Acta Cybernetica, Tom. 9, Fasc. 4, Szeged, 1990

A note on axiomatizing flowchart schemes

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Abstract

A biflow is an equationally presented algebraic structure which completely characterizes flowchart schemes from the algebraic point of view. Usually it is presented using summation, composition, (left or right) feedbackation, identities, and block transpositions. In the present paper we give a new presentation for the biflow structure, without making use of composition and block transpositions, but using an extended feedbackation.

1 Introduction

The axiomatization of flowchart schemes is a basic step toward an algebraic theory of computation (see [CS87b], for example). A series of papers [CU82 & CG84], [BE85], [St86b], [Ba87] and [CS87a & 88b] has lead to an algebraic structure, called biflow, which completely characterizes flowchart schemes from the algebraic point of view. This structure uses a new looping operation, called feedbackation (introduced in [St86a]), which is in some cases better than iteration or repetition, cf. [St86b], [CS88a].

The biflow structure has been introduced in [St86b], without axiomatizing bijections. The bijections were axiomatized in [Ba87] and [CS87a] leading to the actual presentation for the biflow structure.

A usual flowchart scheme is built up from some atomic schemes connected by arrows. Note that a natural structure of biflow may be given on the collection of the sets of arrows used to connect atomic elements in flowchart schemes. The main interest in the biflow structure comes from the following theorem [CS88b, Theorem 8.2] (also presented in Section 4 of [CS87b]):

> *"Flowchart schemes have a universal property similar to that of polynomials, i.e. the flowchart schemes obtained using elements in a biflow T, as connections, and elements in a double-ranked set X, as atomic schemes, form the biflow freely generated by adding X to T."*

Hence in our theory of program schemes the biflow of flowchart schemes have the same role as the ring of polynomials in classical algebra. In other words the axioms which define the biflow structure give a complete axiomatization for the concrete

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Table 1: Axioms B1-16 define a biflow, while B1-5, B6a,b,c, B7a, B8-16 define a flow.

		$B1$	$f\cdot(g\cdot h)=(f\cdot g)\cdot h$
		$B2$	$I_a\cdot f=f=f\cdot I_b$
		$B3$	$f+(g+h)=(f+g)+h$
		$B4$	$I_0+f=f=f+I_0$
		$B5$	$I_a+I_b=I_{ab}$
$B6$	$(f+f')\cdot(g+g')=f\cdot g+f'\cdot g'$	$B6a$	$(f+I_{b'})\cdot(g+g')=f\cdot g+I_{b'}\cdot g'$
		$B6b$	$(f+f')\cdot(I_b+g')=f\cdot I_b+f'\cdot g'$
		$B6c$	$(I_{ab}+f)\cdot({}^aX^b+I_d)=I_{ab}\cdot{}^aX^b+f\cdot I_d$
$B7$	${}^cX^a\cdot(f+g)\cdot{}^bX^d=g+f$ for $f:a\to b, g:c\to d$	$B7a$	${}^cX^a\cdot(f+I_c)\cdot{}^bX^c=I_c+f$ for $f:a\to b$
		$B8$	${}^aX^0=I_a$
		$B9$	${}^aX^{bc}=({}^aX^b+I_c)\cdot(I_b+{}^aX^c)$
		$B10$	$f\cdot g\uparrow^a=((f+I_a)\cdot g)\uparrow^a$
		$B11$	$f\uparrow^a\cdot g=(f\cdot(g+I_a))\uparrow^a$
		$B12$	$f+g\uparrow^a=(f+g)\uparrow^a$
		$B13$	$(f\cdot(I_d+{}^aX^b))\uparrow^{ba}=((I_c+{}^aX^b)\cdot f)\uparrow^{ab}$
		$B14$	$f\uparrow^{ab}=f\uparrow^b\uparrow^a$
		$B15$	$I_a\uparrow^a=I_0$
		$B16$	${}^aX^a\uparrow^a=I_a$

flowchart schemes in the context of Manysorted Equational Logic. An element in an abstract biflow may be regarded as an abstract flowchart scheme.

A comparison between feedbackation, Elgot's iteration and Kleene's repetition is given in [CS88a]. As a by-product there are given certain axiom systems for defining the concept of "biflow over an algebraic theory", in terms of iteration, and the concept of "biflow over a matrix theory", in terms of repetition.

The aim of this paper is to give an axiom system for defining the biflow structure, without making use of composition and block transpositions, but using an extended feedbackation, i.e. we allow a feedback to connect an arbitrary output with an arbitrary, compatible input. (In the previous papers we have used only the right and the left feedbackation.) Consequently the operations we use in this paper are: *summation*, (extended) *feedbackation*, and *identities*.

The inspiration for the present note came from a reading of Parrow's axiomatization [Pa87] for a certain kind of nets, called synchronization flow networks. A comparison between the present axiom system for biflow and Parrow's axiom system is given [CS89].

* * *

Let $(M,+,0)$ be a free monoid. We agree to omit the writing of "+", that is we write, for example, ab insted of $a+b$. The letters a,b,c,d,e,u,v,w will denote elements of M.

Definition 1 *An* M-biflow *(resp.* M-flow*) B is an abstract structure given by:*

- *a family of sets* $\{B(a,b)\}_{a,b\in M}$;

