the set of their indices.

The other class contains the remaining 4-blocks.

The block succeeding some 4-block from this class is then of type 2. Let u_{42} denote the number of the blocks of the second class and I_{42} the set of their indices.

It is now obvious that $u_4=u_{41}+u_{42}$, $I_4=I_{41} \cup I_{42}$, $u_1+u_2+u_3+u_4=r$ and $\bigcup_{j=1}^{4} I_j=I$. Using the introduced notations, by Lemma 3, we obtain

$$s \geqq \sum_{j=1}^{2} \sum_{i \in I_j}\left[(k+2)\frac{2}{3}-(q_i+j)\frac{1}{6}\right]+\sum_{i \in I_3 \cup I_4}\left[(k+2)\frac{2}{3}-(q_i+3)\frac{1}{6}\right],$$

and so

$$s \geqq \sum_{i\in I}(k+2)\frac{2}{3}-\frac{1}{6}\sum_{i\in I}q_i-\frac{1}{6}\sum_{i\in I_1}1-\frac{1}{6}\sum_{i\in I_2}2-\frac{1}{6}\sum_{i\in I_3\cup I_4}3=$$

$$=\frac{2}{3}m-\frac{1}{6}q-\frac{1}{6}u_1-\frac{2}{6}u_2-\frac{3}{6}(u_3+u_4)=$$

$$=\frac{2}{3}m-\frac{1}{6}q-\frac{2}{6}(u_1+u_2+u_3+u_4)+\frac{1}{6}u_1-\frac{1}{6}u_3-\frac{1}{6}u_4=$$

$$=\frac{2}{3}m-\frac{1}{6}q-\frac{1}{3}r+\frac{1}{6}u_1-\frac{1}{6}u_3-\frac{1}{6}u_{41}-\frac{1}{6}u_{42}=$$

$$=\frac{2}{3}m-\frac{1}{6}q-\frac{1}{3}\frac{m}{k+2}+\frac{1}{6}(u_1-u_{41})-\frac{1}{6}u_3-\frac{1}{6}u_{42}.$$

From the definition of u_{41}, it follows that $u_1 \geqq u_{41}$. Thus

$$s \geqq \frac{2}{3}m-\frac{1}{6}q-\frac{1}{3}\frac{m}{k+2}-\frac{1}{6}u_3-\frac{1}{6}u_{42}=$$

$$=\frac{2}{3}m-\frac{1}{6}\Big(1+q+\sum_{i\in I}q_i'\Big)+\frac{1}{6}+\frac{1}{6}\sum_{i\in I}q_i'-\frac{1}{3}\frac{m}{k+2}-\frac{1}{6}u_3-\frac{1}{6}u_{42}=$$

$$=\frac{2}{3}m-\frac{1}{6}(w+1)-\frac{1}{3}\frac{m}{k+2}+\frac{1}{6}+\frac{1}{6}\Big(\sum_{i\in I_3}q_i'-u_3\Big)+\frac{1}{6}\Big(\sum_{i\in I_2}q_i'-u_{42}\Big)+\frac{1}{6}\sum_{i\in I_1\cup I_4}q_i'.$$

From the definition of 3-blocks, we obtain that $\sum_{i\in I_3}q_i'-u_3\geqq 0$, moreover, from Lemma 4 and from the definition of u_{42} it follows that $\sum_{i\in I_2}q_i'-u_{42}\geqq 0$. Therefore,

$$s \geqq \frac{2}{3}m-\frac{1}{6}(w+1)-\frac{1}{3}\frac{m}{k+2}+\frac{1}{6}+\frac{1}{6}\sum_{i\in I_1\cup I_4}q_i'.$$

On the other hand, $\sum_{i\in I_1\cup I_4}q_i'\geqq 0$, and so we obtain the following inequality:

(i) $$s \geqq \frac{2}{3}m-\frac{1}{6}(w+1)-\frac{1}{3}\frac{m-1}{k+2}+\frac{1}{6}-\frac{1}{3(k+2)}.$$

Since $k\geqq 3$, $\frac{1}{6}-\frac{1}{3(k+2)}\geqq 0$. But then

$$s \geqq \frac{2}{3}m-\frac{1}{6}(w+1)-\frac{1}{3}\frac{m-1}{k+2},$$

which completes the proof of this case.

