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Covering Morphisms and Unique Minimal D-Schemes *

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1 Introduction

In this paper we answer the following question: given a D-scheme F, is there a (unique) smallest D-scheme within the strong equivalence class of F? In addition to providing an affirmative answer to this question, we provide a description of a reduction process leading from a D-scheme to its minimization over the class of D-schemes.

Since we are interested in D-schemes in this paper, we restrict our discussion to what, in the terminology of Elgot [CE], could be referred to as biscalar schemes whose outdegrees are bounded by 2. The paper is organized as follows. Section 2 gives the basic definitions of digraphs, schemes, and (homo)morphisms for these two classes and introduces the class of D-schemes. Section 3 recalls the definitions necessary to state the geometric characterization of D-schemes from [BT] and introduces the notion of strong behavior and strong equivalence for schemes. Section 4 develops the basic properties of morphisms between schemes. Section 5 discusses the minimization process over the class of all schemes and states and proves the main theorem of the paper, Theorem 5.1. Section 6 is the final section and details the consequences of the main theorem.

2 Basic Definitions

A *directed graph*, or *digraph*, is a 4-tuple $H = (V, E, s, t)$ where V is a set whose elements are called the *vertices*, or *nodes*, of H; E is a set whose elements are called the *edges* of H; and s and t are functions from E to V called the *source* and *target* functions, respectively, for H. We require that V and E be disjoint. A *subdigraph* of H is a digraph $H' = (V', E', s', t')$ such that V' and E' are subsets, respectively, of V and E and s' and t' are the restrictions, respectively, of s and t. If x is an edge with $s(x) = u$ and $t(x) = v$, then we say that x is an *outedge* of u and an *inedge* of v. The *indegree* (respectively, *outdegree*) in H of a node u is the number of inedges (respectively, outedges) of u in H. A *homomorphism* from digraph $H = (V, E, s, t)$ to digraph $H' = (V', E', s', t')$ is a function h from $V \cup E$ to $V' \cup E'$ such that $h(V) \subseteq V'$, $h(E) \subseteq E'$, and for each edge x of H, $h(s(x)) = s'(h(x))$

*Dedicated to the memory of Calvin C. Elgot and $h(t(x)) = t'(h(x))$. A *digraph isomorphism* is a bijective homomorphism. It is a simple matter to prove that the inverse of a bijective homomorphism is itself a homomorphism.

Let H be a digraph. A *path* in H from node u is an alternating sequence $P = u_0x_1u_1x_2u_2\ldots$ of nodes and edges of H such that $u = u_0$ and for $i \geq 1$, $s(x_i) = u_{i-1}$ and $t(x_i) = u_i$. If P is finite, we require that the last term of the sequence be a node, say $u' = u_n$, and refer to P as a path (of length n) from u to u'; we say that u' is *accessible* in H from u if such a path exists. If $P = u_0x_1\ldots x_mu_m$ and $P' = u'_0x'_1\ldots x'_nu'_n$ are paths in H with $u_m = u'_0$, then the *composite* $P\cdot P'$ of P and P' is the path $u_0x_1\ldots x_mu_mx'_1u'_1\ldots x'_nu'_n$.

The digraph H is *strongly connected* if for any two nodes u, v of H there is a path in H from u to v. For any node u of H, the *strong component* of u is the subdigraph $F[u, v]$ of H made up of all nodes and edges lying on closed paths from u.

For the remainder of the paper we fix a pair $\Gamma = (\Omega, \Pi)$ of disjoint sets, referring to elements of Ω as *operator symbols* and to the elements of Π as *predicate symbols*. A Γ*-flowchart scheme*, or more briefly a *scheme*, is a 6-tuple $F = (V, E, s, t, \lambda, b)$ such that:

1. (V, E, s, t) is a finite digraph (also denoted F).
2. the nodes of F have outdegree at most 2.
3. F has exactly one node e of outdegree 0 (called the *exit* of F).
4. b is a node of F (called the *begin* of F).
5. λ is a function from $V\setminus\{e\}\cup E$ to $\Omega\cup\Pi\cup\{1,2\}$, called the *labeling function*, satisfying:
   a) if $x\in E$ then $\lambda(x)\in\{1,2\}$;
   b) if node u has unique outedge x then $\lambda(u)\in\Omega$ and $\lambda(x) = 1$.
   c) if node u has distinct outedges x and y, then $\lambda(u)\in\Pi$ and $\lambda(x)\neq\lambda(y)$.