Table 2: These axioms define a BIFLOW, while the subset of all the axioms denoted by S.. or F.. define a FLOW

$$S1 \quad f+(g+h)=(f+g)+h$$

$$S2 \quad f+\mathrm{I}_0=f \qquad\qquad S2^\circ \quad f=\mathrm{I}_0+f$$

$$S3 \quad \mathrm{I}_a+\mathrm{I}_b=\mathrm{I}_{ab}$$

$$F0a \quad f\uparrow^{a(0)b}_{c(0)d}=f$$

$$F0b \quad f\uparrow^{a(uv)b}_{c(uv)d}=f\uparrow^{a(u)vb}_{c(u)vd}\uparrow^{a(v)b}_{c(v)d} \qquad\qquad F0b^\circ \quad f\uparrow^{bv(u)a}_{dv(u)c}\uparrow^{b(v)a}_{d(v)c}=f\uparrow^{b(vu)a}_{d(vu)c}$$

$$F1 \quad f\uparrow^{a(u)b}_{c(u)d}+g=(f+g)\uparrow^{a(u)ba'}_{c(u)dc'} \qquad\qquad F1^\circ \quad (g+f)\uparrow^{a'b(u)a}_{c'd(u)c}=g+f\uparrow^{b(u)a}_{d(u)c}$$

$$F2a \quad f\uparrow^{a(u)bvc}_{a'(u)b'vc'}\uparrow^{ab(v)c}_{a'b'(v)c'}=f\uparrow^{aub(v)c}_{a'ub'(v)c'}\uparrow^{a(u)bc}_{a'(v)b'c'}$$

$$F2b \quad f\uparrow^{a(u)bvc}_{a'vb'(u)c'}\uparrow^{ab(v)c}_{a'(v)b'c'}=f\uparrow^{aub(v)c}_{a'(v)b'uc'}\uparrow^{a(u)bc}_{a'b'(u)c'}$$

$$F3 \quad (\mathrm{I}_a+f)\uparrow^{a(a)0}_{0(a)b}=f \qquad\qquad F3^\circ \quad f=(f+\mathrm{I}_a)\uparrow^{0(a)a}_{b(a)0}$$

$$F3_\circ \quad (\mathrm{I}_a+f)\uparrow^{0(a)b}_{a(a)0}=f \qquad\qquad F3^\circ_\circ \quad f=(f+\mathrm{I}_a)\uparrow^{b(a)0}_{0(a)a}$$

$$F4 \quad \mathrm{I}_a\uparrow^{0(a)0}_{0(a)0}=\mathrm{I}_0$$

$$P \quad (f+g)\uparrow^{a(b)0}_{0(b)c}=(g+f)\uparrow^{0(b)a}_{c(b)0}$$

- *two kinds of constants:*
  Identities $\mathrm{I}_a \in B(a,a)$ for $a \in M$ and
  Block Transpositions ${}^a\mathrm{X}^b \in B(ab,ba)$ for $a,b \in M$;
- *three operations:*
  Summation $+ : B(a,b)\times B(c,d) \to B(ac,bd)$ for $a,b,c,d \in M$,
  Composition $\cdot : B(a,b)\times B(b,c) \to B(a,c)$ for $a,b,c \in M$, and
  (Right) Feedbackation $\uparrow^a$: $B(ba,ca) \to B(b,c)$ for $a,b,c \in M$

and fulfilling axioms B1-16 (*resp.* B1-5, B6a,b,c, B7a *and* B8-16) *in Table 1*[1]. □

We declare that feedbackation has the strongest binding power, then composition, then summation. For instance, $f\cdot g+f'\cdot g'$ means $(f\cdot g)+(f'\cdot g')$, while $f+g\uparrow^a$ means $f+(g\uparrow^a)$. The sign of composition " $\cdot$ " is usually omitted.

In an M-(bi)flow B, defined as above, one may define an extended feedbackation $\Uparrow^{a(u)b}_{c(u)d}$ by

$$f\Uparrow^{a(u)b}_{c(u)d}=[(\mathrm{I}_a+{}^b\mathrm{X}^u)\cdot f\cdot(\mathrm{I}_c+{}^u\mathrm{X}^d)]\uparrow^u \quad \text{for } f\in B(aub,cud). \tag{1}$$

Definition 2 *Let us say that B is an* M-BIFLOW (*resp.* M-FLOW) *if it is given by a family of sets $\{B(a,b)\}_{a,b\in M}$, a summation, identities, and an*

Extended Feedbackation $\uparrow^{a(u)b}_{c(u)d}$: $B(aub,cud) \to B(ab,cd)$ for $a,b,c,d,u \in M$

and fulfilling the axioms in Table 2 (resp. all the axioms in Table 2 denoted by strings starting with letter "S" or "F"). □

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[1]The algebra $Fl_{\Sigma,Pfn}$ of representations of Σ-flowcharts over Pfn in [CS87a] is an example of flow which is not a biflow (provided $\Sigma \neq 0$).

Consequently an M-BIFLOW is an M-FLOW which fulfills axiom (P).

In an M-(BI)FLOW B one may define a composition "o" by

$$f \circ g = (f+g) \uparrow^{a(b)0}_{0(b)c} \text{ for } f \in B(a,b), g \in B(b,c), \tag{2}$$

block transpositions ${}^a\mathrm{X}^b$ by

$${}^a\mathrm{X}^b = (\mathrm{I}_a + \mathrm{I}_b + \mathrm{I}_a) \uparrow^{ab(a)0}_{0(a)ba} \text{ for } a, b \in M \tag{3}$$

and a (right) feedbackation $\_ \uparrow^a$ by

$$f \uparrow^a = f \uparrow^{b(a)0}_{c(a)0} \text{ for } f \in B(ba, ca). \tag{4}$$

The aim of this note is to prove that the concept of M-biflow coincides with the concept of M-BIFLOW, i.e. the above definitions give two different presentations of the same algebraic structure. As a by-product we also get the equivalence of the concepts of M-flow and M-FLOW.

More precisely, let us consider the following transformations

$$\mathbf{B} = (B, +, \cdot, \uparrow^a, \mathrm{I}_a, {}^a\underline{\mathrm{X}}^b) \mapsto \alpha(\mathbf{B}) = (B, +, \Uparrow^{a(u)b}_{c(u)d}, \mathrm{I}_a),$$

where $\Uparrow^{a(u)b}_{c(u)d}$ is that defined in $\mathbf{B}$ by formula (1) and

$$\mathbf{C} = (C, +, \uparrow^{a(u)b}_{c(u)d}, \mathrm{I}_a) \mapsto \beta(\mathbf{C}) = (C, +, \circ, \Uparrow^a, \mathrm{I}_a, {}^a\mathrm{X}^b)$$

where $\circ$, ${}^a\mathrm{X}^b$, and $\Uparrow^a$ are those defined in $\mathbf{C}$ by formulas (2), (3), (4), respectively. We shall prove the following theorem.

