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A note on the axiomatization of iteration theories

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Abstract

Iteration theories are a basic underlying structure in many investigations in theoretical computer science. The paper contains some remarks on the aximatization of iteration theories.

Iteration theories were defined in [BEW1] and [BEW2] as the variety generated by pointed iterative theories, which are iterative theories with the operation of iteration made totally defined in an essentially unique way, cf. [E]. Evidence gathered since that time indicates that iteration theories are the basic underlying stucture whenever one is interested in solving fixed point equations (see [BÉs2], [BÉs3], [BÉs4], [BÉs5], [BÉsT] and [St2] for some recent results). One axiomatization of iteration theories was given in [És1]. The purpose of the present note is to present a "scalar axiomtization", one which involves as much as possible morphisms $1 \rightarrow p$. An application of this axiomatization appears in [BÉs5]. We assume the reader is familiar with algebraic theories as defined e.g. in [E], [BÉs1], or [És1]. A preiteration theory is an algebraic theory with an operation of iteration subject to no particular condition. Recall that iteration maps a morphism $f : n \rightarrow n + p$ to $f^\dagger : n \rightarrow p$.

1. Theorem [És1] A preiteration theory is an iteration theory if and only if it satisfies the following identities.
1.1. *Left zero identity*

$$(0_n \oplus f)^\dagger = f, \quad f : n \rightarrow p$$

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1.2. *Right zero identity*

$$(f \oplus 0_q)^\dagger = f^\dagger \oplus 0_q, \qquad f : n \to n+p$$

1.3. *Dual pairing identity*

$$\langle f, g\rangle^\dagger = \langle h^\dagger, (g \cdot (\rho \oplus \mathbf{1}_p))^\dagger \cdot \langle h^\dagger, \mathbf{1}_p\rangle\rangle, \qquad f : n \to n+m+p, \quad g : m \to n+m+p$$

where $\rho = \langle 0_m \oplus \mathbf{1}_n, \mathbf{1}_n \oplus 0_m\rangle$ is the block transposition $n+m \to m+n$ and

$$h = f \cdot \langle \mathbf{1}_n \oplus 0_p, (g \cdot (\rho \oplus \mathbf{1}_p))^\dagger, 0_n \oplus \mathbf{1}_p\rangle : n \to n+p$$

1.4. *Commutative identity*

$$\langle \mathbf{1}_m \cdot \rho \cdot f \cdot (\rho_1 \oplus \mathbf{1}_p), \ldots, m_m \cdot \rho \cdot f \cdot (\rho_m \oplus \mathbf{1}_p)\rangle^\dagger = \rho \cdot (f \cdot (\rho \oplus \mathbf{1}_p))^\dagger$$

where $f : n \to m+p$, $\rho : m \to n$ is surjective base, and where each $\rho_i : m \to m$ is base with $\rho_i \cdot \rho = \rho$.

2. The above identities have a number of consequences. In particular, the following identities hold in any iteration theory.

2.1. *Fixed point identity*

$$f \cdot \langle f^\dagger, \mathbf{1}_p\rangle = f^\dagger, \quad f : n \to n+p$$

2.2. *Pairing identity*

$$\langle f, g\rangle^\dagger = \langle f^\dagger \cdot \langle k^\dagger, \mathbf{1}_p\rangle, k^\dagger\rangle, \quad f : n \to n+m+p, \qquad g : m \to n+m+p$$

where $k = g \cdot \langle f^\dagger, \mathbf{1}_{m+p}\rangle$

2.3. *Permutation identity*

$$(\rho \cdot f \cdot (\rho^{-1} \oplus \mathbf{1}_p))^\dagger = \rho \cdot f^\dagger, \quad f : n \to n+p$$

where $\rho : n \to n$ is a base permutation.

3. Remark The permutation identity is a special case of the commutative identity. The fact that the fixed point identity is implied by the conditions 1.1-1.4 was only recognized in [És3]. A stronger statement, following a suggestion of a referee, is proved in Lemma 5 below. For the pairing identity we mention the following result.