5. $m=r(k+2)+l$, where r and l are positive integers and $1\leqq l\leqq k+1$. We dinstinguish two cases depending on the r-th block.

Case 5/a. Let us suppose that the r-th block is not of type 4. Then disregarding the last l bins, for the remaining $r(k+2)$ bins the same conditions holds as in the previous case. Thus, for the sum $\bar{s}$ of the weights of the items contained in these bins the inequality (i) holds, i.e.

$$\bar{s} \geqq \frac{2}{3}r(k+2)-\frac{1}{6}\Big(1+\sum_{i=1}^{r}(q_i+q_i')\Big)-\frac{1}{3}\frac{r(k+2)-1}{k+2}+\frac{1}{6}-\frac{1}{3(k+2)}.$$

On the other hand, it may be observed that the last l bins form a parcel consisting of l bins. Thus for the sum $\tilde{s}$ of the weights of the items contained in these bins, it holds that

$$\tilde{s} \geqq \frac{2}{3}l-\frac{1}{6}(\bar{q}+\bar{q}'+1)-\frac{1}{3}\frac{l-1}{k+2}$$

where $\bar{q}$ and $\bar{q}'$ denote the numbers of A-bins and C-bins, respectively, for the last l bins. Now, using the above inequalities, we obtain that

$$s=\bar{s}+\tilde{s}\geqq\frac{2}{3}m-\frac{1}{6}(w+1)-\frac{1}{3}\frac{m-1}{k+2}.$$

Case 5/b. Now let us suppose that the r-th block is of type 4. Then, by assertion (III) of Lemma 4, the $(r-1)$-th block is not of type 4, assuming that there exists such a block, i.e. $r>1$. Then, disregarding the last $k+2+l$ bins, for the remaining $(r-1)(k+2)$ bins the same conditions hold as above, and so, for the sum $\bar{s}$ of the weights of the items contained in these bins the inequality (i) holds. Thus,

$$\bar{s} \geqq \frac{2}{3}(r-1)(k+2)-\frac{1}{6}\Big(1+\sum_{i=1}^{r-1}(q_i+q_i')\Big)-\frac{1}{3}\frac{(r-1)(k+2)-1}{k+2}+\frac{1}{6}-\frac{1}{3(k+2)}.$$

It may be observed that the right-hand side of the inequality is equal to 0, if $r=1$. Therefore, we may use it in the case $r=1$, too.

We now investigate the remaining $k+2+l$ bins. Let $B_1^*, \dots B_{k+2+l}^*$ denote them. Since the bins $B_1^*, \dots B_{k+2}^*$ form a 4-block, the bins B_1^*, B_2^*, B_{k+2}^* are D-bins and $B_3^* \dots, B_{k+1}^*$ are C-bins. Let us distinguish two cases depending on l.

If $l \leqq k-1$ then, by Lemma 2, the bins $B_{k+3}^*, \dots B_{k+2+l}^*$ are of type A or C. Thus, for the sum $\tilde{s}$ of the weights of the items contained in the considered $k+2+l$ bins the following inequality holds

$$\tilde{s} \geqq \frac{2}{3}(k+2)-\frac{1}{6}(q_r+3)+\frac{2}{3}l-\frac{1}{6}\bar{q}=\frac{2}{3}(k+2+l)-\frac{1}{6}(q_r+\bar{q}+3),$$

where $\bar{q}$ denotes the number of C-bins with respect to the last l bins.

If $k-1<l\leqq k+1$ then it may be observed that since B_{k+1}^* is a C-bin and B_{k+2}^* is a D-bin, by Lemma 2, the bins $B_{k+3}^*, \dots, B_{2k+1}^*$ are of types A or C.