Theorem 1 (i) *If* $\mathbf{B}$ *is an* M*-biflow, then* $\alpha(\mathbf{B})$ *is an* M*-BIFLOW.*

(ii) *If* $\mathbf{C}$ *is an* M*-BIFLOW, then* $\beta(\mathbf{C})$ *is an* M*-biflow.*

(iii) *If* $\mathbf{B}$ *is an* M*-biflow then* $\beta(\alpha(\mathbf{B})) = \mathbf{B}$*, and if* $\mathbf{C}$ *is an* M*-BIFLOW then* $\alpha(\beta(\mathbf{C})) = \mathbf{C}$.

The proof of this theorem is based on the analogous theorem for flows which is stated below.

Theorem 2 (i) *If* $\mathbf{B}$ *is an* M*-flow, then* $\alpha(\mathbf{B})$ *is an* M*-FLOW.*

(ii) *If* $\mathbf{C}$ *is an* M*-FLOW, then* $\beta(\mathbf{C})$ *is an* M*-flow.*

(iii) *If* $\mathbf{B}$ *is an* M*-flow then* $\beta(\alpha(\mathbf{B})) = \mathbf{B}$*, and if* $\mathbf{C}$ *is an* M*-FLOW then* $\alpha(\beta(\mathbf{C})) = \mathbf{C}$.

The difficult part of the proof is the passing from (BI)FLOWS to (bi)flows. In order to make the proof easier to understand we insert a section with deductions of certain identities that are valid in a (BI)FLOW.

In the sequal we shall use two types of duality, briefly presented here.

Duality. We denote by a° the opposite of the word $a \in M$. Let $t \in B(a,b)$ be a term written with sum, extended feedback and identities.

The $^\circ$*-dual term* $t^\circ \in B(a^\circ, b^\circ)$ is obtained by using the following inductive procedure:

the $^\circ$-*dual term* of a variable $x \in B(a,b)$ is another variable $x^\circ \in B(a^\circ, b^\circ)$, and

$$(f+g)^\circ = g^\circ + f^\circ; (f \uparrow^{a(u)b}_{c(u)d})^\circ = f^\circ \uparrow^{b^\circ(u^\circ)a^\circ}_{d^\circ(u^\circ)c^\circ}; (\mathrm{I}_a)^\circ = \mathrm{I}_{a^\circ}$$

The $_\circ$-*dual term* $t_\circ \in B(b,a)$ is obtained by using the rules:
the $_\circ$-*dual term* of a variable $x \in B(a,b)$ is another variable $x_\circ \in B(b,a)$, and

$$(f+g)_\circ = f_\circ + g_\circ; (f \uparrow^{a(u)b}_{c(u)d})_\circ = f_\circ \uparrow^{c(u)d}_{a(u)b}; (\mathrm{I}_a)_\circ = \mathrm{I}_a$$

Lemma 1 **($^\circ$-duality)**. *If E is an identity which is valid in every M-FLOW, then E° is an identity which is valid in every M-FLOW.*

($_\circ$-duality) *If E is an identity which is valid in every M-FLOW, then $E_\circ$ is an identity which is valid in every M-FLOW.*

Proof. It is enough to see that the $^\circ$-dual (resp. $_\circ$-dual) identity corresponding to a FLOW-axiom in Table 2 is also in Table 2, and the rules for deduction of valid identities (i.e. Manysorted Equational Logic) are invariant under $^\circ$-duality (resp. under $_\circ$-duality). □

Clearly, for every E we have $(E^\circ)_\circ = (E_\circ)^\circ$; in the sequel we shall denote it simply by $E^\circ_\circ$. From Lemma 3 it follows that:

Corollary **($^\circ_\circ$-duality)**. *If E is an identity which is valid in every M-FLOW, then $E^\circ_\circ$ is an identity which is valid in every M- FLOW.* □

In order to simplify the calculation we shall use the following convention.

Convention. The writing of 0, which denotes the neutral element of the underlying monoid M, is usualy omitted.

If we are given two strings of letters, namely $\alpha = a_0(u_{\sigma(1)})a_1 \ldots a_{n-1}(u_{\sigma(n)})a_n$ and $\beta = b_0(u_1)b_1 \ldots b_{n-1}(u_n)b_n$ where each letter denotes an element in M, if moreover $\sigma : [n] \longrightarrow [n]$[2] is a bijection and $u_1, \ldots, u_n$ are all *distinct* letters, then by $\uparrow^\alpha_\beta$ we mean the multiple feedbackation computed, say, from right to left with respect to β. (By axioms (F2a) & (F2b) the order in which the feedbacks are computed is without importance.)For instance

$$f \uparrow^{(a)(v)b'(u)c'}_{(a)(u)b(v)c} \ (= f \uparrow^{0(a)0(v)b'(u)c'}_{0(a)0(u)b(v)c}) \text{ means } f \uparrow^{a(v)b'uc'}_{aub(v)c} \uparrow^{ab'(u)c'}_{a(u)bc} \uparrow^{(a)b'c'}_{(a)bc} .$$

This rule is ambiguous when some letters in $u_1, \ldots, u_n$ are equal. In that case we use numbers to indicate the correct feedbacks (and *not* the order in which the feedbacks are computed, which is without importance). For instance

$$f \uparrow^{(2,u)b(1,u)a}_{(1,u)d(2,u)c} \text{ means } f \uparrow^{(u)bua}_{ud(u)c} \uparrow^{b(u)a}_{(u)dc} .$$

□

With this convention the axioms (F0b) and (F0b$^\circ$), which are equivalent in the presence of (F2a), may be easily written as $f \uparrow^{a(uv)b}_{c(uv)d} = f \uparrow^{a(u)(v)b}_{c(u)(v)d}$.