4. Lemma Let T be a preiteration theory which satisfies the permutation identity. Then the pairing identity holds in T if and only if the dual pairing identity holds.

Proof. We only prove that the dual pairing identity is implied by the pairing identity and the permutation identity. Let $f : n \to n+m+p$, $g : m \to n+m+p$ and, as before, denote by ρ the block transposition $n+m \to m+n$. By the permutation identity and the pairing identity we have

$$\begin{aligned}
\langle f, g\rangle^\dagger &= (\rho \cdot \rho^{-1} \cdot \langle f, g\rangle \cdot (\rho \oplus \mathbf{1}_p) \cdot (\rho^{-1} \oplus \mathbf{1}_p))^\dagger \\
&= \rho \cdot (\rho^{-1} \cdot \langle f, g\rangle \cdot (\rho \oplus \mathbf{1}_p))^\dagger \\
&= \rho \cdot \langle g \cdot (\rho \oplus \mathbf{1}_p), f \cdot (\rho \oplus \mathbf{1}_p)\rangle^\dagger \\
&= \rho \cdot \langle (g \cdot (\rho \oplus \mathbf{1}_p))^\dagger \cdot \langle k^\dagger, \mathbf{1}_p\rangle, k^\dagger\rangle \\
&= \langle k^\dagger, (g \cdot (\rho \oplus \mathbf{1}_p))^\dagger \cdot \langle k^\dagger, \mathbf{1}_p\rangle\rangle,
\end{aligned}$$

where

$$\begin{aligned} k &= f \cdot (\rho \oplus \mathbf{1}_p) \cdot \langle (g \cdot (\rho \oplus \mathbf{1}_p))^\dagger, \mathbf{1}_{n+p} \rangle \\ &= f \cdot \langle \mathbf{1}_n \oplus 0_p, (g \cdot (\rho \oplus \mathbf{1}_p))^\dagger, 0_n \oplus \mathbf{1}_p \rangle. \end{aligned}$$

The proof is complete.

Besides that presented in Theorem 1, several equivalent axiomatizations of iteration theories are known. The following results are taken from [St1],[CSt] and [És2].

5. Lemma [CSt] Let T be a preiteration theory which satisfies the left zero identity 1.1, the dual pairing identity 1.3 and the permutation identity 2.3. Then the fixed point identity 2.1 holds in T.

Proof. Let $f : n \to n + p$ and define $g = \langle 0_n \oplus f, \mathbf{1}_n \oplus 0_{n+p} \rangle$. Using 1.1 and 1.3 we obtain

$$\begin{aligned} (\mathbf{1}_n \oplus 0_n) \cdot g^\dagger &= ((0_n \oplus f) \cdot \langle \mathbf{1}_n \oplus 0_p, (0_n \oplus \mathbf{1}_n \oplus 0_p)^\dagger, 0_n \oplus \mathbf{1}_p \rangle)^\dagger \\ &= ((0_n \oplus f) \cdot \langle \mathbf{1}_n \oplus 0_p, \mathbf{1}_n \oplus 0_p, 0_n \oplus \mathbf{1}_p \rangle)^\dagger \\ &= f^\dagger. \end{aligned}$$

Similarly, by 1.1 and 2.2,

$$\begin{aligned} (\mathbf{1}_n \oplus 0_n) \cdot g^\dagger &= (0_n \oplus f)^\dagger \cdot \langle ((\mathbf{1}_n \oplus 0_{n+p}) \cdot \langle (0_n \oplus f)^\dagger, \mathbf{1}_{n+p} \rangle)^\dagger, \mathbf{1}_p \rangle \\ &= f \cdot \langle ((\mathbf{1}_n \oplus 0_{n+p}) \cdot \langle f, \mathbf{1}_{n+p} \rangle)^\dagger, \mathbf{1}_p \rangle \\ &= f \cdot \langle f^\dagger, \mathbf{1}_p \rangle. \end{aligned}$$

The proof is completed using Lemma 4.

