Determination of the structure of the class $\mathscr{A}(R, S)$ of (0, 1)-matrices

A. KUBA

Kalmár Laboratory of Cybernetics József Attila University
Szeged, Árpád tér 2. H-6720 Hungary

Summary

The class $\mathscr{A}(R, S)$ contains the (0, 1)-matrices having row and column sum vectors R and S, respectively. The problem of the structure of $\mathscr{A}(R, S)$ is considered, that is the problem of determining the sets of invariant 1's, invariant 0's and variant positions. Two methods are given, whereby the structure can be determined if an element of $\mathscr{A}(R, S)$ or the vectors R and S are known. Furthermore, a new proof is given to Ryser's theorem constructing the variant and invariant positions of the class $\mathscr{A}$.

1. Definitions

Let A be a (0, 1)-matrix of size n by m. The sum of row i of A is denoted by r_i:

$$r_i = \sum_{j=1}^{m} a_{ij} \quad (i = 1, 2, ..., n),$$

and the sum of column j of A is denoted by s_j:

$$s_j = \sum_{i=1}^{n} a_{ij} \quad (j = 1, 2, ..., m).$$

We call $R=(r_1, r_2, ..., r_n)$ the *row sum vector* and $S=(s_1, s_2, ..., s_m)$ the *column sum vector* of A. R and S are also called the *projections* of A. There is an extensive literature on different questions concerning binary matrices and their projections (for surveis see e.g. [9] and [1]). Let $\mathscr{A}(R, S)$ denote the class of $n\times m$ (0, 1)-matrices with row sum vector R and column sum vector S. Gale [2] and Ryser [6] have proved that the class $\mathscr{A}(R, S)$ is non-empty if and only if

$$\sum_{j=1}^{k} \bar{s}_j \geqq \sum_{j=1}^{k} s_j$$

for all $k=1, 2, \ldots, m$, where $\bar{S}=(\bar{s}_1, \bar{s}_2, \ldots, \bar{s}_m)$ is the column sum vector of binary matrix $\bar{A}$ defined as

$$\bar{A} = \begin{pmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_n \end{pmatrix},$$

where

$$\delta_i = (1, 1, \ldots, 1, 0, 0, \ldots, 0)$$

with r_i number of 1's and $(m-r_i)$ number of 0's $(0 \leqq r_i \leqq m)$. There is exactly one matrix in $\mathscr{A}(R, S)$ if and only if

$$\sum_{j=1}^{k} \bar{s}_j = \sum_{j=1}^{k} s_j$$

for all $k=1, 2, \ldots, m$ (see e.g. [10]).

Consider the matrices

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

An *interchange* is a transformation of the elements of A that changes a minor of type A_1 into type A_2 or vica versa and leaves all other elements of A unaltered. We say that the four elements of the minor form a *switching component* in A. The interchange theorem of Ryser [6] says that if A and A' are in $\mathscr{A}(R, S)$, then A is transformable into A' by a finite sequence of interchanges.

Let $A \in \mathscr{A}(R, S)$. A is *ambiguous* (with respect to R and S) if there is a different $A' \in \mathscr{A}(R, S)$ $(A' \neq A)$. In the other case, A is *unambiguous*. It is easy to prove (see e.g. [2]) that A is ambiguous if and only if it has a switching component.

An element $a_{ij}=1$ (or 0) of A is called an *invariant* 1 (or 0) if there is no sequence of interchanges which, when applied to A, replaces it by 0 (or 1). Otherwise, a_{ij} is a *variant* element of A. By the interchange theorem, if a_{ij} is an invariant 1 (or 0) of $A \in \mathscr{A}(R, S)$, then a'_{ij} is also an invariant 1 (or 0) of every $A' \in \mathscr{A}(R, S)$. In this sense, we can speak about the *invariant 1, invariant 0* and *variant (i, j) positions* of the class $\mathscr{A}(R, S)$.

Without loss of generality, we can suppose that

$$r_1 \geqq r_2 \geqq \ldots \geqq r_n > 0 \tag{1.1}$$

and

$$s_1 \geqq s_2 \geqq \ldots \geqq s_m > 0, \tag{1.2}$$

because this situation can be reached by excluding zero rows and zero columns and by permuting rows and columns so that the row-sums and the column-sums are nonincreasing. A non-empty class $\mathscr{A}(R, S)$ with R and S satisfying (1.1) and (1.2) is said to be *normalized.*