If there exists at least one A-bin among $B_{k+3}^*, \dots, B_{2k+1}^*$, then $\bar{q}'\geqq 1$, where $\bar{q}'$ denotes the number of A-bins for the last l bins. On the other hand, the bins $B_{k+4}^*, \dots$ $\dots, B_{k+2+l}^*$ form the last $l-1$ bins of the parcel, and so, by (I) of Lemma 4, there exists at most one D-bin among them. Therefore, we obtain that there is at most one D-bin among the last l bins. Thus for $\tilde{s}$ we have

$$\tilde{s} \geqq \frac{2}{3}(k+2)-\frac{1}{6}(q_r+3)+\frac{2}{3}l-\frac{1}{6}(\bar{q}+1) =$$

$$= \frac{2}{3}(k+2+l)-\frac{1}{6}(q_r+\bar{q}+\bar{q}'+3)+\frac{1}{6}(\bar{q}'-1) \geqq$$

$$\geqq \frac{2}{3}(k+2+l)-\frac{1}{6}(q_r+\bar{q}+\bar{q}'+3).$$

If the bins $B^*_{k+3}, \ldots, B^*_{2k+1}$ are all C-bins, then after the packing B^*_{2k+1} has empty room with weight $\frac{1}{3}+\Delta$, where $\Delta>0$. From this, similarly as in the proof of assertion (I) of Lemma 4, we obtain that the remaining bins (B^*_{2k+2} or B^*_{2k+2}, B^*_{2k+3}) are not of type D. But then there is no D-bin among the last l bins, and so

$$\tilde{s} \geqq \frac{2}{3}(k+2)-\frac{1}{6}(q_r+3)+\frac{2}{3}l-\frac{1}{6}\bar{q} = \frac{2}{3}(k+2+l)-\frac{1}{6}(q_r+\bar{q}+3)$$

where $\bar{q}$ denotes the number of C-bins for the last l bins again.

Now, using the common lower bound, we obtain the following inequalities:

$$s = \bar{s}+\tilde{s} \geqq \frac{2}{3}m-\frac{1}{6}\Big(1+q_r+\bar{q}+\bar{q}'+\sum_{i=1}^{r-1}(q_i+q_i')\Big)-\frac{3}{6}-\frac{1}{3}\frac{(r-1)(k+2)}{k+2}+\frac{1}{6} =$$

$$= \frac{2}{3}m-\frac{1}{6}(w+1)+\frac{1}{6}q_r'-\frac{1}{3}\frac{r(k+2)}{k+2} =$$

$$= \frac{2}{3}m-\frac{1}{6}(w+1)-\frac{1}{3}\frac{m-1}{k+2}+\frac{l-1}{3(k+2)}+\frac{1}{6}q_r'.$$

Since $q_r'\geqq 0$ and $l\geqq 1$, we obtain that

$$s \geqq \frac{2}{3}m-\frac{1}{6}(w+1)-\frac{1}{3}\frac{m-1}{k+2}$$

which completes the proof of Theorem 1.

Now let L be an arbitrary list and let us pack the elements of L with the NkF algorithm. Let $B_1, \ldots, B_m$ denote the sequence of bins used by NkF and let w denote the number of all bins containing items with weight greater than 1/2. Then, for $s=s(L)$, the following statement holds.

Theorem 2.

$$s \geqq \frac{2}{3}m-\frac{1}{6}w-\frac{m}{3(k+2)}-\frac{5}{6}.$$

Proof. We distinguish two cases, depending on the contents of the bins.

Case 1. Let us suppose that $s(B_i)>1/2$ $(i=1, \dots, m)$. Then, the considered bins form a parcel consisting of m bins, and so, by Theorem 1, we obtain the validity of Theorem 2.

