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A 1.6 lower-bound for the two-dimensional on-line rectangle bin-packing

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Abstract

Examining on-line algorithms for the two dimensional rectangle bin packing problem, Coppersmith asked in [2] whether one can give a better lower bound for this type of algorithms than the Liang's bound which is 1.5364... . In this paper we present a bound of 1.6.

Keywords: two-dimensional bin-packing, worst-case analysis, on-line algorithms, heuristic.

1 Introduction

Let us first consider the one-dimensional bin-packing problem: There is given a list $L(n) = \{a_1, \ldots a_n\}$, and let us suppose that a size l_i belongs to each $a_i \in L(n), (0 < l_i \leq 1)$. The problem is to pack the elements of $L(n)$ into unit-capacity bins, while attempting to minimize the number of bins needed for packing. The problem is NP-hard (cf.[3]) and therefore various heuristic algorithms have been studied for solving this problem. Let us consider the following class of approximation algorithms which produce a near-optimal solution of the problem: An algorithm belonging to this class packs the elements one by one in the order given by list $L(n)$, and after having placed the element into the bin, it will be never moved again. The algorithms belonging to this class are called on-line algorithms.

One possibility to measure the performance of an algorithm A is to give its asymptotic worst case performance ratio R_A : Let L be a list and denote by L^* the minimal number of bins needed to pack the list L. Moreover, let $A(L)$ represent the number of bins that are used by the algorithm A to pack the elements of L. If $R_A(k)$ denotes the supremum of the ratios $A(L)/L^*$ for all lists L with $L^* = k$, then

$$R_A = \limsup_{k\to\infty} R_A(k).$$

The best known lower bound for R_A for the class of on-line algorithms has heen given by Liang (cf.[4]). He proved that there is no on-line algorithm A for which $R_A < 1.5364...$ .To verify this result Liang considered a $k \in \mathbf{N}_+$ and defined the sequence $m_0 = 1, m_{j+1} = m_j(m_j + 1), (1 \leq j \leq k)$. Finally, he considered the lists $L_k, \ldots, L_0$ where L_j $(0 \leq j \leq k)$, represents a block of n elements of size $l_j = \frac{1}{m_j+1} + \epsilon$ with $\epsilon > 0$ suitable small chosen. He proved for the concatenated

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lists $L_k, L_kL_{k-1}, \ldots, L_k \ldots L_0$ that one of the ratios $A(L_k \ldots L_j)/(L_k \ldots L_j)^*, 0 \leq j \leq k$, is at least 1.5364... for every n.

Now let us consider the following two-dimensional generalization of the one-dimensional problem (cf.[1]): We are given a list of $L(n) = \{a_1, \ldots a_n\}$ with an ordered pair of sizes $(w(a_i), h(a_i))$ where $w(a_i)$, resp. $h(a_i)$ is the width, resp. height of a_i and we are given rectangular bins with sizes W and H. (Without loss of generality we can suppose that $W = H = 1$ and $w(a_i) \leq 1, h(a_i) \leq 1$.) We have to pack the rectangles into the minimal number of bins such that

- the sides of the elements are parallel to the corresponding sides of the bins (no rotation allowed);
- no two rectangles in a bin overlap.

The definition of an on-line algorithm for the two-dimensional case is the same as for the one-dimensional case. It is very easy to see for the lists L_j satisfying $h(a_j) = \frac{1}{m_j+1} + \epsilon$ and $w(a_j) = 1$ that we get a trivial lower bound for the asymptotic worst case ratio of an arbitrary two-dimensional on-line algorithm. This means that the following theorem is true:

Theorem 1 *There is no on-line two-dimensional bin-packing algorithm A for which $R_A < 1.5364...$.*

In [2] Coppersmith mentioned that no better lower bound is known. In this paper we restate the trivial argument and obtain a slightly improved, but non-trivial, lower bound of 1.6.

2 Computation of lower bound

In order to prove the lower bound, we introduce the following lists. Let k be an arbitrary integer, we choose $n = 4k$, and consider the lists L_1, L_2, L_3, L_4 with

- L_1 contains n pieces of A-elements with sizes $(\frac{1}{2} - \epsilon, \frac{1}{2} - 2\epsilon)$;
- L_2 contains n pieces of B-elements with sizes $(\frac{1}{2} + \epsilon, \frac{1}{2} - \epsilon)$;
- L_3 contains n pieces of C-elements with sizes $(\frac{1}{2} - 2\epsilon, \frac{1}{2} + 2\epsilon)$;
- L_4 contains n pieces of D-elements with sizes $(\frac{1}{2} + 2\epsilon, \frac{1}{2} + \epsilon)$;

