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Product hierarchies of automata and homomorphic simulation

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Abstract

A ν_i-product is a network of automata such that each automaton is fed back to at most i of the component automata. We show that the ν_i-hierachy is proper with respect to homomorphic simulation.

For all notions and notations not defined here, see [2], [3] or [6]. An *automaton* $\mathbf{A} = (A, X, \delta)$ is a finite automaton with *state set* A, *input set* X and *transition* $\delta : A \times X \to A$. The transition is also used in the extended sense, i.e. as a function $\delta : A \times X^* \to A$ where X^* is the free monoid of all words over X.

Let $\mathbf{A} = \mathbf{A}_1 \times \ldots \times \mathbf{A}_n(X, \varphi)$ be a *general product* (or *g-product*) of automata $\mathbf{A}_j = (A_j, X_j, \delta_j)$, $j = 1, \ldots, n$, $n \geq 1$. A function

$$\gamma : \{1, \ldots, n\} \to 2^{\{1,\ldots,n\}}$$

is a *neighbourhood function* of $\mathbf{A}$ if each feedback function φ_j is independent of the actual state of any component $\mathbf{A}_k$ with $k \notin \gamma(j)$. Thus the concept of a general product with a neighbourhood function is essentially the same as the automata networks of [7]. A general product $\mathbf{A}$ with a neighbourhood function satisfying $\text{card}(\gamma(j)) \leq i$ for all $j = 1, \ldots, n$, where i is a fixed positive integer, is referred to a *ν_i-product*, cf. [4]. An *$\alpha_0 - \nu_i$-product* is a ν_i-product which is also an α_0-product (i.e. *loop-free product*).

Let $\mathbf{A} = (A, X, \delta)$ and $\mathbf{B} = (B, Y, \delta')$ be automata. We say that $\mathbf{A}$ *homomorphically simulates* $\mathbf{B}$ if there are $A' \subseteq A$ and mappings $h_1 : A' \to B$ and $h_2 : Y \to X^*$ such that h_1 is onto, moreover, $\delta(a, h_2(y)) \in A'$ and

$$h_1(\delta(a, h_2(y))) = \delta'(h_1(a), y)$$

for all $a \in A'$ and $y \in Y$. The function h_2 will be used also in the extended sense, i.e. as a monoid homomorphism $Y^* \to X^*$. Thus $\mathbf{A}$ homomorphically simulates $\mathbf{B}$ if and only if the transformation monoid corresponding to $\mathbf{B}$ is *covered* by the transformation monoid corresponding to $\mathbf{A}$, cf. [5]. If $X = Y$ and $\mathbf{B}$ is a homomorphic image of a subautomaton of $\mathbf{A}$ then $\mathbf{B}$ is *homomorhpically realized* by $\mathbf{A}$, cf. [6].

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Let K be a class of automata and let β refer to one of the above particular cases of the g-product. If an automaton $\mathbf{A}$ is homomorphically realized (simulated) by a β-product of automata from K then we write $\mathbf{A} \in HSP_\beta(K)$ ($\mathbf{A} \in HS^*P_\beta(K)$).

Now let $n \geq 1$ be an integer and let $\mathbf{C}_n = (C_n, \{x\}, \delta_n)$ with $C_n = \{0, \ldots, n-1\}$ and $\delta_n(i, x) = i + 1 \bmod n$, for all $i \in C_n$. Thus $\mathbf{C}_n$ is a *counter* with length n. Let $\mathbf{E} = (E, \{x, y\}, \delta_0)$ be an *elevator*, so that $E = \{0, 1\}$, $\delta_0(0, x) = 0$ and $\delta_0(0, y) = \delta_0(1, x) = \delta_0(1, y) = 1$. We set

$$K = \{\mathbf{E}\} \cup \{\mathbf{C}_p | p > 1 \text{ is a prime}\}$$

and prove that there exists an automaton $\mathbf{M} \in HSP_{\alpha_0 - \nu_{i+1}}(K)$ which does not belong to $HS^*P_{\nu_i}(K)$, where $i \geq 1$ is any fixed integer.

Let m be the product of the first $i + 1$ prime numbers. We define $\mathbf{M} = (M, \{x, y\}, \delta')$ with $M = \{0, \ldots, m\}$ and

$$\delta'(j, x) = \begin{cases} j + 1 \bmod m & \text{if } j = 0, \ldots, m-1 \\ m & \text{if } j = m \end{cases}$$

$$\delta'(j, y) = \begin{cases} j + 1 \bmod m & \text{if } j = 1, \ldots, m-1 \\ m & \text{if } j = 0 \text{ or } j = m. \end{cases}$$

Proof that $\mathbf{M} \notin HS^*P_{\nu_i}(K)$. Assume to the contrary that a ν_i-product with neighbourhood function γ

$$\mathbf{A} = (A, X, \delta) = \mathbf{A}_1 \times \ldots \times \mathbf{A}_n(X, \varphi)$$

of automata form K homomorphically simulates $\mathbf{M}$. We may suppose that n is minimal with this property, i.e., if $\mathbf{B}$ is a ν_i-product of automata from K which homomorphically simulates $\mathbf{M}$, then the number of factors of $\mathbf{B}$ is at least n. Let $A' \subseteq A$ and let $h_1 : A' \to M$, $h_2 : \{x, y\} \to X^*$ be mappings such that h_1 is onto and

$$\delta'(h_1(a), z) = h_1(\delta(a, h_2(z)))$$

for all $a \in A'$ and $z = x, y$, where it is assumed that $\delta(a, h_2(z)) \in A'$. We may choose A' and the functions h_1 and h_2 such that card(A') is minimal.