Case 2. Let us suppose that there exists a bin B_i $(1\leqq i\leqq m)$ with $s(B_i)\leqq 1/2$. Let $i_1, i_2, \dots, i_r$ denote the increasing sequence of indices of all such bins. Let $t\in\{i_1, \dots \dots, i_r\}$ be arbitrary, and let us investigate the contents of B_t and $B_{t+1}, \dots, B_{t+k}$, assuming that there exist such bins. After the packing of L, the relation $s(B_t)\leqq 1/2$ holds; thus, throughout the packing too, $s(B_t)\leqq 1/2$. But then, the weight of the first packed item in B_{t+u} is greater than $1/2$ if $u\in\{1, \dots, k\}$. Therefore, $i_q+k< <i_{q+1}$ $(q=1, \dots, r-1)$ and, if $i_r<m$, then the weight of the first packed item in B_{i_r+u} is greater than $1/2$ for any $1\leqq u\leqq z=\min\{k, m-i_r\}$. We now distinguish further two cases.

Case 2/a. Let us suppose that $i_r+k\leqq m$. Then the weight of the first packed item in B_{i_t+u} is greater than $1/2$ if $1\leqq t\leqq r$; $1\leqq u\leqq k$. Thus, for the sum $\bar{s}_t$ of the weights of the items contained in the bins $B_{i_t}, B_{i_t+1}, \dots, B_{i_t+k}$, the inequality $\bar{s}_t\geqq(k+1)\frac{1}{2}$ holds, since $s(B_{i_t})+s(B_{i_t+1})>1$ and $s(B_{i_t+u})>1/2$ if $2\leqq u\leqq k$. On the other hand, it may be observed that the sequence

$$B_1, \dots, B_{i_1-1};\ B_{i_1+k+1}, \dots, B_{i_2-1};\ \dots;\ B_{i_{r-1}+k+1}, \dots, B_{i_r-1};\ B_{i_r+k+1}, \dots, B_m$$

form parcels consisting of i_1-1, $i_2-i_1-k-1, \dots, i_r-i_{r-1}-k-1$, $m-i_r-k$ bins, respectively, where any parcel of them may be an empty one. Let $m_1=i_1-1$, $m_2= =i_2-i_1-k-1, \dots, m_r=i_r-i_{r-1}-k-1$, $m_{r+1}=m-i_r-k$ and let w_i denote the number of A-bins and C-bins of the i-th parcel for any $i\in\{1, \dots, r+1\}$. Then, by Theorem 1, for the sum s_i of the weights of the items contained in the bins of the i-th parcel the following inequality holds:

$$\text{(a)}\quad s_i\geqq\frac{2}{3}m_i-\frac{1}{6}(w_i+1)-\frac{m_i-1}{3(k+2)}.$$

But then for the sum s of the weights of the items contained in the bins $B_1, \dots, B_m$ we obtain

$$s=\sum_{i=1}^{r+1}s_i+\sum_{t=1}^{r}\bar{s}_t\geqq\sum_{i=1}^{r+1}s_i+r(k+1)\frac{1}{2}\geqq$$

$$\geqq\sum_{i=1}^{r+1}\left(\frac{2}{3}m_i-\frac{1}{6}(w_i+1)-\frac{m_i-1}{3(k+2)}\right)+r(k+1)\frac{1}{2}=$$

$$=\frac{2}{3}\Big(\sum_{i=1}^{r+1}m_i+r(k+1)\Big)-\frac{1}{6}\Big(\sum_{i=1}^{r+1}w_i+rk\Big)-\frac{2r+1}{6}-\frac{\sum_{i=1}^{r+1}(m_i-1)}{3(k+2)}.$$

Since $m=\sum_{i=1}^{r+1} m_i+r(k+1)$ and $w=\sum_{i=1}^{r+1} w_t+rk$, we have

$$s \geqq \frac{2}{3}m-\frac{1}{6}w-\frac{r}{3}-\frac{1}{6}-\frac{\Sigma m_i}{3(k+2)}+\frac{r+1}{3(k+2)}=$$

$$=\frac{2}{3}m-\frac{1}{6}w-\frac{\Sigma m_i+r(k+2)}{3(k+2)}-\frac{1}{6}+\frac{r+1}{3(k+2)}=$$

$$=\frac{2}{3}m-\frac{1}{6}w-\frac{m}{3(k+2)}+\frac{1}{3(k+2)}-\frac{1}{6}.$$

But $\frac{1}{3(k+2)}-\frac{1}{6}\geqq-\frac{5}{6}$, and so,

$$s \geqq \frac{2}{3}m-\frac{1}{6}w-\frac{m}{3(k+2)}-\frac{5}{6}$$

which completes the proof of this case.