6. Theorem [St1], [CSt] A preiteration theory is an iteration theory if and only if the following hold.

6.1. *Parameter identity*

$$(f \cdot (\mathbf{1}_n \oplus g))^\dagger = f^\dagger \cdot g, \qquad f : n \to n + p,\ g : p \to q$$

6.2. *Composition identity*

$$f \cdot \langle (g \cdot \langle f, 0_m \oplus \mathbf{1}_p \rangle)^\dagger, \mathbf{1}_p \rangle = (f \cdot \langle g, 0_n \oplus \mathbf{1}_p \rangle)^\dagger, \qquad f : n \to m + p,\ g : m \to n + p$$

6.3 *Double dagger identity*

$$(f \cdot (\langle \mathbf{1}_n, \mathbf{1}_n \rangle \oplus \mathbf{1}_p))^\dagger = f^{\dagger\dagger}, \qquad f : n \to n + n + p$$

6.4. The commutative identity 1.4.

7. Theorem [És2] A preiteration theory is an iteration theory if and only if the following identities hold.

7.1. The *special left zero identity*, i.e. 1.1 with $n = 1$.
7.2. The *special parameter identity*, i.e. 6.1 with $n = 1$.
7.3. The *special dual pairing identity* 1.3 with $m = 1$.
7.4. *Special permutation identity*

$$f^\dagger \cdot \langle (g \cdot \langle f^\dagger, \mathbf{1}_{1+p}\rangle)^\dagger, \mathbf{1}_p\rangle = (f \cdot \langle \mathbf{1}_1 \oplus 0_p, (g \cdot (\rho \oplus \mathbf{1}_p))^\dagger, 0_1 \oplus \mathbf{1}_p\rangle)^\dagger$$

where $f, g : 1 \to 1 + 1 + p$ and $\rho : 2 \to 2 + p$ is the nontrivial base permutation.
7.5. *Special commutative identity*

$$1_m \cdot \langle 1_m \cdot \rho \cdot f \cdot (\rho_1 \oplus \mathbf{1}_p), \ldots, m_m \cdot \rho \cdot f \cdot (\rho_m \oplus \mathbf{1}_p)\rangle^\dagger = 1_n \cdot (f \cdot (\rho \oplus \mathbf{1}_p))^\dagger$$

where $f : n \to m + p$, $\rho : m \to n$ is a monotone surjective base morphism, and where each $\rho_i : m \to m$ is base with $\rho_i \cdot \rho = \rho$.

8. Remark In addition to 7.1-7.5, the *special fixed point identity* 2.1 with $n = 1$ was also required in [És2]. This is however already implied, for taking $f = 0_1 \oplus h$ and $g = 1_{2+p} = \mathbf{1}_1 \oplus 0_{1+p}$ in the special permutation identity we obtain

$$\begin{aligned} f^\dagger \cdot \langle (g \cdot \langle f^\dagger, \mathbf{1}_{1+p}\rangle)^\dagger, \mathbf{1}_p\rangle &= (0_1 \oplus h)^\dagger \cdot \langle (1_{2+p} \cdot \langle (0_1 \oplus h)^\dagger, \mathbf{1}_{1+p}\rangle)^\dagger, \mathbf{1}_p\rangle \\ &= h \cdot \langle h^\dagger, \mathbf{1}_p\rangle \end{aligned}$$

and

$$\begin{aligned} &(f \cdot \langle \mathbf{1}_1 \oplus 0_p, (g \cdot (\rho \oplus \mathbf{1}_p))^\dagger, 0_1 \oplus \mathbf{1}_p\rangle)^\dagger = \\ &= ((0_1 \oplus h) \cdot \langle \mathbf{1}_1 \oplus 0_p, (0_1 \oplus \mathbf{1}_1 \oplus 0_p)^\dagger, 0_1 \oplus \mathbf{1}_p\rangle)^\dagger \\ &= ((0_1 \oplus h) \cdot \langle \mathbf{1}_1 \oplus 0_p, \mathbf{1}_1 \oplus 0_p, 0_1 \oplus \mathbf{1}_p\rangle)^\dagger \\ &= h^\dagger \end{aligned}$$

by 7.1. Thus $h \cdot \langle h^\dagger, \mathbf{1}_p\rangle = h^\dagger$.