In the determining of the invariant positions of the normalized class $\mathscr{A}(R, S)$, a useful device is the *structure matrix* [8]. Let A be in the normalized class $\mathscr{A}(R, S)$ and let us write

$$A = \begin{pmatrix} W & X \\ Y & Z \end{pmatrix},$$

where W is of size $e \times f$ $(0 \leqq e \leqq n,\ 0 \leqq f \leqq m)$. Let Q be a (0, 1)-matrix, and let $N_0(Q)$ denote the number of 0's in Q, let $N_1(Q)$ denote the number of 1's in Q. Now let

$$t_{ef} = N_0(W) + N_1(Z)$$

$e=0, 1, \ldots, n$; $f=0, 1, \ldots, m$. We call the $(n+1)\times(m+1)$ matrix

$$T = (t_{ef})$$

the structure matrix of $\mathscr{A}(R, S)$. It is easy to see that

$$t_{ef} = e \cdot f + \sum_{i=e+1}^{n} r_i - \sum_{j=1}^{f} s_j.$$

Ryser proved the following

Theorem 1.1 [7]. The normalized class $\mathscr{A}(R, S)$ is with invariant 1's if and only if the matrices in $\mathscr{A}(R, S)$ are of the form

$$A = \begin{pmatrix} J & * \\ * & O \end{pmatrix}.$$

Here O is a zero matrix and J is a matrix of 1's of size $e \times f$ $(0 < e \leqq n,\ 0 < f \leqq m)$ specified by

$$t_{ef} = 0.$$

(The integers e and f are not necessarily unique, but they are determined by R and S and are independent of the particular choice of A in $\mathscr{A}$.)

By Theorem 1.1, one can construct the structure of class $\mathscr{A}(R, S)$ with the help of matrix T. In this paper, another way is given to construct the invariant and variant positions of class $\mathscr{A}$. First, the structure of the variant elements of the (not necessarily normalized) class $\mathscr{A}$ is given. From the determination of the positions of the variant elements, it is also possible to give the whole structure of $\mathscr{A}$. In Section 3, the case of the normalized class is discussed applying the idea of double-projection used earlier in characterization problems of binary matrices [5]. A direct and demonstrative relation between the structure of $\mathscr{A}$ and the vectors R and S is given in Section 4, from which the mode of construction of the structure of $\mathscr{A}$ follows.

2. The structure of the class $\mathscr{A}(R, S)$

First, consider the variant elements of $\mathscr{A}(R, S)$.

Lemma 2.1. Let A be a matrix in $\mathscr{A}(R, S)$, and let

$$a_{ij_1}, a_{ij_2}, \ldots, a_{ij_k},$$

$$a_{i_1 j}, a_{i_2 j}, \ldots, a_{i_l j},$$

be variant elements of A such that $1 \leqq i_1, i_2, \ldots, i_l \leqq n$, $1 \leqq j_1, j_2, \ldots, j_k \leqq m$, $i \in \{i_1, i_2, \ldots, i_l\}$, $j \in \{j_1, j_2, \ldots, j_k\}$, where $1 < l \leqq n$ and $1 < k \leqq m$. Then, $a_{i'j'}$ is variant for all $(i', j') \in \{i_1, i_2, \ldots, i_l\} \times \{j_1, j_2, \ldots, j_k\}$ (see Fig. 1/a).

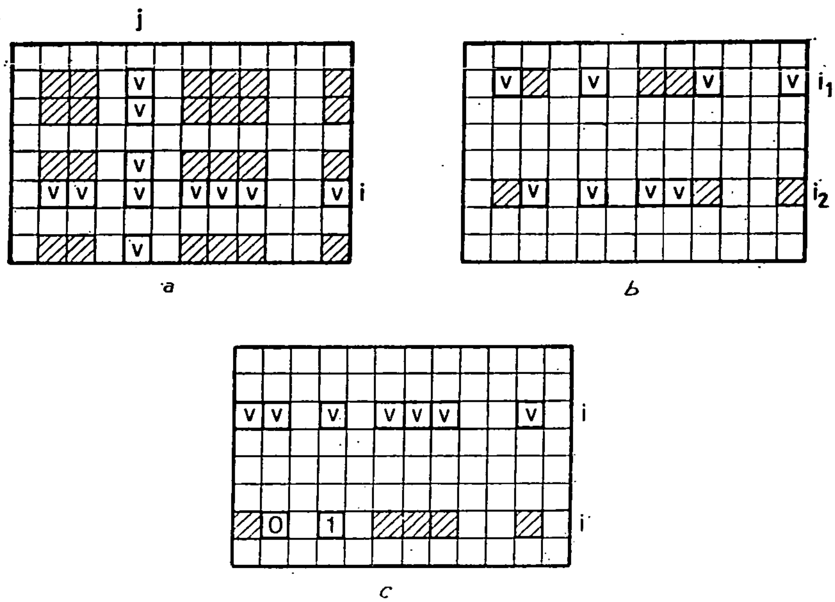

Figure 1. The variant elements ▨ induced by the variant elements ☑ according to *a)* Lemma 2.1, *b)* Lemma 2.2 and *c)* Lemma 2.3