Case 2/b. Let us assume that $i_r+k>m$. Then the number of bins succeeding the bin B_{i_r} is $l=m-i_r$. Moreover, if $l>0$, then the weight of the first packed item in B_{i_r+u} is greater than 1/2 for any $u\in\{1, \ldots, l\}$. Thus, for the sum s^* of the weights of the items contained in the bins $B_{i_r}, \ldots, B_m$ the following inequality holds

$$s^* \geqq s(B_{i_r})+(m-i_r)\frac{1}{2}.$$

On the other hand, for the sum $\bar{s}_t$ of the weights of the items contained in B_{i_t}, $B_{i_t+1}, \ldots, B_{i_t+k}$ again $\bar{s}_t \geqq (k+1)\frac{1}{2}$ holds if $t<r$. Finally, the sequences $B_1, \ldots$ $\ldots, B_{i_1-1}$; $B_{i_1+k+1}, \ldots, B_{i_2-1}$; $\ldots$; $B_{i_{r-1}+k+1}, \ldots, B_{i_r-1}$ again form parcels. Thus, with the notations of the previous case and inequality (a), for the sum s of the weights of the items contained in $B_1, \ldots, B_m$, the following inequalities hold:

$$s=\sum_{i=1}^{r} s_i+\sum_{t=1}^{r-1}\bar{s}_t+s^* \geqq \sum_{i=1}^{r} s_i+(r-1)(k+1)\frac{1}{2}+s^* \geqq$$

$$\geqq \sum_{i=1}^{r}\left(m_i\frac{2}{3}-\frac{1}{6}(w_i+1)-\frac{m_i-1}{3(k+2)}\right)+(r-1)(k+1)\frac{1}{2}+s(B_{i_r})+(m-i_r)\frac{1}{2}=$$

$$=\frac{2}{3}\Big(\sum_{i=1}^{r} m_i+(r-1)(k+1)\Big)-\frac{1}{6}\Big(\sum_{i=1}^{r} w_i+r+(r-1)(k+1)\Big)-$$

$$-\frac{\sum_{i=1}^{r}(m_i-1)}{3(k+2)}+s(B_{i_r})+(m-i_r)\frac{1}{2}=\frac{2}{3}\Big(\sum_{i=1}^{r} m_i+(r-1)(k+1)+(m-i_r)\Big)-$$

$$-\frac{1}{6}\Big(\sum_{i=1}^{r} w_i+(r-1)(k+1)+(m-i_r)+r\Big)-\frac{\sum_{i=1}^{r}(m_i-1)}{3(k+2)}+s(B_{i_r}).$$

Since $\sum_{i=1}^{r} m_i+(r-1)(k+1)+m-i_r=m-1$, we obtain that

$$s \geqq \frac{2}{3}(m-1)-\frac{1}{6}\Big(\sum_{i=1}^{r} w_i+(r-1)k+m-i_r\Big)-\frac{1}{6}(2r-1)-\frac{\sum_{i=1}^{r}(m_i-1)}{3(k+2)}+s(B_{i_r}).$$