We note that it is enough to require the special parameter identity $(f \cdot (\mathbf{1}_1 \oplus g))^\dagger = f^\dagger \cdot g$ in Theorem 7 only for $f : 1 \to 1 + p$ and base $g : p \to q$, cf. [És2]. A result closely related to Theorem 7 was found independently in [CU], see also [Ca]. As a part of the proof of Theorem 7, the following result was established in [És2]. A scalar preiteration theory is an algebraic theory with iteration defined on scalar morphisms $f : 1 \to 1 + p$.

9. Theorem [És2] Let T be a scalar preiteration theory satisfying the special parameter identity and the special permutation identity. Extend the definition of iteration by the special dual pairing identity, i.e. let $0_{1+p}^\dagger = 0_p$ and for $f : n \to n + 1 + p$ and $g : 1 \to n + 1 + p$,

$$\langle f, g\rangle^\dagger = \langle h^\dagger, (g \cdot (\rho \oplus \mathbf{1}_p))^\dagger \cdot \langle h^\dagger, \mathbf{1}_p\rangle\rangle$$

with ρ and h as in 1.3. Then T becomes a preiteration theory in which the identities 1.1-1.3, 2.1 and 2.3 hold.

10. Corollary The preiteration theory of Theorem 9 also satisfies the parameter identity 6.1, the composition identity 6.2 and the double dagger identity 6.3.

Proof. It is proved in [És1] that any preiteration theory T in which 1.1-1.3 and 2.3 hold satisfies the parameter identity. By the main result of [BÉs1], T satisfies any identity valid for flowchart schemes. Therefore the double dagger identity holds in T. A direct proof, starting with $a = \langle f, \mathbf{1}_n \oplus 0_{n+p}\rangle$, $f : n \to n + n + p$, may be obtained by a calculation similar to that given below (cf. [St1]). Now for the composition identity. Given $f : n \to m + p$ and $g : m \to n + p$, define

$$a = \langle 0_n \oplus f, g \cdot (\mathbf{1}_n \oplus 0_m \oplus \mathbf{1}_p)\rangle : n + m \to n + m + p.$$

The pairing identity 2.2, which holds in T by Lemma 4, and the left zero identity 1.1 imply

$$\begin{aligned}
(\mathbf{1}_n \oplus 0_m) \cdot a^\dagger &= (0_n \oplus f)^\dagger \cdot \langle (g \cdot (\mathbf{1}_n \oplus 0_m \oplus \mathbf{1}_p) \cdot \langle (0_n \oplus f)^\dagger, \mathbf{1}_{m+p}\rangle)^\dagger, \mathbf{1}_p\rangle \\
&= f \cdot \langle (g \cdot (\mathbf{1}_n \oplus 0_m \oplus \mathbf{1}_p) \cdot \langle f, \mathbf{1}_{m+p}\rangle)^\dagger, \mathbf{1}_p\rangle \\
&= f \cdot \langle (g \cdot \langle f, 0_m \oplus \mathbf{1}_p\rangle)^\dagger, \mathbf{1}_p\rangle.
\end{aligned}$$

By the dual pairing identity 1.3 and the left zero identity again,

$$\begin{aligned}
&(\mathbf{1}_n \oplus 0_m) \cdot a^\dagger = \\
&= ((0_n \oplus f) \cdot \langle \mathbf{1}_n \oplus 0_p, (g \cdot (\mathbf{1}_n \oplus 0_m \oplus \mathbf{1}_p) \cdot (\rho \oplus \mathbf{1}_p))^\dagger, 0_n \oplus \mathbf{1}_p\rangle)^\dagger \\
&= ((0_n \oplus f) \cdot \langle \mathbf{1}_n \oplus 0_p, (0_m \oplus g)^\dagger, 0_n \oplus \mathbf{1}_p\rangle)^\dagger \\
&= (f \cdot \langle g, 0_n \oplus \mathbf{1}_p\rangle)^\dagger,
\end{aligned}$$

where ρ denotes the block transposition $n + m \to m + n$.