Let us partition A' as $A' = A_0 \cup A_1$ where $A_0 = h_1^{-1}(M - \{m\})$ and $A_1 = h_1^{-1}(m)$. If $a \in A_0$ and $b \in A'$ then, by the minimality of card(A'), there is a word $u \in \{x, y\}^*$ with $\delta(a, h_2(u)) = b$. Therefore, if $pr_j(a_0) = 1$ and $\mathbf{A}_j = \mathbf{E}$ for some $j = 1, \ldots, n$ and $a_0 \in A_0$, then $pr_j(a) = 1$ for all $a \in A'$. (Of course, pr_j denotes the j-th projection.) But then we can get rid of the j-th component obtaining a ν_i-product of $n-1$ factors that homomorphically simulates $\mathbf{M}$. Since this contradicts the minimality of n we have $pr_j(a) = 0$ for all $a \in A_0$ and $j \in \{1, \ldots, n\}$ with $\mathbf{A}_j = E$. By the construction of $\mathbf{A}$ and the minimality of card(A') it is easy to see that for every $a \in A_1$ there exists $j \in \{1, \ldots, n\}$ with $pr_j(a) = 1$ and $\mathbf{A}_j = \mathbf{E}$.

Now let $a \in h_1^{-1}(0)$ be a fixed state. We have $\delta(a, h_2(y)) \in A_1$, so that $pr_j(\delta(a, h_2(y))) = 1$ and $\mathbf{A}_j = \mathbf{E}$ for some $j \in \{1, \ldots, n\}$. Let $\gamma(j) = \{j_1, \ldots j_t\}$, $t \leq i$. For $s = 1, \ldots, t$, define $r_s = p$ if $\mathbf{A}_j = \mathbf{C}_p$ and $r_s = 1$ if $\mathbf{A}_j = \mathbf{E}$. Let r be the product of the integers r_s. It is clear that m is not a divisor of r. Thus, for $u = h_2(x)$, $\delta(a, u^r) = b \in h_1^{-1}(q)$ with $q \in \{1, \ldots, m-1\}$. Since $pr_{j_s}(b) = pr_{j_s}(a)$ for all $s = 1, \ldots, t$, it follows that $pr_j(\delta(b, h_2(y))) = 1$, which contradicts $\delta(b, h_2(y)) \in A_0$.

Proof that $\mathbf{M} \in HSP_{\alpha_0-\nu_{i+1}}(K)$. For each $j = 1, \ldots, i+1$, let p_j denote the j-th prime number. We construct an $\alpha_0 - \nu_{i+1}$-product

$$\mathbf{A} = \mathbf{C}_{p_1} \times \ldots \times \mathbf{C}_{p_{i+1}} \times \mathbf{E}(\{x, y\}, \varphi)$$

with

$$\varphi_j(k_1, \ldots k_{i+1}, k, z) = \begin{cases} y & \text{if } k_1 = \ldots = k_{i+1} = 0, j = i+2 \text{ and } z = y \\ x & \text{otherwise.} \end{cases}$$

It is straightforward to show that $\mathbf{A}$ maps homomorphically onto $\mathbf{M}$.

Theorem 1 *The ν_i-hierarchy is proper with respect to both homomorphic simulation and homomorphic realization. There exists a class K with the following properties, where $i \geq 1$ is any integer:*

(i) $HSP_{\nu_i}(K) \subset HSP_{\nu_{i+1}}(K)$,

(ii) $HS^*P_{\nu_i}(K) \subset HS^*P_{\nu_{i+1}}(K)$,

(iii) $HSP_{\alpha_0-\nu_i}(K) \subset HSP_{\alpha_0-\nu_{i+1}}(K)$,

(iv) $HS^*P_{\alpha_0-\nu_i}(K) \subset HS^*P_{\alpha_0-\nu_{i+1}}(K)$.

Remarks. For the class K exhibited in the proof we even have $HSP_{\alpha_0}(K) = HSP_g(K)$. Consequently

$$HSP_{\nu_i}(K) \subset HSP_{\alpha_0}(K) \text{ and } HS^*P_{\nu_i}(K) \subset HS^*P_{\alpha_0}(K)$$

hold, too. One might wish to modify the definition of homomorphic simulation by requiring that only nonempty words occur in the range of function h_2. Our result holds with the same proof for this notion or homomorphic simulation, too. Part i) has been already proved in [2] and part ii) in [1]. Nevertheless the class K given above is considerably simpler than that in [1] or [2].

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