Lemma 1

$$(L_1 \ldots L_j)^* \leq j\frac{n}{4}. \qquad j = 1, 2, 3, 4.$$

Proof. We leave it to the reader to verify the cases $j = 1, 2, 4$, and we prove only the case $j = 3$:
We give a feasible packing which consists of the following bins:

- $\frac{n}{4}$ times bins with two A-elements and two C-elements;
- $\frac{n}{2}$ times bins with $1A$-elements, $2B$-elements an $1C$-elements. (see Figure 1.) □

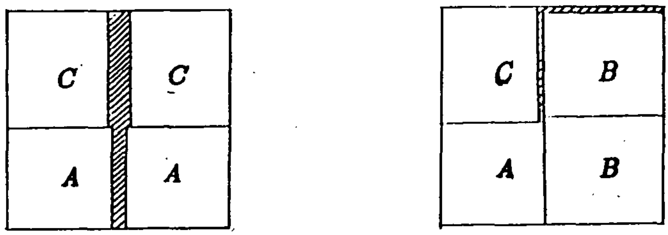

Figure 1: A possible packing of (L_1, L_2, L_3).

Let us now pack the elements of the concatenated list $(L_1 L_2 L_3 L_4)$. We say that a bin has type $\mathbf{t} = (t_1, t_2, t_3, t_4)$ if it contains t_1 pieces of A-elements, t_2 pieces of B-elements etc., and we denote the set of bins after having packed the concatenated list by T. Moreover, if a bin is represented by its content (t_1, t_2, t_3, t_4) we define the following subsets:

$$T_k = \{(t_1, t_2, t_3, t_4) \in T | i_j = 0, \text{ if } j < k \text{ and } i_k > 0\}. \quad 1 \leq k \leq 4.$$

It is clear that $T = \bigcup_{i=1}^{4} T_i$ and $T_i \cap T_j = \emptyset$ if $i \neq j$.

We denote by $a(\mathbf{t})$ the number of bins which contain t_j elements from the list L_j, $(1 \leq j \leq 4)$.

Now we are ready to state our lower bound theorem:

Theorem 2 *There is no on-line two-dimensional bin-packing algorithm A for which $R_A < 1.6$.*

Proof.: We examine how many bins have been used after having packed the list L_j, $(1 \leq j \leq 4)$:

$$A(L_1 \ldots L_j) = \sum_{i=1}^{j} \sum_{\mathbf{t} \in T_i} a(\mathbf{t}) \tag{1}$$

and the number of the packed elements for each j, $1 \leq j \leq 4$:

$$n = \sum_{\mathbf{t} \in T} t_j a(\mathbf{t}), \qquad 1 \leq j \leq 4. \tag{2}$$

Adding all the equations (1) and substracting (2) it follows:

$$A(L_1) + A(L_1 L_2) + A(L_1 L_2 L_3) + A(L_1 L_2 L_3 L_4) - 4n =$$

$$4 \sum_{\mathbf{t} \in T_1} a(\mathbf{t}) + 3 \sum_{\mathbf{t} \in T_2} a(\mathbf{t}) + 2 \sum_{\mathbf{t} \in T_3} a(\mathbf{t}) + \sum_{\mathbf{t} \in T_4} a(\mathbf{t}) - \sum_{\mathbf{t} \in T} a(\mathbf{t}) \sum_{j=1}^{4} t_j. \tag{3}$$

Lemma 2 *The right hand side of (3) is non negative.*

Proof. First we consider a bin which is in T_1. We have to prove that for each such type of bin $\sum_{i=1}^{4} t_i \leq 4$. In other words, we have to prove that in each bin in which there is at least one element of L_1 the maximum number of the elements is 4. And this is trivial.

Similarly we have to prove obvious statements in the other cases as well.

□

We introduce the following notations

$$r_j = \frac{A(L_1 \dots L_j)}{(L_1 \dots L_j)^*}, \qquad 1 \leq j \leq n. \tag{4}$$

and

$$r = \max_{1 \leq j \leq 4} r_j. \tag{5}$$

Now using the Lemmas 1-2 and replacing (4) into the left hand side of (3) we get

$$\sum_{j=1}^{4} j r_j \geq 16.$$

Now using (5) we get the statement of our theorem.

3 Conclusions

Since the best known on-line algorithm has been analysed in [5], and its asymptotic worst case ratio is about 2.86, the gap between the given lower-bound and this value is large. On the one hand we are sure that this very simple construction studied in our paper has a refinement, and we suspect a lower bound near to 2. On the other hand one can show that the examined algorithms do not used out deeply that our problem is "two-dimensional" and most of them are different generalizations of the known - and analysed - one-dimensional algorithms. So we suspect that with a new method the researchers will be able to give better algorithms than the Generalyzed Harmonic Fit which was presented in [5].

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