Except for 7.5, the axiomatization given in Theorem 7 is based on scalar iteration, for the special dual pairing identity can be thought of as a definition of vector iteration (together with $0_p^\dagger = 0_p$ which is already forced by the theory identities). Nevertheless both sides of the special permutation identity can be expressed in terms of scalar iteration. Below we present another axiomatization of this sort.

11. Theorem Let T be a scalar preiteration theory such that the *special parameter identity*, the *special composition identity* 6.2 with $n = m = 1$, and the *special double dagger identity* 6.3 with $n = 1$ hold. If iteration is extended by the special dual pairing identity then T becomes a preiteration theory satisfying the identities 1.1 - 1.3, 2.1 and 2.3. Moreover, T satisfies the parameter identity, the composition identity and the double dagger identity.

Proof. We show that the special fixed point identity and the special permutation identity hold in T. For the special fixed point identity just take $g = \mathbf{1}_1 \oplus 0_p = 1_{1+p}$ in the special composition identity

$$f \cdot \langle (g \cdot \langle f, 0_1 \oplus \mathbf{1}_p\rangle)^\dagger, \mathbf{1}_p\rangle = (f \cdot \langle g, 0_1 \oplus \mathbf{1}_p\rangle)^\dagger.$$

For the special permutation identity first we prove that

$$(g\cdot\langle f^\dagger,\mathbf{1}_{1+p}\rangle)^\dagger=((g\cdot(\rho\oplus\mathbf{1}_p))^\dagger\cdot\langle f^\dagger,0_1\oplus\mathbf{1}_p\rangle)^\dagger, \tag{11.1}$$

where $\rho:2\to 2$ is the nontrivial base permutation. We use the special parameter identity and the special double dagger identity.

$$\begin{aligned}
&((g\cdot(\rho\oplus\mathbf{1}_p))^\dagger\cdot\langle f^\dagger,0_1\oplus\mathbf{1}_p\rangle)^\dagger=\\
&= (g\cdot(\rho\oplus\mathbf{1}_p)\cdot(\mathbf{1}_1\oplus\langle f^\dagger,0_1\oplus\mathbf{1}_p\rangle))^{\dagger\dagger}\\
&= (g\cdot(\rho\oplus\mathbf{1}_p)\cdot\langle\mathbf{1}_1\oplus 0_{1+p},0_1\oplus f^\dagger,0_2\oplus\mathbf{1}_p\rangle)^{\dagger\dagger}\\
&= (g\cdot\langle 0_1\oplus f^\dagger,\mathbf{1}_1\oplus 0_{1+p},0_2\oplus\mathbf{1}_p\rangle)^{\dagger\dagger}\\
&= (g\cdot\langle 0_1\oplus f^\dagger,\mathbf{1}_1\oplus 0_{1+p},0_2\oplus\mathbf{1}_p\rangle\cdot(\langle\mathbf{1}_1,\mathbf{1}_1\rangle\oplus\mathbf{1}_p))^\dagger\\
&= (g\cdot\langle 0_1\oplus f^\dagger,\mathbf{1}_1\oplus 0_{1+p},0_2\oplus\mathbf{1}_p\rangle\cdot\langle\mathbf{1}_1\oplus 0_p,\mathbf{1}_1\oplus 0_p,0_1\oplus\mathbf{1}_p\rangle)^\dagger\\
&= (g\cdot\langle f^\dagger,\mathbf{1}_{1+p}\rangle)^\dagger
\end{aligned}$$

Next, by the special composition identity,

$$f^\dagger\cdot\langle((g\cdot(\rho\oplus\mathbf{1}_p))^\dagger\cdot\langle f^\dagger,0_1\oplus\mathbf{1}_p\rangle)^\dagger,\mathbf{1}_p\rangle=(f^\dagger\cdot\langle(g\cdot(\rho\oplus\mathbf{1}_p))^\dagger,0_1\oplus\mathbf{1}_p\rangle)^\dagger. \tag{11.2}$$