Lemma 2 *In the presence of (F0-2) the axioms (F3) & (F3 $^\circ_\circ$) are equivalent to (F3$_\circ$) & (F3 $^\circ$).*

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[2] $[n] = \{1, 2, \ldots, n\}$

Table 3: These identities are valid in a FLOW

$F5\ (\mathrm{I}_a + f) \uparrow^{a(a)b}_{(a)c} = f$	$F5^\circ\quad f = (f + \mathrm{I}_a) \uparrow^{b(a)a}_{c(a)}$
$F5_\circ\ (\mathrm{I}_a + f) \uparrow^{(a)c}_{a(a)b} = f$	$F5^\circ_\circ\quad f = (f + \mathrm{I}_a) \uparrow^{c(a)}_{b(a)a}$
$F6\ (\mathrm{I}_u + f) \uparrow^{(2,u)b(1,u)a}_{(1,u)d(2,u)c} = f \uparrow^{b(u)a}_{d(u)c}$	$F6^\circ\quad f \uparrow^{a(u)b}_{c(u)d} = (f + \mathrm{I}_u) \uparrow^{a(1,u)b(2,u)}_{c(2,u)d(1,u)}$

$F7\ (= F7^\circ_\circ)(f+\mathrm{I}_u+g) \uparrow^{a(1,u)v(2,u)w}_{b(1,u)c(2,u)d} = (f+g) \uparrow^{av(u)w}_{b(u)cd}$ for $f : a \to buc, g : vuw \to d$

$F7^\circ\ (= F7_\circ)(g+f) \uparrow^{w(u)va}_{dc(u)b} = (g+\mathrm{I}_u+f) \uparrow^{w(2,u)v(1,u)a}_{d(2,u)c(1,u)b}$ for $f : a \to cub, g : wuv \to d$

$F8\quad (\mathrm{I}_b + f) \uparrow^{ba(b)}_{(b)c} = (f + \mathrm{I}_a) \uparrow^{(a)ba}_{c(a)}$

$F8_\circ\quad (\mathrm{I}_b + f) \uparrow^{(b)c}_{ba(b)} = (f + \mathrm{I}_a) \uparrow^{c(a)}_{(a)ba}$

Proof. Suppose (F3) and $(F3^\circ_\circ)$ hold. Then $(F3^\circ)$ may be proved as follows:

$(f + \mathrm{I}_a) \uparrow^{(a)a}_{b(a)} =$ (by $F3^\circ_\circ$, then $F3$)$(\mathrm{I}_a + ((f + \mathrm{I}_a) \uparrow^{(a)a}_{b(a)} + \mathrm{I}_b) \uparrow^{a(b)}_{(b)b}) \uparrow^{a(a)}_{(a)b}$

$= (\mathrm{I}_a + f + \mathrm{I}_{ab}) \uparrow^{a(1,a)(2,a)(b)}_{(2,a)(b)(1,a)b} = (\mathrm{I}_a + f + \mathrm{I}_{ab}) \uparrow^{aa(ab)}_{(ab)ab} \uparrow^{a(a)}_{(a)b} =$ (by $F3^\circ_\circ$, then $F3$)f.

We have proved that

(x) (F3) & $F3^\circ_\circ \Longrightarrow F3^\circ$.

By applying $^\circ_\circ$-duality to (x) it follows that $(F3^\circ_\circ)$& (F3) $\Longrightarrow F3_\circ$. By applying $^\circ$-duality to (x) it follows that $(F3^\circ)$&$(F3_\circ) \Longrightarrow (F3)$, and by applying $_\circ$-duality to (x) it follows that $(F3_\circ)$&$(F3^\circ) \Longrightarrow (F3^\circ_\circ)$. □

Lemma 3 *The identities in Table 3 are valid in a FLOW.*

Proof. Proof of (F5):

$(\mathrm{I}_a + f) \uparrow^{a(a)b}_{(a)c} =$ (by $F3^\circ$, and $F3_\circ$)$[\mathrm{I}_c + ((\mathrm{I}_a + f) \uparrow^{a(a)b}_{(a)c} + \mathrm{I}_{ab}) \uparrow^{(ab)ab}_{c(ab)}] \uparrow^{(c)ab}_{c(c)}$

$= (\mathrm{I}_{ca} + f + I_{ab}) \uparrow^{(c)(2,a)(1,a)(b)ab}_{c(1,a)(c)(2,a)(b)} = [(\mathrm{I}_{ca} + f + \mathrm{I}_a) \uparrow^{(ca)aba}_{ca(ca)} + \mathrm{I}_b] \uparrow^{(ab)ab}_{c(ab)}$

=(by $F3_\circ$ and $F3^\circ$) f.

Proof of (F6):

$(\mathrm{I}_u + f) \uparrow^{(2,u)b(1,u)a}_{(1,u)d(2,u)c} =$ (by $F5$)$(\mathrm{I}_b + (\mathrm{I}_u + f) \uparrow^{(2,u)b(1,u)a}_{(1,u)d(2,u)c}) \uparrow^{b(b)a}_{(b)dc}$

$= (\mathrm{I}_{bu} + f) \uparrow^{b(2,u)(b)(1,u)a}_{(b)(1,u)d(2,u)c} = (\mathrm{I}_{bu} + f) \uparrow^{bu(bu)a}_{(bu)duc} \uparrow^{b(u)a}_{d(u)c} =$ (by $F5$)$f \uparrow^{b(u)a}_{d(u)c}$.

Proof of (F7):

$(f + \mathrm{I}_u + g) \uparrow^{a(1,u)v(2,u)w}_{b(1,u)c(2,u)d} =$ (by $F6^\circ$)$(f + \mathrm{I}_u + g + \mathrm{I}_u) \uparrow^{a(3,u)v(2,u)w(4,u)}_{b(4,u)c(2,u)d(3,u)}$

$= (f + (\mathrm{I}_u + g + \mathrm{I}_u) \uparrow^{(3,u)v(2,u)wu}_{(2,u)d(3,u)}) \uparrow^{avw(4,u)}_{b(4,u)cd}$

$=$ (by $F6$)$(f + (g + \mathrm{I}_u) \uparrow^{v(5,u)wu}_{d(5,u)}) \uparrow^{avw(4,u)}_{b(4,u)cd}$

$= (f + g + \mathrm{I}_u) \uparrow^{av(5,u)w(4,u)}_{b(4,u)cd(5,u)} =$ (by $F6^\circ$)$(f + g) \uparrow^{av(u)w}_{b(u)cd}$.