Let $F = (V, E, s, t, \lambda, b)$ and $F' = (V', E', s', t', \lambda', b')$ be schemes and let e and e' be the exits of F and F', respectively. A *scheme morphism* from F to F' is a digraph homomorphism h from F to F' preserving begins, ends and labels — that is, such that $h(b) = b'$, $h(e) = e'$ and for each $z\in(V\cup E)\setminus\{e\}$, $\lambda'(h(z)) = \lambda(z)$. A *scheme isomorphism* is a bijective scheme homomorphism. If v is the target of an edge with source u and label i, then we shall refer to node v as the i*-successor* of node u in scheme F. We shall also use the notation b_F and e_F for the begin and exit, respectively, of a scheme F.

The simplest examples of schemes are the trivial scheme and the atomic schemes. The *trivial scheme* T consists of a single point (which is thus the begin and the exit) and no edges. For each $f\in\Omega$, the *atomic scheme* determined by f (which we also perversely denote by f) consists of a single edge whose source, labeled by f, is the begin and whose target is the exit.

We now introduce basic operations used in building schemes which are analogous to the programming operations of concatenation, if-then-else, and while-do. The *composite* $F\cdot G$ of schemes F and G is the scheme obtained from the disjoint union of F and G by identifying the exit of F with the begin of G; all labels remain the same, with the point of identification retaining the label of the begin of G if such a label exists, which is to say provided that $G\neq T$. The begin of $F\cdot G$ is the begin of F; since the exit of G is now the only point of outdegree 0, it is the exit of $F \cdot G$. It should be observed that the trivial scheme is an identity for the composition operation. For each $\pi \in \Pi$, the *alternation* by π of schemes F,G is the scheme $\pi(F, G)$ constructed as follows: form the disjoint union of F, G, and a new vertex u labeled π; add new edges x,y labeled 1 and 2, respectively, with $s(x) = s(y) = u$, $t(x)$ the begin of F, and $t(y)$ the begin of G; and then identify the exit of F with that of G. Finally, given $\pi \in \Pi$ and $i \in \{1, 2\}$, the *while* by π, i of scheme F, denoted $wh(\pi, i, F)$, is constructed as follows: add to F two new points u and v and two new edges x and y with $s(x) = s(y) = u$, $t(x) = v$, and $t(y)$ the begin of F; identify the exit of F with u; label u by π, y by i, and x by $3 - i$; and designate u as the begin.

A scheme G is a *D-scheme*, or "simple while scheme", if it may be constructed from the trivial scheme and atomic schemes by a finite sequence of applications of the composition, alternation and while operations.

3 The characterization theorem for D-schemes

The definition given above for D-schemes was essentially algebraic in that it specified the generators (T and the atomic schemes) and operations (composition, alternations, and whiles) for the class of such schemes. We now develop definitions which will be used throughout the paper and state the geometric characterization of D-schemes due to the author and Bloom [BT].

The *trace* of a path

$$P = u_0 x_1 u_1 \dots$$

in scheme F is the string

$$\lambda(u_0)\lambda(x_0)\lambda(u_1)\dots$$

if P is infinite, and the string

$$\lambda(u_0)\lambda(x_1)\dots\lambda(u_{n-1})\lambda(x_n)$$

if P is of length n. Notice that we did not include the label of the final vertex of a finite path, and thus the trace is well defined even if the final vertex is the (unlabeled) exit node. A *successful* path of F is a path from the begin to the exit. The *strong behavior* of a scheme F is the set of traces of both the successful paths and the infinite paths from the begin of F. Two schemes are *strongly equivalent* if they have the same strong behavior.

A scheme is *accessible* if every node is accessible from the begin, is *coaccessible* if the exit is accessible from every node, and is *biaccessible* if it is both accessible and coaccessible. We point out the obvious fact that a biaccessible scheme is one in which every node lies on a successful path. One may establish by a straightforward induction on the number of operations used in building a D-scheme that every D-scheme is biaccessible. Let A be a subdigraph of a scheme F. A path $P = u_0 x_1 \dots x_n u_n$ is said to be *A-simple* if there is at most one value of i, $0 \le i \le n$, for which u_i is in A. If the A-simple path P as above has initial vertex the begin of F and final vertex in A, then P is an *entry path* to A and u_n is an *entry point* of A; if on the other hand P has initial vertex in A and final vertex the exit of F, P is an *exit path* from A and u_0 is an *exit point* of A.

A *cycle* in a scheme F is a subdigraph of F made up of the edges and nodes of some positive length *simple closed path*, by which we mean a path in which the only repetition of nodes is due to the initial and final nodes being the same. A *bipath* in F from node u to node $v \neq u$ is a subdigraph of F made up of the nodes and edges lying on some pair of simple paths in F from u to v which overlap only at u and v. A *simple path* is one in which there is no repetition of nodes.