Finally, we observe that

$$(f\cdot\langle\mathbf{1}_1\oplus 0_p,(g\cdot(\rho\oplus\mathbf{1}_p))^\dagger,0_1\oplus\mathbf{1}_p\rangle)^\dagger=(f^\dagger\cdot\langle(g\cdot(\rho\oplus\mathbf{1}_p))^\dagger,0_1\oplus\mathbf{1}_p\rangle)^\dagger. \tag{11.3}$$

Indeed, by the special parameter identity and the special double dagger identity,

$$\begin{aligned}
&(f^\dagger\cdot\langle(g\cdot(\rho\oplus\mathbf{1}_p))^\dagger,0_1\oplus\mathbf{1}_p\rangle)^\dagger=\\
&= (f\cdot(\mathbf{1}_1\oplus\langle(g\cdot(\rho\oplus\mathbf{1}_p))^\dagger,0_1\oplus\mathbf{1}_p\rangle))^{\dagger\dagger}\\
&= (f\cdot\langle\mathbf{1}_1\oplus 0_{1+p},0_1\oplus(g\cdot(\rho\oplus\mathbf{1}_p))^\dagger,0_2\oplus\mathbf{1}_p\rangle)^{\dagger\dagger}\\
&= (f\cdot\langle\mathbf{1}_1\oplus 0_{1+p},0_1\oplus(g\cdot(\rho\oplus\mathbf{1}_p))^\dagger,0_2\oplus\mathbf{1}_p\rangle\cdot(\langle\mathbf{1}_1,\mathbf{1}_1\rangle\oplus\mathbf{1}_p))^\dagger\\
&= (f\cdot\langle\mathbf{1}_1\oplus 0_p,(g\cdot(\rho\oplus\mathbf{1}_p))^\dagger,0_1\oplus\mathbf{1}_p\rangle)^\dagger.
\end{aligned}$$

The proof is now easily completed. By 11.1-11.3,

$$\begin{aligned}
(f\cdot\langle\mathbf{1}_1\oplus 0_p,(g\cdot(\rho\oplus\mathbf{1}_p))^\dagger,0_1\oplus\mathbf{1}_p\rangle)^\dagger&=(f^\dagger\cdot\langle(g\cdot(\rho\oplus\mathbf{1}_p))^\dagger,0_1\oplus\mathbf{1}_p\rangle)^\dagger\\
&=f^\dagger\cdot\langle((g\cdot(\rho\oplus\mathbf{1}_p))^\dagger\cdot\langle f^\dagger,0_1\oplus\mathbf{1}_p\rangle)^\dagger,\mathbf{1}_p\rangle\\
&=f^\dagger\cdot\langle(g\cdot\langle f^\dagger,\mathbf{1}_{1+p}\rangle)^\dagger,\mathbf{1}_p\rangle.
\end{aligned}$$

Theorem 11 implies Theorem 2, Chapter 13 in [C] for matrix theories, see also [BÉs5]. A consequence of Theorem 11 and the previous results is given below.

12. Theorem A preiteration theory is an iteration theory if and only if the special parameter identity, the special composition identity, the special double dagger identity, the special dual pairing identity and the special commutative identity hold.

It is interesting to compare Theorem 12 with the following result, essentially taken from [És2].

13. Theorem [És2] A preiteration theory is an iteration theory if and only if the special left zero identity, the special parameter identity, the special pairing identity and the following variant of the commutative identity hold:

$$\mathbf{1}_m \cdot \langle \mathbf{1}_m \cdot \rho \cdot f \cdot (\rho_1 \oplus \mathbf{1}_p), \ldots, m_m \cdot \rho \cdot f \cdot (\rho_m \oplus \mathbf{1}_p) \rangle^\dagger = \mathbf{1}_m \cdot \rho \cdot (f \cdot (\rho \oplus \mathbf{1}_p))^\dagger \qquad 13.1$$

where f, ρ and $\rho_i, i = 1, \ldots, m$, are as in the commutative identity 1.4.