Proof of (F8):

$(\mathrm{I}_b + f) \uparrow^{ba(b)}_{(b)c} =$ (by $F5^\circ$)$((\mathrm{I}_b + f) \uparrow^{ba(b)}_{(b)c} + \mathrm{I}_a) \uparrow^{b(a)a}_{c(a)} = (\mathrm{I}_b + f + \mathrm{I}_a) \uparrow^{b(a)(b)a}_{(b)c(a)}$

$= (\mathrm{I}_b + (f + I_a) \uparrow^{(a)ba}_{c(a)}) \uparrow^{b(b)a}_{(b)c} =$ (by $F5$)$(f + \mathrm{I}_a) \uparrow^{(a)ba}_{c(a)}$.

The proof af the remaining identities in Table 3 may be obtained from these ones by using $^\circ$-duality, $_\circ$- duality, or $^\circ_\circ$-duality. □

Proof of Theorem 1.(i) (resp. of Theorem 2.(i)). Clearly the identities in Table 2 hold in the algebras of flowchart schemes (resp. of flowchart scheme representations), hence by Theorem 8.2 in [CS88b] (resp. by Theorem 2.b.5. in [CS87a]) they hold in every biflow (resp. flow). Hence every biflow (resp. flow) is a BIFLOW (resp. FLOW). Direct deductions are given in [CS89]. □

Lemma 4 *The following identities hold in a FLOW.*

(T1) $f \circ (\mathrm{I}_b + {}^c\mathrm{X}^d) = (f + \mathrm{I}_c) \uparrow^{a(c)}_{b(c)dc}$ *for* $f : a \to bcd$

$(T1^\circ_\circ)$ $(\mathrm{I}_c + f) \uparrow^{cd(c)b}_{(c)a} = ({}_d\mathrm{X}_c + \mathrm{I}_b) \circ f$ *for* $f : dcb \to a$

(T2) $f \circ ({}^b\mathrm{X}^c + \mathrm{I}_d) = (\mathrm{I}_c + f) \uparrow^{(c)a}_{cb(c)d}$ *for* $f : a \to bcd$

$(T2^\circ_\circ)$ $(f + \mathrm{I}_c) \uparrow^{d(c)bc}_{a(c)} = (\mathrm{I}_d + {}_c\mathrm{X}_b) \circ f$ *for* $f : dcb \to a$

"$\circ$" and "X" are those defined by formulas (2) and (3), respectively; ${}_a\mathrm{X}_b := {}^b\mathrm{X}^a$).

Proof. First we prove (T1):

$f \circ (\mathrm{I}_b + {}^c\mathrm{X}^d) = (f + \mathrm{I}_{bcdc}) \uparrow^{a(bcd)(c)}_{(bcd)b(c)dc} = ((f + \mathrm{I}_{bcd}) \uparrow^{a(bcd)}_{(bcd)bcd} + \mathrm{I}_c) \uparrow^{a(c)}_{b(c)dc}$

$= (\text{ by } F3^\circ_\circ)(f + \mathrm{I}_c) \uparrow^{a(c)}_{b(c)dc}$.

For (T2) note that

$f \circ ({}^b\mathrm{X}^c + \mathrm{I}_d) = (f + \mathrm{I}_{bcbd}) \uparrow^{a(bc)(b)(d)}_{(bc)(d)(b)cbd} = (\text{ by } F7)(f + \mathrm{I}_{cbd}) \uparrow^{a(c)(b)(d)}_{(b)(c)(d)cbd}$

$= (\text{ by } F5_\circ)(\mathrm{I}_c + (f + \mathrm{I}_{cbd}) \uparrow^{a(c)(b)(d)}_{(b)(c)(d)cbd}) \uparrow^{(c)a}_{c(c)bd}$

$= ((\mathrm{I}_c + f + \mathrm{I}_c) \uparrow^{(1,c)a(2,c)}_{cb(2,c)d(1,c)} + \mathrm{I}_{bd}) \uparrow^{a(bd)}_{c(bd)bd}$

$= (\text{ by } F6^\circ \text{ and } F5^\circ_\circ)(\mathrm{I}_c + f) \uparrow^{(c)a}_{cb(c)d}$.

The other identities $(T1^\circ_\circ)$ and $(T2^\circ_\circ)$ follows by using $^\circ_\circ$-duality. □

Proof of Theorem 2. (ii). Let $\mathbf{C} = (C, +, \uparrow^{a(u)b}_{c(u)d}, \mathrm{I}_a)$ be an M-FLOW. We have to show that $(C, +, \circ, \Uparrow^a, \mathrm{I}_a, {}^a\mathrm{X}^b)$ is an M-flow, where "$\circ$", "${}^a\mathrm{X}^b$", and "$\Uparrow^a$" are those defined in $\mathbf{C}$ by formulas (2), (3), and (4), respectively. That is, we have to verify the validity of the axioms (B1-5), (B6a,b,c), (B7a), (B8-16) in Table 1.

By using (F1), $(F1^\circ)$, and (F2a) one may easily see that (B1) is valid, and by using (F3) and $(F3^\circ_\circ)$ it follows that (B2) is valid. (B3-5) are common axioms. For (B6a) suppose $f : a \longrightarrow b, g : b \longrightarrow c$ and $g' : b' \longrightarrow c'$. Then

$(f + \mathrm{I}_{b'}) \circ (g + g') = (f + \mathrm{I}_{b'} + g + g') \uparrow^{ab'(bb')}_{(bb')cc'}$

$= (\text{ by } F3^\circ)((f + \mathrm{I}_{b'} + g + g') \uparrow^{ab'(bb')}_{(bb')cc'} + \mathrm{I}_{ab'}) \uparrow^{(ab')ab'}_{cc'(ab')}$

$= (f + (\mathrm{I}_{b'} + g + g' + I_{ab'}) \uparrow^{(1,b')b(2,b')ab'}_{(2,b')cc'a(1,b')}) \uparrow^{(a)(b)ab'}_{(b)cc'(a)}$

$= (\text{ by } F6)(f + (g + g' + \mathrm{I}_{ab'}) \uparrow^{b(b')ab'}_{cc'a(b')}) \uparrow^{(a)(b)ab'}_{(b)cc'(a)}$

$= [((f + g) \uparrow^{a(b)}_{(b)c} + g') + \mathrm{I}_{ab'}] \uparrow^{(ab')ab'}_{cc'(ab')} = (\text{ by } F3^\circ) f \circ g + g' = f \circ g + \mathrm{I}_{b'} \circ g'$.