Let F be a scheme. Then F is said to be *reducible* if every cycle in F has a unique entry point, F is *coreducible* if every cycle has a unique exit point, and is *bireducible* if each cycle contains a point which is both its unique entry point and its unique exit point. Finally, F has the *bipath exit property* if for any bipath B from u to v in F, v is the unique exit point of B.

We are now prepared to state the geometric characterisation theorem for D-schemes.

Theorem 3.1 (BT) *A scheme G is a D-scheme if and only if G is biaccessible, bireducible and satisfies the bipath exit property.*

A D-scheme may be viewed as having a natural "block structure", which we will make precise after the development of some additional definitions and intermediate results. Let F be a coaccessible scheme, u a node of F, and S a set of nodes of F. S is a *bottleneck set* rel u if S contains neither u nor the exit and every path from u to the exit contains at least one point of S. If no proper subset of S is a bottleneck set rel u we say that S is a *minimal bottleneck set* rel u. If S consists of a single point v, then v is said to be a *bottleneck* rel u. If u is the begin of F, we omit reference to u and speak of a bottleneck or bottleneck set without further qualification. If v is a bottleneck rel u in F, or v is the exit of F and $u \neq v$, the *segment of F from u to v* is the subdigraph of F made up of the points and edges lying on $\{v\}$-simple paths in F from u to v. It is clear that, except possibly for the node v, every node of $F[u,v]$ has the same outdegree in $F[u,v]$ as in F, and v has outdegree 0 in $F[u,v]$. Thus $F[u,v]$ may be viewed as a scheme by designating u as the begin and labeling the nodes as in F, with the exception that the exit v of $F[u,v]$ recieves no label. Note that a segment is always nontrivial. A segment $F[u,v]$ is a *block* of F if for any edge x of F, $t(x) \in F[u,v] - \{u,v\}$ implies $s(x) \in F[u,v] - \{v\}$; and if b_F is in $F[u,v]$ then b_F is either u or v. We will use the notation $F[u]$ for $F[u,e_F]$. Note that bottleneck sets, segments and blocks are defined only for coaccessible schemes.

The next proposition, whose simple proof we omit, deals with the "inheritance" of geometric properties by segments and blocks.

Proposition 3.1 *Let v be a bottleneck rel u in F. Then*

a) v is the unique exit of $F[u,v]$;

b) if $F[u,v]$ is a block, then u is the unique entry point of $F[u,v] - \{v\}$;

c) if F is a reducible, then $F[u,v]$ is reducible.

d) if F is a D-scheme and $F[u,v]$ is a block, then $F[u,v]$ is a D-scheme.

As noted above, every block of a D-scheme is itself a D-scheme. More can be shown. We will say that scheme G is obtained by a *block substitution* from a coaccessible scheme F if for some coaccessible scheme H and block $F[u,v]$ of F, G is isomorphic to the scheme constructed as follows: delete $F[u,v] - \{u,v\}$ from F and form the disjoint union of the result with H; identify the begin of H with u and the exit of H with v; label the edges and the nonexit nodes of H as in H; and label the nodes and edges from $(F - F[u,v]) \cup \{v\}$ as in F. Note that if H is the trivial scheme T, then u and v are identified and the resulting node retains the label of v in F. The definition of "block" ensures that the result is well-defined as a scheme; in particular, the target function on edges will not attempt to map an edge of F which was retained in the new scheme to a point of $F[u,v] - \{u,v\}$. If F and H are members of some class of coaccessible schemes, we say that G is obtained by a block substitution within the class; if the result of a block substitution within a given coaccessible class always produces another scheme within the class, we say that the class is *closed under block substitution.* Note that the class of all coaccessible schemes is closed under block substitution. The characterisation theorem may be used to show that the class of D-schemes is closed under block substitution. In fact, the class of D-schemes may be characterised as the smallest class of schemes closed under block substitution and containing T, f, $f \cdot g$, $\pi(f, g)$, $wh(\pi, 1, f)$, and $wh(\pi, 2, f)$ for all $\pi \in \Pi$ and $f, g \in \Omega$.

4 Coverings

We now turn to the basic properties of morphisms between schemes.

Proposition 4.1 *Let H be a morphism from scheme F to scheme G and let u and v be points of F and G respectively. If $h(u) = v$, then H maps the outedges of u in F bijectively onto those of v in G.*

Proof. Immediate from the following direct consequences of definitions: u and $v = h(u)$ must have the same outdegrees in their respective schemes; h preserves edge labels; the outedges of a node in a scheme are enumerated by their labels.