Proof. The assumptions of Lemma 2.1 include that a_{ij} is a variant element of A. Let (i', j') $(i' \neq i, j' \neq j)$ be an otherwise arbitrary element of $\{i_1, i_2, \ldots, i_l\} \times \times \{j_1, j_2, \ldots, j_k\}$. If

$$a_{i'j'} = a_{ij}, \tag{2.1}$$

$$a_{i'j'} = 1 - a_{i'j}, \tag{2.2}$$

$$a_{i'j'} = 1 - a_{ij'}, \tag{2.3}$$

then a_{ij}, $a_{i'j}$, $a_{ij'}$ and $a_{i'j'}$ form a switching component in A, and hence $a_{i'j'}$ is variant. If any of the equalities (2.1)—(2.3) is not satisfied, then, since a_{ij}, $a_{i'j}$ and $a_{ij'}$ are variant, it is possible to alter any of them (occasionally all of them) by a suitable interchange in order to get a switching component at $\{i', i'\}\times\{j, j'\}$. That is, (i', j') is a variant position in $\mathcal{A}(R, S)$. A simple consequence of Lemma 2.1 is the following

Lemma 2.2. Let A be a matrix in $\mathcal{A}(R, S)$, and let

$$J = \{j_1, j_2, \ldots, j_k\}, \quad J' = \{j_1', j_2', \ldots, j_l'\},$$

$$J \cap J' \neq \emptyset,$$

such that $a_{i_1 j_1}, a_{i_1 j_2}, \ldots, a_{i_1 j_k}$ and $a_{i_2 j_1'}, a_{i_2 j_2'}, \ldots, a_{i_2 j_l'}$ are variant. Then, a_{ij} is variant for all $(i, j) \in \{i_1, i_2\}\times(J \cup J')$ (see Fig. 1/*b*).

Proof. By Lemma 2.1, the elements of A at $\{i_1, i_2\}\times J$ and $\{i_1, i_2\}\times J'$ are variant.

Lemma 2.3. Let A be a matrix in $\mathcal{A}(R, S)$, and let $J=\{j_1, j_2, \ldots, j_k\}$ be the indices of variant elements in row i $(1 \leqq i \leqq n)$. If there is a row i' $(i' \neq i)$ such that the elements $a_{i'j_1}, a_{i'j_2}, \ldots, a_{i'j_k}$ also include 0 and 1, then $a_{i'j_1}, a_{i'j_2}, \ldots, a_{i'j_k}$ are variant elements (see Fig. 1/c).

Proof. Let us suppose that $a_{i'j_1}=0$ and $a_{i'j_2}=1$ (by a suitable rewriting of the indices, we can always reach such a situation). We shall construct a switching component at $\{i, i'\}\times\{j_1, j_2\}$: If $a_{ij_1}=1$ and $a_{ij_2}=0$, then we are ready. If $a_{ij_1}=1$ and $a_{ij_2}=1$, then, since a_{ij_2} is variant, there is a switching component whereby a_{ij_2} will be 0 (a_{ij_1} and $a_{i'j_2}$ remain unchanged). Similarly, if $a_{ij_1}=0$ and $a_{ij_2}=0$, then there is a switching component whereby a_{ij_1} will be 1 (in this case a_{ij_2} and $a_{i'j_1}$ remain unchanged). In the last case, if $a_{ij_1}=0$ and $a_{ij_2}=1$, then we can change a_{ij_1} and a_{ij_2} by at most two interchanges (without changing $a_{i'j_1}$ and $a_{i'j_2}$).

Theorem 2.1. The variant positions of class $\mathcal{A}(R, S)$, if there are any, are in sets $T_1, T_2, \ldots, T_p$ $(p=0$ is also possible) such that

$$T_s = I_s \times J_s,$$

$s=1, 2, \ldots, p$, where I_s are pairwise disjunct subsets of $\{1, 2, \ldots, n\}$ and J_s are pairwise disjunct subsets of $\{1, 2, \ldots, m\}$.

Proof. Consider the set of column indices of the variant elements in row i, denoted by J_i. Let

$$I_i = \{l \mid J_l \cap J_i \neq \emptyset\},$$

and let

$$\bar{J}_i = \bigcup_{l \in I_i} J_l.$$

By Lemma 2.2, every position (i, j) is variant for which $(i, j) \in I_i \times \bar{J}_i$. By definition, it is clear that (i, j), $(i', j') \in I_i \times \bar{J}_i$ if and only if

$$I_i \times \bar{J}_i = I_{i'} \times \bar{J}_{i'}.$$

That is, by applying the procedure for all $i=1, 2, ..., n$, we get disjoint subsets $I_1, I_2, ..., I_p$ and $J_1, J_2, ..., J_p$, and the sets

$$T_s = I_s \times J_s,$$

$s=1, 2, ..., p$, contain all of the variant positions of $\mathscr{A}(R, S)$.

3. The structure of the normalized class $\mathscr{A}(R, S)$

Henceforth, we take $\mathscr{A}(R, S)$ normalized.