Now, it may be observed that $w=\sum_{i=1}^{r} w_i+(r-1)k+m-i_r$, and so

$$s \geqq \frac{2}{3}(m-1)-\frac{1}{6}w-\frac{r-1}{3}-\frac{1}{6}-\frac{\sum_{i=1}^{r} m_i}{3(k+2)}+\frac{r}{3(k+2)}+s(B_{i_r}) =$$

$$=\frac{2}{3}(m-1)-\frac{1}{6}w-\frac{(r-1)(k+2)+\sum_{i=1}^{r} m_i}{3(k+2)}+\frac{r}{3(k+2)}-\frac{1}{6}+s(B_{i_r}) =$$

$$=\frac{2}{3}(m-1)-\frac{1}{6}w-\frac{\sum_{i=1}^{r} m_i+(r-1)(k+1)}{3(k+2)}+\frac{1}{3(k+2)}-\frac{1}{6}+s(B_{i_r}) =$$

$$=\frac{2}{3}(m-1)-\frac{1}{6}w-\frac{\sum_{i=1}^{r} m_i+(r-1)(k+1)+m-i_r+1}{3(k+2)}+\frac{m-i_r+2}{3(k+2)}-\frac{1}{6}+s(B_{i_r}) =$$

$$=\frac{2}{3}m-\frac{1}{6}w-\frac{m}{3(k+2)}+\frac{m-i_r+2}{3(k+2)}-\frac{5}{6}+s(B_{i_r}).$$

But $\frac{m-i_r+2}{3(k+2)}-\frac{5}{6}+s(B_{i_r}) \geqq -\frac{5}{6}$, and so

$$s \geqq \frac{2}{3}m-\frac{1}{6}w-\frac{m}{3(k+2)}-\frac{5}{6}$$

which completes the proof of Theorem 2.

We can now prove the following result.

Theorem 3.

$$R_{NkF} \leqq \frac{7}{4}+\frac{7}{4}\cdot\frac{1}{2k+3}.$$

Proof. Let L be an arbitrary list and let us pack its elements with the *NkF* algorithm. Let m denote the number of bins used by *NkF* and let s denote the sum of the weights of the items contained in these bins. Moreover, let w denote the number of those bins which contain some item with weight greater than 1/2. Now, depending on w we distinguish three cases.

Case 1. Let us suppose that $w=0$. Then, by Theorem 2, we obtain

$$s \geqq \frac{2}{3}m - \frac{m}{3(k+2)} - \frac{5}{6};$$

On the other hand $s \leqq \mathrm{OPT}(L)$, and so

$$\frac{m}{\mathrm{OPT}(L)} \leqq \frac{m}{s} \leqq \frac{1}{\frac{2}{3} - \frac{1}{3(k+2)} - \frac{5}{6m}} \leqq$$

$$\frac{1}{\frac{2}{3} - \frac{1}{6}\cdot\frac{4k+6}{7(k+2)} - \frac{1}{3(k+2)} - \frac{5}{6m}} = \frac{7(k+2)}{4k+6-\frac{7(k+2)}{m}\frac{5}{6}}.$$

Case 2. Let us assume that $w \neq 0$ and $\frac{m}{w} \leqq \frac{7(k+2)}{4k+6}$. From the definition of w it follows that $w \leqq \mathrm{OPT}(L)$. But then

$$\frac{m}{\mathrm{OPT}(L)} \leqq \frac{m}{w} \leqq \frac{7(k+2)}{4k+6}.$$

Case 3. Let us suppose that $w \neq 0$ and $\frac{7(k+2)}{4k+6} < \frac{m}{w}$. Again, by Theorem 2,

$$s \geqq \frac{2}{3}m - \frac{1}{6}w - \frac{m}{3(k+2)} - \frac{5}{6},$$

and so,

$$\frac{m}{\mathrm{OPT}(L)} \leqq \frac{m}{s} \leqq \frac{m}{\frac{2}{3}m - \frac{1}{6}w - \frac{m}{3(k+2)} - \frac{5}{6}} = \frac{1}{\frac{2}{3} - \frac{1}{6}\frac{w}{m} - \frac{1}{3(k+2)} - \frac{5}{6m}}.$$

By our assumption on m/w, $\frac{w}{m} < \frac{4k+6}{7(k+2)}$, and so

$$\frac{m}{\mathrm{OPT}(L)} \leqq \frac{1}{\frac{2}{3} - \frac{1}{6}\frac{4k+6}{7(k+2)} - \frac{1}{3(k+2)} - \frac{5}{6m}} = \frac{7(k+2)}{4k+6-\frac{7(k+2)}{m}\frac{5}{6}}.$$

Now let $k \geqq 3$ be a fixed integer. It may be observed that if $\mathrm{OPT}(L) \to \infty$ then $m \to \infty$, and so, under the fixed k, $\frac{7(k+2)}{m}\frac{5}{6} \to 0$.