If h is a morphism from F to G and $P = u_0 x_1 u_1 x_2 \ldots$ is a path in F, the *image* of P under h is the path $h(P) = h(u_0)h(x_1)h(u_1)h(x_2)\ldots$ in G. We shall then refer to P as a u_0*-lift* of the path $h(P)$. The following consequence of Proposition 4.1 is the direct analogue of a basic property of covering spaces in topology.

Proposition 4.2 *If h is a morphism from F to G then for any node v of G, any path Q from v in G, and any point u of F with $h(u) = v$, Q has a unique u-lift in F.*

Proof. When Q is of finite length n, Proposition 4.1 provides the basis step $(n = 1)$ for a proof by induction on n; moreover, since every path of length $n > 1$ may be written as a path of length $n - 1$ followed by a path of length 1, the inductive step is immediate. If $Q = v_0 y_1 \ldots$ is infinite then for each $i \geq 1$ let Q_i be the initial segment $v_0 y_1 \ldots y_i v_i$ of Q. If P_i is the unique u-lift of Q_i, then for any $j < i$, the initial segment of P_i of path length j is a u-lift of Q_j and hence equals P_j. Thus the sequence $P_1, P_2, \ldots$ is an infinite sequence of paths with P_{i+1} an extension of P_i for $i \geq 1$ and hence determines an infinite path P which is the unique u-lift of Q.

Proposition 4.3 *If there is a morphism from F to G then F and G have the same strong behaviors.*

Proof. Let h be a morphism from F to G. Since morphisms must preserve begins, P is a path from the begin in F if and only if $h(P)$ is a path from the begin in G. Moreover, since the exit of a scheme is its unique node of outdegree 0 and morphisms preserve outdegrees, P terminates at the exit of F if and only if $h(P)$ terminates at the exit of G. Thus P is a successful path in F if and only if its image is a successful path in G. Since P and $h(P)$ have the same trace, the proof is complete.

Proposition 4.4 *If F and G are accessible schemes, then there is at most one morphism from F to G, and any such morphism must be surjective.*

Proof. Let g and h be morphisms from F to G. For any node u of F, there exists at least one path from the begin to u and thus we may define the *distance* from the begin of F to u as the length of a shortest path in F from the begin to u. We now prove by induction on the distance from the begin to u that $g(u) = h(u)$ for all nodes u of F. The only node of distance 0 from the begin is the begin itself and by the definition of morphisms, g and h both map the begin of F onto the begin of G; so the basis step is established. For the inductive step, let $P = u_0x_1 \ldots x_nu_n$ be a shortest path from the begin of F to u. Then, since u_{n-1} has smaller distance from the begin than $u = u_n$, $g(u_{n-1}) = h(u_{n-1})$. If i is the label of x_n in F, then $g(x_n)$ and $h(x_n)$ must both be outedges of the same node in G with the same label in G and hence must be the same edge y of G. It then follows that g and h map the target u_n of edge x_n onto the target of the edge y and thus $g(u) = h(u)$, and the proof of the uniqueness of morphisms is complete.

We now prove that every morphism h from F to G is surjective. Let v be a point of G. Since G is accessible, there is a path Q in G from the begin to v and thus Q has a unique b_F-lift P. It then follows that h maps the terminal point of P onto v so that v is in the image of h. If y is an edge of G, then since the source of y in G is in the image of h, Proposition 4.1 implies that y is in the image of h, and we have shown that h is surjective.

Proposition 4.5 *Let h be a surjective morphism from F to G and let C be a cycle in G. Then there exists a cycle C' in F with $h(C') = C$.*

Proof. Let $Q = v_0y_1 \ldots y_nv_0$ be a simple closed path determining C, let u_0 be a point of F with $h(u_0) = v_0$, and let m be the number of nodes in F. If Q' is the m-fold composition of Q with itself, then the u_0-lift P' of Q' cannot be a simple path since its length exceeds the number of nodes in F. We may then choose a contiguous subsequence P of P' such that P is a simple closed path in F. It then follows that $h(P)$ is a closed path all of whose nodes and lines appear in Q. If u is a node occuring in P, it has a unique outedge x in P and $h(x)$ is an outedge of $v = h(u)$ in G. Since $h(x)$ is an edge appearing in Q and u has a unique outedge in Q, $h(x)$ is the outedge of u in Q. It then follows that if u is in $h(P)$ the unique successor of u in Q is in $h(P)$ and thus that the subgraph of F determined by $h(P)$ is the same as that determined by Q, and the proof is complete.

We now turn to the question of which of the geometric properties of accessible schemes defined in section 3 are preserved by morphisms. It will be convenient to refer to a surjective morphism h from F to G as a *covering* of G by F and hence to say that F *covers* G and that F is a *cover* of G. In the situations where the domain scheme F is a D-scheme, we shall refer to h as a *D-covering* and to F as a *D-cover* of G.