Lemma 3.1. Let A be a binary matrix in the normalized class $\mathscr{A}(R, S)$, and let

$$u_i = \begin{cases} \max\{j | a_{ij} = 1\}, & \text{if } a_{ij} = 1 \text{ for some } j = 1, 2, ..., m \\ 0, & \text{if } a_{ij} = 0 \text{ for all } j = 1, 2, ..., m \end{cases}$$

and

$$z_i = \begin{cases} \min\{j | a_{ij} = 0\}, & \text{if } a_{ij} = 0 \text{ for some } j = 1, 2, ..., m \\ m+1, & \text{if } a_{ij} = 1 \text{ for all } j = 1, 2, ..., m \end{cases}$$

for all $i=1, 2, ..., n$. If $z_i < u_i$ for some i, then a_{ij} is variant for all j, $z_i \leqq j \leqq u_i$.

Proof. If there is an i, $1 \leqq i \leqq n$, such that $a_{ij}=0$, $a_{ij'}=1$, $j \leqq j'$, then, since $s_j \geqq s_{j'}$, there is an i', $1 \leqq i' \leqq n$, such that $a_{i'j}=1$, $a_{i'j'}=0$. That is, $a_{ij}, a_{ij'}, a_{i'j}$ and $a_{i'j'}$ form a switching component. Therefore, all of the positions between z_i and u_i are variant.

An analogous lemma is true for the columns:

Lemma 3.2. Let A be a (0, 1)-matrix in the normalized class $\mathscr{A}(R, S)$, and let

$$v_j = \begin{cases} \max\{i | a_{ij} = 1\}, & \text{if } a_{ij} = 1 \text{ for some } i = 1, 2, ..., n \\ 0, & \text{if } a_{ij} = 0 \text{ for all } i = 1, 2, ..., n \end{cases}$$

and

$$w_j = \begin{cases} \min\{i | a_{ij} = 0\}, & \text{if } a_{ij} = 0 \text{ for some } i = 1, 2, ..., n \\ n+1, & \text{if } a_{ij} = 1 \text{ for all } i = 1, 2, ..., n \end{cases}$$

for all $j=1, 2, ..., m$. If $w_j < v_j$ for some j, then a_{ij} is variant for all i, $w_j \leqq i \leqq v_j$.

Theorem 3.1. The variant positions of the normalized class $\mathscr{A}(R, S)$ are in the sets $T_1, T_2, ..., T_p$ ($p=0$ is also possible) such that

$$T_s = I_s \times J_s,$$

$s=1, 2, ..., p$, where

$$I_s = \{i'_s, i'_s+1, ..., i''_s\}, \quad 1 \leqq i'_1 < i''_1 < i'_2 < i''_2 ... < i'_p < i''_p \leqq n,$$

$$J_s = \{j'_s, j'_s+1, ..., j''_s\}, \quad 1 \leqq j'_p < j''_p < j'_{p-1} < j''_{p-1} < ... < j'_1 < j''_1 \leqq m.$$

Proof. We know that the variant elements of $\mathscr{A}(R, S)$, which are recognized by Lemmas 3.1 and 3.2, follow in rows and in columns consecutively. Following the same idea as in the Proof of Theorem 2.1, we have that the sets $T_s = I_s \times J_s$, $s=1, 2, ..., p$, are the places of variant elements, where I_s and J_s contain the indices of consecutive rows and columns, respectively. Furthermore, $I_s \cap I_{s'} = \emptyset$ and $J_s \cap J_{s'} = \emptyset$ if $s \neq s'$. From this construction, it is clear that $(\{1, 2, \dots, i_s' - 1\} \times J_s) \cup (I_s \times \{1, 2, \dots, j_s' - 1\})$ contains only 1's and $(\{i_s'' + 1, i_s'' + 2, \dots, m\} \times J_s) \cup (I_s \times \{j_s'' + 1, j_s'' + 2, \dots, n\})$ contains only 0's. Since the elements of R and S are in decreasing order, $i_r'' < i_s'$, $1 \leq r, s \leq p$, if and only if $j_r' > j_s''$. That is, if $T_1, T_2, \dots, T_p$ are indexed so that

$$1 \leqq i_1' < i_1'' < i_2' < i_2'' < \dots < i_p' < i_p'' \leqq n,$$

then

$$1 \leqq j_p' < j_p'' < j_{p-1}' < j_{p-1}'' < \dots < j_1' < j_1'' \leqq m.$$

It is easy to see that the set $\{1, 2, \dots, n\} \times \{1, 2, \dots, m\} \setminus \bigcup_s T_s$ contains only invariant positions and so $\bigcup_s T_s$ is the set of the variant positions of the normalized class $\mathscr{A}(R, S)$.

The following algorithm can be used to determine the sets of the indices of the variant elements, $I_s = \{i_s', i_s' + 1, \dots, i_s''\}$ and $J_s = \{j_s', j_s' + 1, \dots, j_s''\}$:

Step 1: First, the indices z_i and u_i are computed for each row i. It is clear that $z_i \leqq u_i + 1$ $(1 \leqq i \leqq n)$.