A proof of (F6b) may be obtained from the above proof of (F6a) by using $^\circ_\circ$-duality.

In order to prove the axioms for block transpositions we use the identities (T1), $(T1^\circ_\circ)$ and (T2) in Lemma 6. For (B6c) note that

$(\mathrm{I}_{ab} + f) \circ ({}^a\mathrm{X}^b + I_d) = (\text{ by } T2)(\mathrm{I}_b + \mathrm{I}_{ab} + f) \uparrow^{(b)abc}_{ba(b)d}$

$= \mathrm{I}_{bab} \uparrow^{(b)ab}_{ba(b)} + f = (\text{ by } F8)\mathrm{I}_{aba} \uparrow^{ab(a)}_{(a)ba} + f = {}^a\mathrm{X}^b + f = \mathrm{I}_{ab} \circ {}^a\mathrm{X}^b + f \circ \mathrm{I}_d$.

For (B7a) note that

$$ {}^c\mathrm{X}^a \circ (f+\mathrm{I}_c) \circ {}^b\mathrm{X}^c = (\text{ by } T1) {}^c\mathrm{X}^a \circ (f+\mathrm{I}_c+\mathrm{I}_b) \uparrow^{ac(b)}_{(b)cb} $$
$$ = (\text{ by } T1^\circ_\circ)(\mathrm{I}_c + (f+\mathrm{I}_{cb}) \uparrow^{ac(b)}_{(b)cb}) \uparrow^{ca(c)}_{(c)cb} = ((\mathrm{I}_c+f)+I_{cb}) \uparrow^{ca(cb)}_{(cb)cb} = (\text{ by } F3^\circ_\circ)\mathrm{I}_c+f. $$

The proof of (B8) is obvious. For (B9) note that

$$ ({}^a\mathrm{X}^b+\mathrm{I}_c) \circ (\mathrm{I}_b + {}^a\mathrm{X}^c) = (\text{ by } T1)({}^a\mathrm{X}^b+\mathrm{I}_c+\mathrm{I}_a) \uparrow^{abc(a)}_{b(a)ca} $$
$$ = (\mathrm{I}_{aba}+\mathrm{I}_{ca}) \uparrow^{ab(1,a)c(2,a)}_{(1,a)b(2,a)ca} = (\text{ by } F7)\mathrm{I}_{abca} \uparrow^{abc(a)}_{(a)bca} = {}^a\mathrm{X}^{bc}. $$

The right feedback $\_ \Uparrow^a$ is defined by formula (4), i.e.

$$ f \Uparrow^a = f \uparrow^{b(a)}_{c(a)} \text{ for } f : ba \longrightarrow ca. $$

Axiom (B10) is valid. Indeed, if $f : d \longrightarrow b$ and $g : ba \longrightarrow ca$, then

$$ ((f+\mathrm{I}_a) \circ g) \Uparrow^a = (f+\mathrm{I}_a+g) \uparrow^{da(ba)}_{(ba)ca} \uparrow^{d(a)}_{c(a)} = (f + (\mathrm{I}_a+g) \uparrow^{(2,a)b(1,a)}_{(1,a)c(2,a)}) \uparrow^{d(b)}_{(b)c} $$
$$ = (\text{ by } F6)(f + g \uparrow^{b(a)}_{c(a)}) \uparrow^{d(b)}_{(b)c} = f \circ (g \Uparrow^a). $$

The proof of (B11) is similar. Axiom (B12) is a particular case of $(F1^\circ)$. For (B13) suppose $f : cba \longrightarrow dab$. Then

$$ (f \circ (\mathrm{I}_d +{}^a \mathrm{X}^b)) \Uparrow^{ba} = (\text{ by } T1)(f+\mathrm{I}_a) \uparrow^{cba(a)}_{d(a)ba} \uparrow^{c(ba)}_{d(ba)} = (\text{ by } F6^\circ) f \uparrow^{c(b)(a)}_{d(a)(b)}, $$

and similarly

$$ ((\mathrm{I}_c +{}^a \mathrm{X}^b) \circ f) \Uparrow^{ab} = (\text{ by } T2^\circ_\circ)(f+\mathrm{I}_b) \uparrow^{c(b)ab}_{dab(b)} \uparrow^{c(ab)}_{d(ab)} = (\text{ by } F6^\circ) f \uparrow^{c(b)(a)}_{d(a)(b)}, $$

hence (B13) is valid. Axiom (B14) is a particular case of $(F0b^\circ)$ and (B15) is (F4). For (B16) note that

$$ {}^a\mathrm{X}^a \Uparrow^a = \mathrm{I}_{aaa} \uparrow^{aa(a)}_{(a)aa} \uparrow^{a(a)}_{a(a)} = (\mathrm{I}_{aa}+\mathrm{I}_a) \uparrow^{a(1,a)(2,a)}_{(2,a)a(1,a)} = (\text{ by } F6^\circ)\mathrm{I}_{aa} \uparrow^{a(a)}_{(a)a} = (\text{ by } F3)\mathrm{I}_a. $$