We next note that neither the bipath exit property nor reducibility is preserved by coverings. Let F be the scheme having vertices b_F, v_1, v_2, e_F and satisfying

1. $\lambda(b_F) = \nu \in \Pi, \lambda(v_1) = \pi \in \Pi$, and $\lambda(v_2) = f \in \Omega$;
2. v_i is the i-successor of b_F for $i = 1, 2$;
3. v_1 has 1-successor e_F and 2-successor v_2;
4. e_F is the 1-successor of v_2.

Then F is covered by the D-scheme $\nu(\pi(T, f), f)$ and F does not satisfy the bipath exit property. If G is the scheme obtained from F by changing the 1-successor of v_2 from e_F to v_1, then G is not reducible and is covered by the D-scheme $\nu(T, f) \cdot wh(\pi, 2, f)$. The property of coreducibility is however preserved by coverings.

Proposition 4.6 *If F is coreducible and h is a covering of G by F, then G is coreducible.*

Proof. Let C be a cycle in G. Then by 4.5 there is a cycle C' in F with $h(C') = C$. If P is an exit path from C with initial point u, then for any u' in C' with $h(u') = u$, the u'-lift of P is an exit path from C'. Since C' has a unique exit point in F, it follows that C has a unique exit point in G.

One obvious consequence of proposition 4.6 is that coreducibility is a necessary condition for a biaccessible scheme to be covered by a D-scheme; it is also sufficient. Proposition 4.6 and its converse for biaccessible schemes is a generalisation of Kasai's theorem which uses coreducibility to characterise the schemes with the same strong behavior as a D-scheme [K]. We will not give a separate proof of the converse of 4.6 since it is a corollary of the main theorem of this paper, Theorem 5.1 of the next section. We now establish an important lemma and state without proof some straightforward results detailing relationships between morphisms and the D-operations.

Lemma 4.1 *Let h be a covering of G by a coaccessible scheme F and let v be a bottleneck rel u in F. Then $h(v)$ is a bottleneck rel $h(u)$ in G and $h(F[u,v]) = G[h(u),h(v)]$.*

Proof. To show that $h(v)$ is a bottleneck rel $h(u)$ in G, let Q be a path from $h(u)$ to the exit of G and consider the u-lift Q' of Q. Since Q' is a path from u to the exit of F, v must lie on Q' and therefore $h(v)$ lies on $h(Q') = Q$.

We now turn to showing that $h(F[u,v]) = G[h(u),h(v)]$. Since the u-lift of any $\{h(v)\}$-simple path in G from $h(u)$ is a $\{v\}$-simple path in F, $G[h(u),h(v)]$ is a subdigraph of $h(F[u,v])$. To prove that $h(F[u,v])$ is equal to $G[h(u),h(v)]$, we need only show that the image under h of any $\{v\}$-simple path in F from u to v is $\{h(v)\}$-simple. Suppose to the contrary that $P = v_0x_1\ldots x_nv_n$ is $\{v\}$-simple, $v_0 = u$, $v_n = v$, and $h(v_i) = h(v)$ for some $i < n$. If Q is a simple path from $h(v)$ to e_G, Q' is the v_i-lift of Q, and $P' = v_0x_1\ldots x_iv_i$, then $P' \cdot Q'$ is a path from u to the exit of F which does not contain v. This contradiction completes the proof.

Proposition 4.7 *Let π be a predicate symbol and j an element of $\{1,2\}$. If F_i covers G_i, $i = 1,2$, then*

a) $wh(\pi, j, F_1)$ covers $wh(\pi, j, G_1)$;

b) $\pi(F_1, F_2)$ covers $\pi(G_1, G_2)$; and

c) $F_1 \cdot F_2$ covers $G_1 \cdot G_2$.

Proposition 4.8 *If $F = wh(\pi, j, F')$ covers G then there is a scheme G' such that F' covers G' and $G = wh(\pi, j, G')$.*

Proposition 4.9 *If F_1 and F_2 are nontrivial coaccessible schemes such that $F_1 \cdot F_2$ covers G, then there is a bottleneck v of G such that F_1 covers $G[b_G, v]$ and F_2 covers $G[v]$.*

Proposition 4.10 *Let v be a bottleneck of coaccessible scheme G. If F_1 covers $G[b_G, v]$ and F_2 covers $G[v]$, then $F_1 \cdot F_2$ covers G.*