Step 2: The sequence of indices u_i is modified taking the rows from down to up such that if $u_{i+1} > u_i$ then let $u_i = u_{i+1}$ $(n-1 \geqq i > 1)$.

Step 3: The rows are scanned one by one from $i=1$ to $i>n$ with an initial value $s=0$. If $z_i > u_i$ then there is no variant element in the row i. In the other case, i.e. if $z_i \leqq u_i$, then there are variant elements in this row and let $s = s+1$, $i_s' = i$, $j_s' = z_i$ (initially) and $j_s'' = u_i$. The indices j_s' and i_s'' can be determined by scanning the rows further while $j_s' \leqq u_i$ such that meanwhile if $j_s' > z_i$ then let $j_s' = z_i$. In the row, where $j_s' > u_i$, let $i_s'' = i-1$ (this condition will be satisfied at least once if we set $u_{n+1} = z_{n+1} = -1$ at the beginning of the procedure).

Let us see two examples:

Example 3.1. Let the (0, 1)-matrix A be defined as

$$a_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

$i = 1, 2, \dots, n$, $j = 1, 2, \dots, n$. In this case $u_i = i$, $z_i = 1$, $i = 1, 2, \dots, n$ (with the exception that $z_1 = 2$). Applying the algorithm, we get that the set T_1 containing the indices of the variant elements is

$$T_1 = \{1, 2, \dots, n\} \times \{1, 2, \dots, n\},$$

that is the whole matrix.

Example 3.2. Let A be given by Figure 2. Then

$$u_1 = 13, \quad z_1 = 14, \quad u_2 = 11, \quad z_2 = 12, \quad u_3 = 10, \quad z_3 = 11,$$

$$u_4 = 10, \quad z_4 = 11, \quad u_5 = 11, \quad z_5 = 9, \quad u_6 = 7, \quad z_6 = 8,$$

$$u_7 = 5, \quad z_7 = 3, \quad u_8 = 6, \quad z_8 = 4, \quad u_9 = 1, \quad z_9 = 2,$$

$$i_1' = 3, \quad i_1'' = 5, \quad j_1' = 9, \quad j_1'' = 11$$

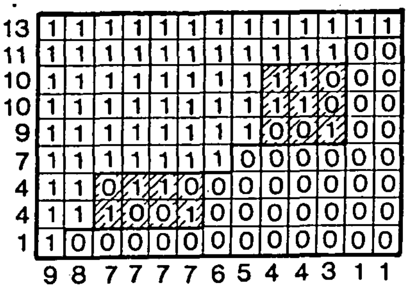

Figure 2. The structure of the normalized class $\mathscr{A}(R, S)$ of Example 3.2

and

$$i_2' = 7, \quad i_2'' = 8, \quad j_2' = 3, \quad j_2'' = 6.$$

That is,

$$T_1 = \{3, 4, 5\}\times\{9, 10, 11\}, \quad T_2 = \{7, 8\}\times\{3, 4, 5, 6\}.$$

4. Determination of the structure of class $\mathscr{A}(R, S)$ from the projections

Consider the matrices $A^{(x)}$ and $A^{(y)}$ defined by R and S as

$$a_{ij}^{(x)} = \begin{cases} 0, & \text{if } \ j > r_i; \\ 1, & \text{otherwise,} \end{cases}$$

and (4.1)

$$a_{ij}^{(y)} = \begin{cases} 0, & \text{if } \ i > s_j; \\ 1, & \text{otherwise,} \end{cases}$$

$i=1, 2, \ldots, n$, $j=1, 2, \ldots, m$ (see [5]). The projections of $A^{(x)}$ are $(R^{(x)}, S^{(x)})$, where $R^{(x)}=R$. The projections of $A^{(y)}$ are $(R^{(y)}, S^{(y)})$, where $S^{(y)}=S$. Similarly, the matrices $A^{(xy)}$ and $A^{(yx)}$ are defined by $S^{(x)}$ and $R^{(y)}$ as

$$a_{ij}^{(xy)} = \begin{cases} 0, & \text{if } \ i > s_j^{(x)}; \\ 1, & \text{otherwise,} \end{cases}$$

and (4.2)

$$a_{ij}^{(yx)} = \begin{cases} 0, & \text{if } \ j > r_i^{(y)}; \\ 1, & \text{otherwise} \end{cases}$$

$i=1, 2, \ldots, n$, $j=1, 2, \ldots, m$. The projections of $A^{(xy)}$ and $A^{(yx)}$ are denoted by $(R^{(xy)}, S^{(xy)})$, and $(R^{(yx)}, S^{(yx)})$, respectively. It is easy to see that $A^{(xy)}$ and $A^{(yx)}$ are unambiguous (they have no switching component). From the construction, it follows that $R^{(xy)}$ consists of the elements of R in decreasing order and $S^{(yx)}$ consists of the elements of S in decreasing order. That is, by constructing a (0, 1)-matrix B with projections $(R^{(xy)}, S^{(yx)})$ and making a suitable permutation of its rows and columns, we get a binary matrix of $\mathscr{A}(R, S)$.