Therefore

$$\limsup_{n\to\infty}\left\{\frac{NkF(L)}{\mathrm{OPT}(L)} : \mathrm{OPT}(L) = n\right\} \leqq \frac{7(k+2)}{4k+6} = \frac{7}{4} + \frac{7}{4}\cdot\frac{1}{2k+3},$$

which completes the proof of Theorem 3. □

We now improve the lower bound given by Johnson. For this purpose, we define a sequence of lists such that $OPT(L_i) \to \infty$ and the lists have bad behaviour on NkF packing. Let j now be a fixed positive integer.

Let $n(j)$ denote the number of elements in the j-th list and let

$$n(j) = 30j(k-2)+30j.$$

Let

$$\delta \ll 18^{-j(k-2)}$$

and let $L_{n(j)}$ denote the j-th list in the sequence. We divide the items into three parts:

(1) In the first part there are elements about $1/6$; there are $j(k-2)$ blocks, with 10 items in each (thus, in the first part there are $10j(k-2)$ items). Let us denote the items of the i-th block by

$$a_{0i}, a_{1i}, \ldots, a_{9i}.$$

The exact definition of the weights is as follows. Let

$$\delta_i = \delta \cdot 18^{j(k-2)-i} \quad (1 \leqq i \leqq j(k-2))$$

and

$$a_{0i} = 1/6+33\delta_i,$$

$$a_{1i} = 1/6-3\delta_i,$$

$$a_{2i} = 1/6-7\delta_i = a_{3i},$$

$$a_{4i} = 1/6-13\delta_i,$$

$$a_{5i} = 1/6+9\delta_i,$$

$$a_{6i} = 1/6-2\delta_i = a_{7i} = a_{8i} = a_{9i}.$$

Then, the first $10j(k-2)$ items of the list are $a_{01}, a_{11}, \ldots, a_{91}, a_{02}, a_{12}, \ldots, a_{92}, \ldots$ $\ldots, a_{0,j(k-2)}, \ldots, a_{9,j(k-2)}$. Clearly

$$a_{0i}+a_{1i}+a_{2i}+a_{3i}+a_{4i} = 5/6+3\delta_i,$$

$$a_{5i}+a_{6i}+a_{7i}+a_{8i}+a_{9i} = 5/6+\delta_i,$$

and thus we fill $2j(k-2)$ bins with this part.

(2) In the second part, there are elements about $1/3$; there are also $j(k-2)$ blocks, with 10 items in each. Let us denote the items of the i-th block by

$$b_{0i}, b_{1i}, \ldots, b_{9i},$$

and the items

$$b_{01}, b_{11}, \ldots, b_{91}, b_{02}, b_{12}, \ldots, b_{92}, \ldots, b_{0,j(k-2)}, \ldots, b_{9,j(k-2)}$$

follow the items of the first part.

The exact definition of these items is as follows:

$b_{0i} = 1/3+46\delta_i,$

$b_{1i} = 1/3-34\delta_i,$

$b_{2i} = 1/3+6\delta_i = b_{3i},$

$b_{4i} = 1/3+12\delta_i,$

$b_{5i} = 1/3-10\delta_i,$

$b_{6i} = 1/3+\delta_i = b_{7i} = b_{8i} = b_{9i}.$

Clearly

$b_{0i}+b_{1i} = 2/3+12\delta_i,$

$b_{2i}+b_{3i} = 2/3+12\delta_i,$

$b_{4i}+b_{5i} = 2/3+2\delta_i,$

$b_{6i}+b_{7i} = 2/3+2\delta_i,$

$b_{8i}+b_{9i} = 2/3+2\delta_i,$

and thus we fill $5j(k-2)$ bins with the second part.