5 Minimal Schemes

In this section we consider the question of the existence of unique minimal schemes within strong equivalence classes. If one is minimizing over the class of all coaccessible schemes, then such schemes exist since a scheme F may be viewed as a finite automaton over the alphabet $\Omega \times \{1\} \cup \Pi \times \{1,2\}$ as follows: let b_F be the start and e_F be the unique final state, replace the label i of an edge x by the ordered pair (α, i), where α is the label of the source of x, and remove the labels of the nodes. Since the infinite strings in the strong behavior of a coaccessible scheme are determined by the finite strings, it is clear that strong equivalence is then the same as automaton equivalence. Thus the construction of the minimum scheme $F\mathbb{M}$ strongly equivalent to F is essentially the same as for finite automata and has as vertices the behavior-equivalence classes of nodes of F. Moreover, the function sending a node to its equivalence class induces a morphism of F onto $F\mathbb{M}$. As in the case of finite automata, two schemes are strongly equivalent if and only if they have isomorphic minimal schemes.

As noted in the previous section, there are D-schemes which cover schemes which are not D-schemes. In fact, the two examples given there were "minimization" coverings, so that minimization (over the class of all schemes) does not map the class of D-schemes into itself. The primary motivation for the present paper is the question of whether the class of D-schemes has unique minimum elements. We obtain the strongest possible such theorem: for each D-scheme F there is a D-scheme $\mathbb{D}F$ which is covered by F and is such that any D-scheme strongly equivalent to F also covers $\mathbb{D}F$. Furthermore, we show that $\mathbb{D}F$ may be obtained from F by two simple types of reductions.

In order to state our main theorem, we must develop a few more definitions. It is simple to see that for any scheme F and predicate symbol π, there are coverings of $wh(\pi, 1, F)$ by $\pi(F \cdot wh(\pi, 1, F), T)$ and $wh(\pi, 2, F)$ by $\pi(T, F \cdot wh(\pi, 2, F))$. We will refer to such a morphism as a *wrap-around morphism*, or more simply as a *wrap-around*. Moreover if G and H are schemes, then there is a covering of $\pi(G, H) \cdot F$ by $\pi(G \cdot F, H \cdot F)$, which we shall refer to as a *pull-through morphism*, or simply as a *pull-through*. It should be clear that the domain of a wrap-around or pull-through is a D-scheme if and only if the same is true of its range. Let us say that a covering h of G by F has *support* $F[u, v]$ if $F[u, v]$ is a block of F and G is isomorphic to the result of substituting $h(F[u, v])$ for $F[u, v]$ in F. An *elementary reduction morphism* is a morphism with support a block on which it is a wrap-around or a pull-through. A *reduction morphism* is morphism which is a finite composition of elementary reductions. We will also say that G is a *reduction* of F is there is a reduction morphism of F onto G; if in addition, F and G both cover some scheme H, we will say that G is a *reduction over* H of F. If F covers H and there exists no reduction G of F over H with G not isomorphic to F, then we say that F is a *reduced cover* of H. We note that a reduction of a D-scheme is also a D-scheme.

We are now prepared to state the main theorem of the present paper. Recall that a D-scheme which covers a scheme H is referred to as a D-cover of H.

Theorem 5.1 *Every coaccessible, coreducible scheme H has a unique reduced D-cover $H\mathbb{D}$.*

Before proceeding to the proof of 5.1 we establish two lemmas concerning bottleneck sets in D-schemes.

Lemma 5.1 *Let S be a minimal bottleneck set for a D-scheme $G = G_1 \cdot G_2$. Then either S consists precisely of the begin of G_2 or S is a minimal bottleneck set for exactly one of G_1, G_2.*

Proof. By the minimality of S, if the begin of G_2 is a point of S it is the only point in S, so we assume that this is not the case. Let $S_i = S \cap G_i$ $(i = 1, 2)$. There must be an $i \in \{1, 2\}$ such that S_i is a bottleneck set for G_i, for otherwise S is not a bottleneck set for G. Since the begin of G_2 is a bottleneck for G, every path in G is made up of a path in G_1 followed by a path in G_2 and thus must contain a point of S_i. Therefore S_i is also a bottleneck set for G. The minimality of S then implies that $S_i = S$ and the proof of the lemma is complete.

Lemma 5.2 *Let S be a minimal bottleneck set for a D-scheme G with $|S| \geq 2$. Then there is an alternation block*

$$G[y, z] = \nu(G[y_1, z], G[y_2, z])$$

of G and points s_i of S, $i = 1, 2$, such that

$$G[y_i, z] = G[y_i, s_i] \cdot G[s_i, z].$$

Proof. We proceed by induction on the number of points of G. Since the begin and exit of G are separated by S and $|S| \geq 2$, G must have at least 4 points. If G has exactly 4 points it must be the alternation of two atomic schemes and S must consist of the begins of the two atomic schemes. In this case we may set y and z equal to the begin and exit, respectively, of G to establish the conclusion.