If $A^{(xy)}=A^{(yx)}$, then let $B=A^{(xy)}(=A^{(yx)})$. As B is uniquely determined by its projections, it has no variant element, and so there is no variant element of A that can be constructed from B by suitable row and column permutations.

If $A^{(xy)}\neq A^{(yx)}$, then from matrix $A^{(xy)}$ the matrix B can be constructed by successively shifting the 1's from the left to the right in the rows of $A^{(xy)}$, similarly as in [10]:

Procedure to construct (0, 1)-matrix B:

Step 1: $j:=1$, $B:=A^{(xy)}$.

Step 2: Consider the jth column of B. If the number of 1's in this column is greater than $s_j^{(yx)}$, then find the first row, begin from the bottom position upward, which contains a 1 in the jth column and a 0 nearest to the right. Interchange the 1 and the 0 in B. Repeat in this fashion until only $s_j^{(yx)}$ 1's are left in this column.

Step 3: $j:=j+1$. If $j=m$, stop. Otherwise, go to Step 2.

The result of this Procedure is a (0, 1)-matrix B having row and column projections $R^{(xy)}$ and $S^{(yx)}$, respectively.

If $A^{(xy)}\neq A^{(yx)}$, then $S^{(xy)}\neq S^{(yx)}$, but even in this case

$$\sum_{j=1}^{k} s_j^{(xy)} \geqq \sum_{j=1}^{k} s_j^{(yx)}$$

for all k, $1\leqq k\leqq m$, so that there is inequality for at least one k. Let $1\leqq j_p'<j_p''<<j_{p-1}'<j_{p-1}''<\ldots<j_1'<j_1''\leqq m$ $(p\geqq 1)$ be the column indices such that

$$\sum_{j=1}^{k} s_j^{(xy)} > \sum_{j=1}^{k} s_j^{(yx)} \tag{4.3}$$

if $j_s'\leqq k<j_s''$ for all $s=1, 2, \ldots, p$, and

$$\sum_{j=1}^{k} s_j^{(xy)} = \sum_{j=1}^{k} s_j^{(yx)}$$

otherwise. It is easy to see that during the Procedure only the jth columns of B can be modified, where $j_s'\leqq j\leqq j_s''$. It is also clear that, if $a_{i_s''j_s'}^{(xy)}=1$ was the bottom 1 in the j_s' th column, then finally it will be in the j_s'' column of B: $b_{i_s''j_s'}=0$ and $b_{i_s''j_s''}=1$. Applying Lemma 3.1, we have $u_{i_s''}=j_s''$ and $z_{i_s''}=j_s'$. Hence, the elements of

$$T_s = I_s\times J_s$$

are invariant, where

$$I_s = \{i_s', i_s'+1, \ldots, i_s''\}$$

and

$$J_s = \{j_s', j_s'+1, \ldots, j_s''\}.$$

During the Procedure, the column j is unaltered if $j_s'\leqq j\leqq j_s''$ is not satisfied for any j_s' and j_s'', $1\leqq s\leqq p$. These columns of B are the same as these columns of $A^{(xy)}$. Therefore, all of the variant elements of $\mathcal{A}(R^{(xy)}, S^{(yx)})$ are in the columns j, where

$$j_s' \leqq j \leqq j_s''$$

for an s, $1 \leqq s \leqq p$. From the definition of $A^{(xy)}$, it follows that

$$w_{j''_s} = s^{(xy)}_{j''_s} + 1 = i'_s$$

and (4.4)

$$v_{j''_s} = s^{(xy)}_{j'_s} = i''_s$$

where $w_{j''_s}$ and $v_{j''_s}$ are defined for the class $\mathcal{A}(R^{(xy)}, S^{(yx)})$, as in Lemma 3.2. An analogous procedure and philosophy for the rows gives that all of the variant elements of $\mathcal{A}(R^{(xy)}, S^{(yx)})$ are in the rows i, where

$$i'_s \leqq i \leqq i''_s,$$

$s=1, 2, \dots, p$, where $1 \leqq i'_1 < i''_1 < i'_2 < i''_2 < \dots < i'_p < i''_p \leqq n$ $(p \geqq 1)$ are the row indices such that

$$\sum_{i=1}^{k} r_i^{(yx)} > \sum_{i=1}^{k} r_i^{(xy)} \tag{4.5}$$

if $i'_s \leqq k < i''_s$, $s=1, 2, \dots, p$, and

$$\sum_{i=1}^{k} r_i^{(yx)} = \sum_{i=1}^{k} r_i^{(xy)}$$

otherwise. That is, from the projections $S^{(xy)}$ and $S^{(yx)}$ we can give the sets of the variant elements of B, T_s, $s=1, 2, \dots, p$, by (4.3) and (4.4) (or equivalently by (4.3) and (4.5)) explicitly, as they are described in Theorem 3.1.