□

Proof of Theorem 2. (iii). Suppose $\mathbf{B} = (B, +, \cdot, \uparrow^a, \mathrm{I}_a, {}^a\underline{\mathrm{X}}^b)$ is an M-flow. The new composition "$\circ$" in $\beta(\alpha(\mathbf{B}))$ acts as follows

$$ f \circ g = (f+g) \Uparrow^{a(b)}_{(b)c} = [(\mathrm{I}_a + {}^0\underline{\mathrm{X}}^b) \cdot (f+g) \cdot (\mathrm{I}_0 + {}^b\underline{\mathrm{X}}^c)] \uparrow^b $$
$$ = ((f+g) \cdot {}^b\underline{\mathrm{X}}^c) \uparrow^b = f \cdot [(\mathrm{I}_b+g) \cdot {}^b\underline{\mathrm{X}}^c] \uparrow^b = f \cdot ({}^b\underline{\mathrm{X}}^b \cdot (g+\mathrm{I}_b)) \uparrow^b $$
$$ = f \cdot ({}^b\underline{\mathrm{X}}^b \uparrow^b) \cdot g = f \cdot \mathrm{I}_b \cdot g = f \cdot g. $$

Hence $f \circ g = f \cdot g$.

The new block transposition ${}^a\mathrm{X}^b$ in $\beta(\alpha(\mathbf{B}))$ is

$$ {}^a\mathrm{X}^b = \mathrm{I}_{aba} \Uparrow^{ab(a)}_{(a)ba} = [(\mathrm{I}_{ab} + {}^0\underline{\mathrm{X}}^a) \cdot \mathrm{I}_{aba} \cdot (\mathrm{I}_0 + {}^a\underline{\mathrm{X}}^{ba})] \uparrow^a $$
$$ = {}^a\underline{\mathrm{X}}^{ba} \uparrow^a = [({}^a\underline{\mathrm{X}}^b+\mathrm{I}_a) \cdot (\mathrm{I}_b + {}^a\underline{\mathrm{X}}^a)] \uparrow^a = {}^a\underline{\mathrm{X}}^b \cdot (\mathrm{I}_b + {}^a\mathrm{X}^a \uparrow^a) = {}^a\underline{\mathrm{X}}^b \cdot (\mathrm{I}_b+\mathrm{I}_a) = {}^a\underline{\mathrm{X}}^b. $$

Hence ${}^a\mathrm{X}^b = {}^a\underline{\mathrm{X}}^b$.

The new right feedbackation $\_ \Uparrow^a$ in $\beta(\alpha(\mathbf{B}))$ acts as follows. For $f : ba \rightarrow ca$, $f \Uparrow^a = [(\mathrm{I}_b + {}^0\underline{\mathrm{X}}^a) \cdot f \cdot (\mathrm{I}_c + {}^a\underline{\mathrm{X}}^0)] \uparrow^a = f \uparrow^a$.
Hence $\_ \Uparrow^a = \_ \uparrow^a$.

Conversely, suppose $\mathbf{C} = (C, +, \uparrow^{a(u)b}_{c(u)d}, \mathrm{I}_a)$ is an M-FLOW. The new extended feedbackation $\Uparrow^{a(u)b}_{c(u)d}$ in $\alpha(\beta(\mathbf{C}))$ acts as follows. For $f : aub \longrightarrow cud$

$$ f \Uparrow^{a(u)b}_{c(u)d} = [(\mathrm{I}_a + {}^b\underline{\mathrm{X}}^u) \circ f \circ (\mathrm{I}_c + {}^u\underline{\mathrm{X}}^d)] \Uparrow^u $$
$$ = (\text{ by } T1)[(\mathrm{I}_a + {}^b\mathrm{X}^u) \circ (f+\mathrm{I}_u) \uparrow^{aub(u)}_{c(u)du}] \Uparrow^u $$
$$ = (\text{ by } T2^\circ_\circ)[(f+\mathrm{I}_u) \uparrow^{aub(u)}_{c(u)du} + \mathrm{I}_u] \uparrow^{a(u)bu}_{cdu(u)} \uparrow^{ab(u)}_{cd(u)} $$
$$ = (\text{ by } F6^\circ)(f+\mathrm{I}_u) \uparrow^{aub(u)}_{c(u)du} \uparrow^{a(u)b}_{cd(u)} = (\text{ by } F6^\circ) f \uparrow^{a(u)b}_{c(u)d}. $$

Hence $\Uparrow^{a(u)b}_{c(u)d} = \uparrow^{a(u)b}_{c(u)d}$. □

Table 4: These axioms define a scalar BIFLOW, while the subset of all the axioms denoted by SS.. or SF.. define a scalar FLOW

$SS1$	$f+(g+h)=(f+g)+h$		
$SS2$	$f+\mathrm{I}_s\uparrow^1_1=f$	$SS2^\circ$	$f=\mathrm{I}_s\uparrow^1_1+f$
$SF1$	$f\uparrow^i_j+g=(f+g)\uparrow^i_j$	$SF1^\circ$	$(g+f)\uparrow^{\lvert c\rvert+i}_{\lvert d\rvert+j}=g+f\uparrow^i_j$ for $g:c\longrightarrow d$
$SF2a$	$f\uparrow^{\lvert a\rvert+1}_{\lvert a'\rvert+1}\uparrow^{\lvert ab\rvert+1}_{\lvert a'b'\rvert+1}=f\uparrow^{\lvert ab\rvert+2}_{\lvert a'b'\rvert+2}\uparrow^{\lvert a\rvert+1}_{\lvert a'\rvert+1}$		for $f:asbtc\longrightarrow a'sb'tc'$
$SF2b$	$f\uparrow^{\lvert a\rvert+1}_{\lvert a'b'\rvert+2}\uparrow^{\lvert ab\rvert+1}_{\lvert a'\rvert+1}=f\uparrow^{\lvert ab\rvert+2}_{\lvert a'\rvert+1}\uparrow^{\lvert a\rvert+1}_{\lvert a'b'\rvert+1}$		for $f:asbtc\longrightarrow a'tb'sc'$
$SF3$	$(\mathrm{I}_{s_1}+\ldots\mathrm{I}_{s_n}+f)\uparrow^{n+n}_n\uparrow^{n+n-1}_{n-1}\ldots\uparrow^{n+1}_1=f$		for $f:s_1\ldots s_n\longrightarrow a$
$SF3^\circ_\circ$	$f=(f+\mathrm{I}_{s_n}+\ldots+\mathrm{I}_{s_1})\uparrow^{\lvert a\rvert+n}_n\uparrow^{\lvert a\rvert+n-1}_{n-1}\ldots\uparrow^{\lvert a\rvert+1}_1$		for $f:a\longrightarrow s_n\ldots s_1$
SP	$(f+g)\uparrow^{\lvert a\rvert+n}_n\uparrow^{\lvert a\rvert+n-1}_{n-1}\ldots\uparrow^{\lvert a\rvert+1}_1=(g+f)\uparrow^n_{\lvert b\rvert+n}\uparrow^{n-1}_{\lvert b\rvert+n-1}\ldots\uparrow^1_{\lvert b\rvert+1}$		for $f:a\longrightarrow s_1\ldots s_n, g:s_1\ldots s_n\longrightarrow b$