We thus move to the inductive step. Under the hypothesis, the exit cannot be an immediate successor of the begin in G and thus G is not a while-do. If G is a nontrivial composite $G_1 \cdot G_2$ then by lemma 5.2, S is a nontrivial bottleneck set for either G_1 or G_2. In either case the inductive hypothesis holds and the conclusion follows. Thus we assume that $G = \nu(G_1, G_2)$. Let b_i denote the begin of G_i and S_i the intersection of S with G_i $(i = 1, 2)$. Let i be one of 1,2. The i-outedge of b_G followed by a successful path in G_i is a successful path in G. Thus S_i is nonempty and contains a point of any given successful path in G_i; moreover S_i is minimal with respect to the latter property, so that either $S_i = b_i$ or S_i is a minimal bottleneck set for G_i. If S_i is nontrivial we may apply the inductive hypothesis to conclude the proof. Therefore, we suppose that for each $i \in \{1, 2\}$, S_i is a singleton set $\{s_i\}$ and thus that $G_i = G[b_i, s_i] \cdot G[s_i]$. Notice that the latter equation is valid regardless of whether $b_i = s_i$. The proof of the lemma is now complete as we may set $y = b_G$ and $z = e_G$.

We now turn to establishing Theorem 5.1, which states that every coaccessible, coreducible scheme H has a unique reduced D-cover $H\mathbb{D}$. The proof is by induction on the number of nodes in H, the basis step of which is trivial. Thus we henceforth fix $n > 1$, let H be a biaccessible, coreducible scheme with n points and assume the following inductive hypothesis:

> every biaccessible, coreducible scheme H' with fewer than n points has a unique reduced D-covering $H'\mathbb{D}$.

Lemma 5.3 *If G is a reduced D-cover of H and w is a bottleneck of H, then $G = G_1 \cdot G_2$ for reduced D-coverings G_1 and G_2 of $H[b_H, w]$ and $H[w]$, respectively.*

Proof. Let g be the morphism from G to H. Since every successful path of G is the b_G-lift of a successful path of H and w is a bottleneck of H, $g^{-1}(w)$ is a bottleneck set for G. Let S be a smallest minimal bottleneck set contained in $g^{-1}(w)$.

If S is nontrivial we may find a block of G of the form

$$G[y, z] = \nu(G[y_1, s_1] \cdot G[s_1, z], G[y_2, s_2] \cdot G[s_2, z])$$

as in lemma 5.3. Since z is a bottleneck rel s_i in G, $g(z)$ is a bottleneck rel $w = g(s_i)$ in H and $g(G[s_i, z]) = H[w, g(z)]$, $i = 1, 2$. Therefore $G[s_1, z]$ and $G[s_2, z]$ are both reduced covers of $H[w, g(z)]$ and hence, by the inductive hypothesis, are isomorphic. Moreover, we now see that the block $G[y, z]$ of G admits a pull-through morphism over its image under g, contradicting the assumption that G is a reduced D-covering of H. Thus we may assume that S consists of a single point u. Since u is a bottleneck of G, $G = G[b_G, u] \cdot G[u]$. Since G is a reduced D-cover of H, $G[b_G, u]$ and $G[u]$ are reduced covers of $H[b_H, w]$ and $H[w]$, respectively, so the proof of the lemma is complete.

Lemma 5.4 *If H is a while-do of some scheme H_1, then the while-do of $H_1\mathbf{D}$ is the unique reduced D-cover of H.*

Proof. Without loss of generality we assume $H = wh(\pi, 1, H_1)$. If G' is the corresponding while-do of $H_1\mathbf{D}$, then clearly G' is a reduced D-cover of H. We now prove uniqueness by showing that any reduced D-cover of H is isomorphic to G'. Let G be a reduced D-cover of H. Then, since H has no bottlenecks, G is not a nontrivial composition. If G is a while-do, then it is the while-do of a reduced D-cover of H_1 and hence is isomorphic to G', as claimed.

Suppose now that there is a reduced D-cover of H distinct from G' and let G be a smallest such scheme. By the above, G must be an alternation and thus, since the 2-successor of b_H is the exit of H, $G = \pi(F, T)$ for some scheme F. If x is the 1-successor of b_H — i.e., the begin of H_1 – then F must be a reduced D-cover of the coreducible scheme $H' = H[x]$ whose begin is x. Since the begin of H is a bottleneck rel x in H, it is a bottleneck of H'. By lemma 5.4, $F = F_1 \cdot F_2$, where F_1 is a reduced D-covering of $H'[x, b_H] = H[x, b_H] = H_1$ and F_2 is a reduced D-covering of $H'[b_H] = H$. By the inductive hypothesis, F_1 is isomorphic to $H_1\mathbf{D}$. Now F_2 has fewer points than G and is a reduced cover of H; since G was the smallest reduced cover of H distinct from G', we conclude that F_2 is isomorphic to $G' = wh(\pi, 1, H_1\mathbf{D})$. But then G admits a pull-through over H. This contradiction completes the proof of lemma 5.5.