Let π_x denote a permutation of $S^{(yx)}$ such that $\pi_x(S^{(yx)})=S$, and let π_y denote a permutation of $R^{(xy)}$ such that $\pi_y(R^{(xy)})=R$. Let

$$\pi_y(I_s) = \{\pi_y(i'_s), \pi_y(i'_s+1), \dots, \pi_y(i''_s)\}$$

and

$$\pi_x(J_s) = \{\pi_x(j'_s), \pi_x(j'_s+1), \dots, \pi_x(j''_s)\}.$$

Since the sets $T_s = I_s \times J_s$, $s=1, 2, \dots, p$, contain the indices of the variant elements of $\mathcal{A}(R^{(xy)}, S^{(yx)})$, the sets

$$\pi(T_s) = \pi_y(I_s) \times \pi_x(J_s), \tag{4.6}$$

$s=1, 2, \dots, p$, contain the indices of the variant elements of the class $\mathcal{A}(R, S)$.

Theorem 4.1. The variant elements of the class $\mathcal{A}(R, S)$, if there are any, are in the sets $\pi(T_s)$, $s=1, 2, \dots, p$ $(p=0$ is also possible), defined by (4.1)—(4.6).

Let us see two examples.

Example 4.1. Let $R=(1, 1, \dots, 1)$ and $S=(1, 1, \dots, 1)$. Then

$$S^{(xy)} = (n, 0, 0, \dots, 0), \quad S^{(yx)} = (1, 1, \dots, 1), \quad j'_1 = 1, \quad j''_1 = n, \quad i'_1 = 1, \quad i''_1 = 1,$$

$$p = 1, \quad I_1 = \{1, 2, \dots, n\}, \quad J_1 = \{1, 2, \dots, n\}, \quad \pi = (1, 2, \dots, n),$$

$$\pi_y(I_1) = \{1, 2, \dots, n\}, \quad \pi_x(J_1) = \{1, 2, \dots, n\}, \quad \pi(T_1) = \{1, 2, \dots, n\} \times \{1, 2, \dots, n\}.$$

Example 4.2 (see Figure 3).

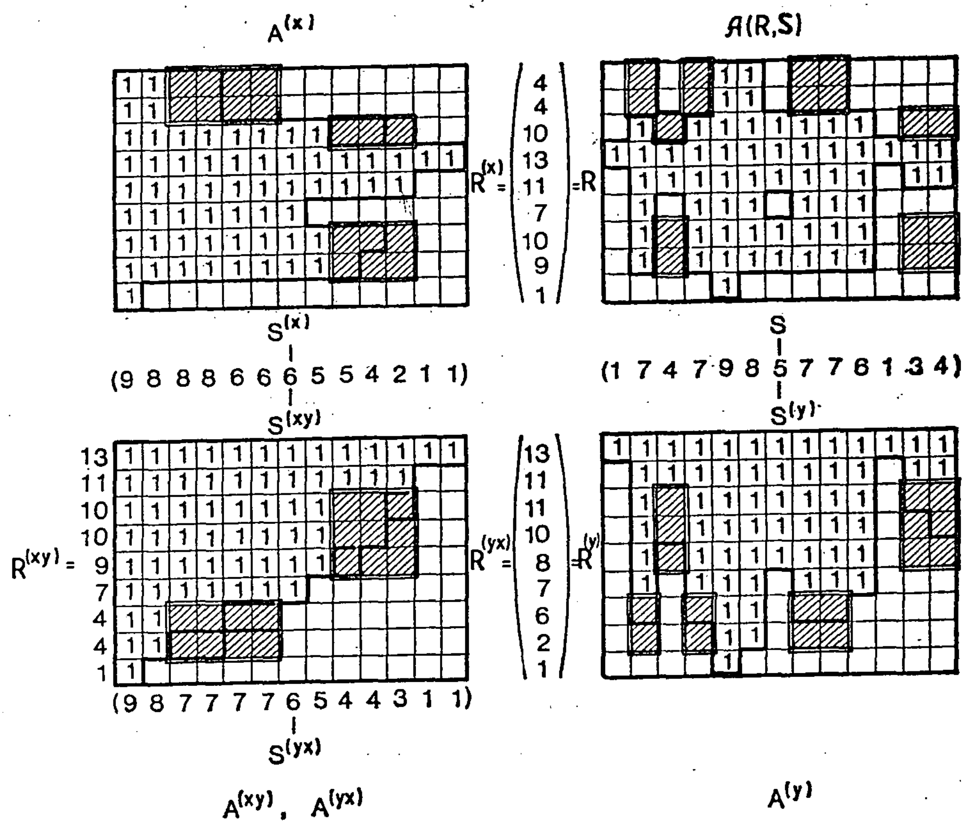

Figure 3. Determination of the structure of $\mathscr{A}(R, S)$ from the projections R and S, □ and ▨ denote the invariant 0's and the variant positions, respectively