14. Remark By Lemma 4, the dual pairing identity 1.3 can be replaced by the pairing identity 2.2 in Theorem 1. Similarly, we may use the special pairig identity 2.2 with $m = 1$ (or $n = 1$) in Theorems 7, 9, 12 and 13. In Theorems 7, 9, 11 and 13 one can also use the special symmetric pairing identity

$$\langle f, g \rangle^\dagger = \langle h^\dagger, k^\dagger \rangle,$$

where $f : n \to n + 1 + p,\ g : 1 \to n + 1 + p$, and where h and k are defined as in 1.3 and 2.2. In Theorem 12, instead of the special commutative identity, we may require 13.1 for monotone surjective $\rho : m \to n$.

Let T be a preiteration theory such that the the parameter identity, the permutation identity and the dual pairing identity (or pairing identity) hold. Suppose that

$$f \cdot (\alpha \oplus \mathbf{1}_p) = \alpha \cdot g$$

for $f : n \to n + p,\ g : m \to m + p$ and an injective base morphism $\alpha : n \to m$. It is a routine calculation to prove that

$$f^\dagger = \alpha \cdot g^\dagger.$$

A preiteration theory has a *weak functorial dagger* if

$$f \cdot (\rho \oplus \mathbf{1}_p) = \rho \cdot g \Rightarrow f^\dagger = \rho \cdot g^\dagger \qquad 14.1$$

for all $f : n \to n + p,\ g : m \to m + p$ and surjective base morphism $\rho : n \to m$. It is known that the commutative identity holds in any preiteration theory with weak functorial dagger, cf. [És1]. Most known iteration theories have weak functorial dagger. The existence of an iteration theory not satisfying 14.1 was pointed out in [És4].

15. Proposition Let T be a preiteration theory such that the parameter identity, the permutation identity and the dual pairing (or pairing) identity hold. Then T has weak functorial dagger if and only if 14.1 holds with $m = 1$.

Proof. Since the permutation identity holds in T, it suffices to prove 14.1 for monotone surjective base morphisms $\rho : n \to m$. Our argument uses induction on m. The basis case $m = 1$ holds by assumption. Supposing the statement holds for $m \geq 1$, let $f : n \to n + p,\ g : m + 1 \to m + 1 + p$ and $\rho : n \to m + 1$ be such that

$$f \cdot (\rho \oplus \mathbf{1}_p) = \rho \cdot g, \qquad 15.1$$

where $\rho : n \to m+1$ is a monotone surjective base morphism. We can write

$$\begin{array}{lll} f = & \langle f_1, f_2 \rangle, & f_1 : n_1 \to n+p,\ f_2 : n_2 \to n+p \\ g = & \langle g_1, g_2 \rangle, & g_1 : m \to m+1+p,\ g_2 : 1 \to m+1+p \\ \rho = & \rho_1 \oplus \rho_2, & \rho_1 : n_1 \to m,\ \rho_2 : n_2 \to 1 \end{array}$$

where ρ_1 and ρ_2 are monotone surjective base morphisms with

$$f_i \cdot (\rho_1 \oplus \rho_2 \oplus \mathbf{1}_p) = \rho_i \cdot g_i, \quad i = 1, 2. \qquad 15.2$$

The induction hypothesis and the parameter identity yield

$$f_1^\dagger \cdot (\rho_2 \oplus \mathbf{1}_p) = \rho_1 \cdot g_1^\dagger. \qquad 15.3$$