Lemma 5.5 *If H has a bottleneck, then there is a unique reduced D-cover of H.*

Proof. It is simple to see that if w is a bottleneck of H, then $H[b_H, w] \cdot H[w]$ covers H. Since $H[b_H, w]$ has fewer points than H, it has a unique reduced D-cover G_1, and thus $G_1 \cdot H[w]$ covers H. If b_H is not in $H[w]$, then $H[w]$ also has a unique reduced D-cover G_2, and thus $G = G_1 \cdot G_2$ is a D-cover of H. Since the support of an elementary reduction of a composition $G_1 \cdot G_2$ must be contained in one of G_1, G_2, G is also reduced. Uniqueness then follows directly from lemma 5.4.

We thus assume that b_H is accessible from every bottleneck of H. Let w be a bottleneck of H. Then b_H is in $H[w]$, from which it is obvious that w must be the exit point for the strong component K of b_H. Let v be the immediate successor of w not in K. Since every successful path must contain v and b_H is not accessible from v, v must be the exit of H. But then $H[w]$ is a while-do and thus, by lemma 5.6, has a unique reduced D-cover G_2. As shown in the previous paragraph, $G = G_1 \cdot G_2$ is then the unique reduced D-cover of H, and the proof of lemma 5.6 is complete.

We are now prepared to complete the proof of theorem 5.1 by establishing the following.

Lemma 5.6 *If H has no bottlenecks and is not a while-do, then H has a unique reduced D-cover.*

Proof. If b_H has nontrivial strong component K, then K has a unique exit u. Since H has no bottlenecks, $u = b_H$ and the successor of b_H not in K must be the exit of H. Since in this latter case H would be a while-do, we may conclude that b_H has trivial strong component and hence has indegree zero. Moreover, b_H may not be labeled by an operator symbol since that would imply that H is a nontrivial composition and hence has a bottleneck.

We thus have that b_H has no in-edges and is labeled by some predicate symbol π. It then follows that no while-do scheme can cover H. Since H has no bottlenecks, no nontrivial composition can cover H. Thus every D-cover of H is an alternation. For $i = 1, 2$, let x_i be the i-successor of b_H in H; then b_H is not in $H[x_i]$ and thus, by the inductive hypothesis, there is a unique reduced D-cover G_i of $H[x_i]$. Letting $G = \pi(G_1, G_2)$, we see that G is a D-cover of H which is either reduced or admits a global reduction over H. But the latter is not possible since H may not be covered by either a nontrivial composition or a while-do. Uniqueness is now a simple matter since any reduced covering of H must be the alternation of reduced coverings of $H[x_1]$ and $H[x_2]$ and thus is isomorphic to G, the alternation of their unique reduced D-covers.

6 Corollaries to the main theorem

Theorem 5.1 is a condensation of many distinct results into one compact statement, which we now unravel as corollaries. Since we did not assume the converse of proposition 4.6, it follows as the following corollary of 5.1.

Corollary 6.1 *Every biaccessible, coreducible scheme is covered by a D-scheme.*

Since a D-scheme is its own unique reduced D-cover the next result is immediate.

Corollary 6.2 *Every morphism between D-schemes is a composition of morphisms each of which has support a block on which it is a wrap-around or a pull-through.*

We remark that Douglas Troeger has used corollary 6.2 as the basis for an axiomatization of the algebra of strongly equivalence classes of D-schemes[DT]. The next corollary could be viewed as the "unique minimum D-schemes" theorem referred to earlier in the paper. For any D-scheme F, let $\mathbf{D}(F)$ be the unique reduced D-cover of the minimization FM of F over the class of all schemes.

Corollary 6.3 *Let F and G be D-schemes. Then*

a) F is strongly equivalent to $\mathbf{D}(F)$;

b) $\mathbf{D}(F)$ has the fewest nodes among all D-schemes strong equivalent to F;

c) G is strongly equivalent to F if and only if $\mathbf{D}(F)$ is isomorphic to $\mathbf{D}(G)$; and

d) if G is strongly equivalent to F and has the same number of nodes as $\mathbf{D}(F)$, then G is isomorphic to $\mathbf{D}(F)$.

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