Consequence 4.1. The (i, j) elements, $i=1, 2, \ldots, n$, $j=1, 2, \ldots, m$, can be divided into three sets: the positions of invariant 0's, invariant 1's and variant elements. From the construction of $T_1, T_2, \ldots, T_p$ from $R^{(xy)}$ and $S^{(yx)}$, it follows that the set of invariant 1's of the class $\mathscr{A}(R^{(xy)}, S^{(yx)})$ is

$$\{(i, j) | a_{ij}^{(xy)} = 1\} \setminus \bigcup_{s=1}^{p} T_s;$$

the set of variant elements of the class $\mathscr{A}(R^{(xy)}, S^{(yx)})$ is

$$\bigcup_{s=1}^{p} T_s;$$

the set of invariant 0's of the class $\mathscr{A}(R^{(xy)}, S^{(yx)})$ is

$$\{1, 2, \ldots, n\} \times \{1, 2, \ldots, m\} \setminus \{(i, j) | a_{ij}^{(xy)} = 1\} \setminus \left(\bigcup_{s=1}^{p} T_s\right).$$

Similarly, the set of invariant 1's of the class $\mathcal{A}(R, S)$ is

$$\{(i, j)|a_{ij} = 1\}\setminus\bigcup_{s=1}^{p} \pi(T_s);$$

the set of variant elements of the class $\mathcal{A}(R, S)$ is

$$\bigcup_{s=1}^{p} \pi(T_s);$$

the set of invariant 0's of the class $\mathcal{A}(R, S)$ is

$$\{1, 2, ..., n\}\times\{1, 2, ..., m\}\setminus\{(i, j)|a_{ij} = 1\}\setminus(\bigcup_{s=1}^{p} \pi(T_s)),$$

where A is an arbitrary element of the class $\mathcal{A}(R, S)$.

Consequence 4.2. From Ryser's Theorem [6], we know that if $A, A'\in\mathcal{A}(R, S)$, then A is transformable into A' by a finite sequence of interchanges. From the structure of $\mathcal{A}(R, S)$ given by Theorem 4.1, it is also clear that the four elements of an interchange are in one of the sets $\pi(T_s)$. That is, if $A, A'\in\mathcal{A}(R, S)$, then A is transformable into A' by a finite sequence of separate interchanges in $\pi(T_1)$, $\pi(T_2)$, ..., $\pi(T_p)$. Let n_s denote the number of different binary matrices generated from an $A\in\mathcal{A}(R, S)$ by interchanges only in $\pi(T_s)$, $s=1, 2, ..., p$. The number of elements of $\mathcal{A}(R, S)$ is an interesting unsolved problem (see [4] and [11]), which can be reduced to the determination of the numbers n_s, $s=1, 2, ..., p$, in the following way:

$$|\mathcal{A}(R, S)| = \prod_{s=1}^{p} n_s.$$

The author thanks Mrs. S. Silóczki and Mrs. E. Vida for the technical assistance in the preparing of the manuscript.

References

[1] R. A. Brualdi: Matrices of zeros and ones with fixed row and column sum vectors Linear Algebra and Its Applications 33, 159—231 (1980).
[2] S.-K. Chang: The reconstruction of binary patterns from their projections Comm. ACM 14, 21—25 (1971).
[3] D. Gale: A theorem on flows in networks Pacific J. Math. 7, 1073—1082 (1957).
[4] W. Honghui: Structure and cardinality of the class $\mathcal{A}(R, S)$ of (0, 1)-matrices J. Math. Res. Exposition 4, 87—93 (1984).
[5] A. Kuba: On the reconstruction of binary matrices Pubbl. Ist. Anal. Globale Apply., Serie „Problemi non ben posti ed inversi", No. 28. Firenze (1986).
[6] H. J. Ryser: Combinatorical properties of matrices of zeros and ones Canad. J. Math. 9, 371—377 (1957).
[7] H. J. Ryser: The term rank of a matrix Canad. J. Math. 10, 57—65 (1958).
[8] H. J. Ryser: Traces of matrices of zeros and ones Canad. J. Math. 12, 463—476 (1960).
[9] H. J. Ryser: Matrices of zeros and ones Bull. Amer. Math. Soc 66, 442—464 (1960).
[10] Y. R. Wang: Characterization of binary patterns and their projections IEEE Trans. Computers C-24, 1032—1035 (1975).
[11] W.-D. Wei: The class $\mathcal{A}(R, S)$ of (0, 1)-matrices Discrete Mathematics 39, 301—305 (1982).

(Received February 6, 1989)