It has to be noted that a biflow is a flow fulfilling (B7). Indeed, (B6) may be proved by using (B7) as follows: if $f':a'\longrightarrow b'$ and $g:b\longrightarrow c$, then

$$g+f' = (\text{ by } B7)\,{}^b\mathrm{X}^{a'}(f'+g)^{b'}\mathrm{X}^c = (\text{ by } B6a)\,{}^b\mathrm{X}^{a'}(f'+\mathrm{I}_b)(\mathrm{I}_{b'}+g)^{b'}\mathrm{X}^c$$
$$= (\text{ by } B7a)(\mathrm{I}_b+f')^b\mathrm{X}^{b'}\cdot{}^{b'}\mathrm{X}^b(g+\mathrm{I}_{b'}) = (\text{ by } B7a)(\mathrm{I}_b+f')(g+\mathrm{I}_{b'})$$

therefore if moreover $f:a\longrightarrow b$ and $g':b'\longrightarrow c'$, then

$$(f+f')(g+g') = (\text{ by } B6a)(f+\mathrm{I}_{a'})(\mathrm{I}_b+f')(g+\mathrm{I}_{b'})(\mathrm{I}_c+g')$$
$$= (f+\mathrm{I}_a)(g+f')(\mathrm{I}_c+g') = (\text{ by } B6a)(fg+f')(\mathrm{I}_c+g') = (\text{ by } B6b)fg+f'g'.$$

Proof of Theorem 1. The proof of this theorem is based on Theorem 2 and we shall use the notation in the above proof of Theorem 2. Suppose **B** is an M-biflow. We still have to show that $\Uparrow^{a(u)b}_{c(u)d}$ fulfils axiom (P). Indeed, if $f:a\longrightarrow b$ and $g:b\longrightarrow c$ then

$$(f+g)\Uparrow^{a(b)}_{(b)c}=[(\mathrm{I}_a+{}^0\underline{\mathrm{X}}^b)(f+g)(\mathrm{I}_0+{}^b\underline{\mathrm{X}}^c)]\uparrow^b=((f+g)^b\underline{\mathrm{X}}^c)\uparrow^b$$
$$= (\text{ by } B7)({}^a\underline{\mathrm{X}}^b(g+f))\uparrow^b=[(\mathrm{I}_0+{}^a\underline{\mathrm{X}}^b)(g+f)(\mathrm{I}_c+{}^b\underline{\mathrm{X}}^0)]\uparrow^b=(g+f)\Uparrow^{(b)a}_{c(b)}.$$

(ii) Suppose **C** is an M-BIFLOW. We still have to show that axiom (B7) holds. Indeed, if $f:a\longrightarrow b$ and $g:c\longrightarrow d$ then

$${}^c\mathrm{X}^a\circ(f+g)\circ{}^b\mathrm{X}^d = (\text{ by } T1 \text{ and } T1^\circ_\circ)(\mathrm{I}_c+f+g+\mathrm{I}_b)\uparrow^{ca(c)(b)}_{(c)(b)db}$$
$$= [(\mathrm{I}_c+f)+(g+\mathrm{I}_b)]\uparrow^{ca(cb)}_{(cb)db} = (\text{ by } P)[(g+\mathrm{I}_b)+(\mathrm{I}_c+f)]\uparrow^{(cb)ca}_{db(cb)}$$
$$= (\text{ by } F3^\circ)(g+\mathrm{I}_{bc}+f+\mathrm{I}_{ca})\uparrow^{(cb)(ca)ca}_{db(cb)(ca)} = (g+(\mathrm{I}_{bc}+f+\mathrm{I}_{ca})\uparrow^{(bc)aca}_{bc(bc)a})\uparrow^{(ca)ca}_{db(ca)}$$
$$= (\text{ by } F5_\circ)(g+f+\mathrm{I}_{ca})\uparrow^{(ca)ca}_{db(ca)} = (\text{ by } F3^\circ)g+f. \qquad \square$$

Since the monoid M is a free monoid S^*, feedbackation may be restricted to letters. More precisely a

Scalar Extended Feedbackation is a family of operations

$$\uparrow^i_j: B(s_1\ldots s_m,t_1\ldots t_n)\longrightarrow B(s_1\ldots s_{i-1}s_{i+1}\ldots s_m,t_1\ldots t_{j-1}t_{j+1}\ldots t_n),$$

where $s_1,\ldots,s_m,t_1,\ldots,t_n\in S, i\in[m], j\in[n]$ are such that $s_i=t_j$.

The resulting axiom system is presented in Table 4. In this table the letters s ant t range over S, while the others over S^*, and $\lvert a\rvert$ denotes the lenght of a word $a\in S^*$. Axioms (SS1) and (SS2 & $SS2^\circ$) show the sum is associative with neutral element $\mathrm{I}_s\uparrow^1_1$. Axioms (SF1 & $SF1^\circ$) show the feedback commutes with sum. Axioms (SF3 & $SF3^{\circ}_{0}$) show identities behave in a natural way, and finally axiom (SP) is a permutability axiom.

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(Received October 17, 1989)