(3) In the third part, there are elements about 1/2. We have here $10j$ blocks, with $k+1$ items in each. In the i-th block, the first item is $1/2-\delta/(i+1)$, and the second is $1/2+\delta/i$. Then, we have a number $(k-2)$ of $1/2+\delta$ items and the last item of this block is a δ. Thus, with this part we exactly fill $10jk$ bins.

On summing the number of bins in the three parts, we obtain:

$$NkF(L_{n(j)}) = 2j(k-2)+5j(k-2)+10jk = 17jk-14j.$$

In the optimal packing of $L_{n(j)}$, we have to pack all $1/2+\delta$ items in separate bins. Thus, we pair the items from the first and second part in the following way:

i) $a_{l,i}+b_{l,i}$, if $2 \leqq l \leqq 9$, $1 \leqq i \leqq j(k-2)$,

ii) $a_{0i}+b_{1i}$, if $1 \leqq i \leqq j(k-2)$,

iii) $a_{1i}+b_{0,(i+1)}$, if $1 \leqq i \leqq j(k-2)-1$.

Clearly, we can pack all pairs with a $1/2+\delta$ element together. Accordingly, we fill $10j(k-2)-1$ bins, and b_{01}, $a_{1,j(k-2)}$ are not used. From the third part, one $1/2+\delta$ item, a number $10j$ of δ items and the following items are not used:

$$\frac{1}{2}-\frac{\delta}{2},\ \frac{1}{2}+\frac{\delta}{1},$$

$$\frac{1}{2}-\frac{\delta}{3},\ \frac{1}{2}+\frac{\delta}{2},$$

$$\vdots$$

$$\frac{1}{2}-\frac{\delta}{10j+1},\ \frac{1}{2}+\frac{\delta}{10j}.$$

Here $1/2-\delta/i$ and $1/2+\delta/i$ fill a bin $(i=2, 3, \ldots, 10j)$ and so we have a further $10j-1$ bins. All other items can be packed into three bins, if δ is small enough. Thus,

$$\mathrm{OPT}(L_{n(j)}) \leqq 10j(k-2)-1+10j-1+3 = 10jk-10j+1.$$

Then

$$\frac{NkF(L_{n(j)})}{\mathrm{OPT}(L_{n(j)})} \geqq \frac{17jk-14j}{10jk-10j+1},$$

and hence

$$R_{NkF} \geqq \liminf_{j\to\infty} \frac{NkF(L_{n(j)})}{\mathrm{OPT}(L_{n(j)})} = \frac{17k-14}{10k-10}.$$

We have obtained

Theorem 4. For $k \geqq 3$

$$R_{NkF} \geqq 1.7+\frac{3}{10(k-1)}.$$

From Theorem 3 and Theorem 4, we conclude our

Main results. For $k \geqq 3$

$$1.7+\frac{3}{10(k-1)} \leqq R_{NkF} \leqq \frac{7}{4}+\frac{7}{4}\cdot\frac{1}{2k+3}.$$

To conclude this paper, we give R_{N2F}. For this, we define a sequence of lists as follows. Here the j-th list has a number $n(j)=30j$ of items. Let

$$L_{n(j)} = \left(\frac{1}{2}-\frac{\delta}{2}, \frac{1}{2}+\frac{\delta}{1}, \delta, \frac{1}{2}-\frac{\delta}{3}, \frac{1}{2}+\frac{\delta}{2}, \delta, \ldots, \frac{1}{2}-\frac{\delta}{10j+1}, \frac{1}{2}+\frac{\delta}{10j}, \delta\right).$$

Then we use $20j$ bins in the $N2F$ packing, and $10j+1$ bins in the optimal packing. Thus, we get:

Corollary 1. $R_{N2F}=2$.

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