Now let $h = f_2 \cdot \langle f_1^\dagger, \mathbf{1}_{n_2+p} \rangle : n_2 \to n_2 + p$ and $k = g_2 \cdot \langle g_1^\dagger, \mathbf{1}_{1+p} \rangle : 1 \to 1 + p$. We have

$$h \cdot (\rho_2 \oplus \mathbf{1}_p) = \rho_2 \cdot k. \qquad 15.4$$

Indeed,

$$\begin{array}{rcl} h \cdot (\rho_2 \oplus \mathbf{1}_p) & = & f_2 \cdot \langle f_1^\dagger, \mathbf{1}_{n_2+p} \rangle \cdot (\rho_2 \oplus \mathbf{1}_p) \\ & = & f_2 \cdot \langle f_1^\dagger \cdot (\rho_2 \oplus \mathbf{1}_p), \rho_2 \oplus \mathbf{1}_p \rangle \\ & = & f_2 \cdot \langle \rho_1 \cdot g_1^\dagger, \rho_2 \oplus \mathbf{1}_p \rangle, \end{array}$$

by 15.3,

$$\begin{array}{rl} = & f_2 \cdot (\rho_1 \oplus \rho_2 \oplus \mathbf{1}_p) \cdot \langle g_1^\dagger, \mathbf{1}_{1+p} \rangle \\ = & \rho_2 \cdot g_2 \cdot \langle g_1^\dagger, \mathbf{1}_{1+p} \rangle \\ = & \rho_2 \cdot k, \end{array}$$

by 15.4. From 15.4, by the induction hypothesis, we obtain

$$h^\dagger = \rho_2 \cdot k^\dagger. \qquad 15.5$$

The proof is completed by using the pairing identity.

$$\begin{array}{rcl} f^\dagger & = & \langle f_1^\dagger \cdot \langle h^\dagger, \mathbf{1}_p \rangle, h^\dagger \rangle \\ & = & \langle f_1^\dagger \cdot \langle \rho_2 \cdot k^\dagger, \mathbf{1}_p \rangle, \rho_2 \cdot k^\dagger \rangle, \end{array}$$

by 15.5,

$$\begin{array}{rl} = & \langle f_1^\dagger \cdot (\rho_2 \oplus \mathbf{1}_p) \cdot \langle k^\dagger, \mathbf{1}_p \rangle, \rho_2 \cdot k^\dagger \rangle \\ = & \langle \rho_1 \cdot g_1^\dagger \cdot \langle k^\dagger, \mathbf{1}_p \rangle, \rho_2 \cdot k^\dagger \rangle, \end{array}$$

by 15.3,

$$\begin{aligned} &= (\rho_1 \oplus \rho_2) \cdot \langle g_1^\dagger \cdot \langle k^\dagger, \mathbf{1}_p \rangle, k^\dagger \rangle \\ &= \rho \cdot g^\dagger. \end{aligned}$$

An equivalent statement is proved independently in [B]. Proposition 15 also appears in [BÉs4].

16. Corollary An iteration theory has weak functional dagger if and only if 16.1 $f \cdot (\rho \oplus \mathbf{1}_p) = \rho \cdot g \Rightarrow 1_n \cdot f^\dagger = g^\dagger$ for all $f : n \to n + p, g : 1 \to 1 + p$ with $n \geq 1$, where ρ is the unique base morphisms $n \to 1$.

By an example given in [És4], it is not possible to impose an upper bound on the integer n appearing in 16.1. Nevertheless it is enough to require 16.1 for any infinite set of integers n. Combining Corollary 16 with Theorems 1, 5, 6, or 11, one obtains axiomatizations of the quasivariety of iteration theories with weak functorial dagger studied under the name of strong iteration theories in [St1]. Thus we have e.g. the following statement.

17. Corollary A preiteration theory is an iteration theory with weak functorial dagger if and only if it satisfies 16.1 and the special parameter identity, the special composition identity, the special double dagger identity and the special dual pairing identity.

Finally we mention some simplifications of the commutative identity. It is implicit in Lemmas 1.1 and 3.2 in [És1] that the commutative identity reduces to the special case that each ρ_i is a bijective base morphism or that each one is an aperiodic base morphism. Similarly, it suffices to require the special commutative identity in Theorems 7 and 12 above in one of these two cases. However it is not known if it is enough to require the commutative identity for $n = 1$.

Open problem Find an essential simplification of the commutative identity.

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