Acta Cybernetica, Vol. 10, Nr. 3, Szeged, 1992

On the randomized complexity of monotone graph properties

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1 Introduction

Let $C^R(P)$ be the number of questions of the form 'Does the graph G contain the edge $e(i,j)$?' that have to be asked in the worst case by any randomized decision tree algorithm for computing an n-vertex graph property P. For non-trivial, monotone graph properties it is known, that the deterministic complexity is $\Omega(n^2)$ (see [4]). R. Karp [5] conjectured, that this bound holds for randomized algorithms as well. As far as this conjecture we know the following results. The best uniform lower bound for all non-trivial, monotone graph properties is $\Omega(n^{4/3})$ due to P. Hajnal [1].

No non-trivial, monotone graph property is known having a randomized complexity of less than $n^2/4$. Some properties have been proven to have complexity of $\Omega(n^2)$ (see A. Yao [6]).

In this paper we refine the idea of Yao. This leads to a further improvement in the reductions of arbitrary graph properties to bipartite graph properties. (see [1], [3]) and yields a uniform lower bound for the subgraph isomorphism properties of $\Omega(n^{3/2})$. Furthermore we show, that a large variety of isomorphism properties as well as k-colourability require $\Omega(n^2)$ questions.

2 Preliminaries, notations

A decision tree is a rooted binary tree with labels on each node and edge. Each inner node is labeled by a variable symbol and the two edges leaving the node are labeled by 0 and 1. Each leaf is also labeled by 0 or 1. Obviously, any truth-assignment of the variables determines a unique path from the root to a leaf.

A decision tree A *computes a boolean function* f if for all input $\underline{x}$ the corresponding path in A leads to a leaf labeled by $f(\underline{x})$.

Let $cost(A, \underline{x})$ be the number of questions asked when the decision tree A is executed on input $\underline{x}$. This is the length of the path induced by $\underline{x}$. The *deterministic decision tree complexity* of a boolean function f is $C(f) = \min_A \max_{\underline{x}} cost(A, \underline{x})$, where the minimum is taken over all decision trees A computing the function f.

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In a *randomized decision tree* the question asked next not only depends on the answers it got so far but also on the outcome of a trial. Since all trials can be done in advance we can view a randomised decision tree as a probability distribution on the set of deterministic trees. *A randomized decision tree computes a boolean function* f iff the distribution is non-sero only on deterministic trees computing f.

Definition 2.1 *Let* $\{A_1, \ldots, A_N\}$ *be the set of all deterministic decision trees computing* f. *Let* $R = \{p_1, \ldots, p_N\}$ *be a randomized decision tree, where* p_i *denotes the probability of* A_i. *The* cost of R on input $\underline{x}$ *is* $cost(R,x) = \sum_i p_i \cdot cost(A_i, \underline{x})$. *The* randomized decision tree complexity *of a function* f *is*

$$C^R(f) = \min_R \max_{\underline{x}} cost(R, \underline{x}),$$

where the minimum is taken over all randomized decision trees computing the function f. *The following lemma yields the base of all lower bound proofs for randomized decision tree complexity so far.*

Lemma 2.2 *(A. Yao [6]) Let* d *be a probability distribution on the set of all possible inputs and let* $d(\underline{x})$ *be the probability of input* $\underline{x}$. *We define the* average case *performance of a deterministic tree* A *computing* f *as* $av(A,d) = \sum_{\underline{x}})d(\underline{x})cost(A, \underline{x})$.

Then for any boolean function f

$$C^R(f) = \max_d \min_A av(A, d),$$

where the minimum is taken over all deterministic decision trees computing f.

A boolean function f is called *non-trivial, monotone* iff $f(\underline{0}) = f(\underline{1}) = 1$ and $f(\underline{x}_1) \leq f(\underline{x}_2)$ for all $\underline{x}_1 \leq \underline{x}_2$. Here we mean component wise less or equal. In this paper we deal only with graph properties and bipartite graph properties. Since a graph on n vertices can be identified with a $(0,1)$-string of length $\binom{n}{2}$, a graph property can be given by a boolean function which takes equal values on isomorphic graphs. So, by *graph property* we mean a suitable boolean function and sometimes instead of the function we give the property by the set of all graphs having this property. A graph property is called *non-trivial, monotone* iff the corresponding boolean function is non-trivial, monotone.

Let us denote the set of all n-vertex by $\mathcal{G}_n$ and the set of all non-trivial, monotone graph properties defined on $\mathcal{G}_n$ by $\mathcal{P}_n$. Clearly, a property $P \in \mathcal{P}_n$ can be characterized by the set of minimal graphs having that property. Let $\min(P)$ be the list of minimal graphs for P. If $\min(P)$ contains up to isomorphism only one graph G, we call P a *subgraph isomorphism property* and denote it by P_G.

Let us denote by $d_G(x)$ the degree of a node x in G, by $D(G)$ the maximal degree of G, by $\delta(G)$ the minimal degree of G and by $\bar{d}(G)$ the average degree of G. Furthermore, denote $V(G)$ the set of vertices with non-sero degree of G, $E(G)$ the set of edges of G and K_n, E_n the complete and the empty graph on n nodes, respectively. Sometimes we use the disjoint union of K_{n-r} and E_r, and this graph is denoted by K^*_{n-r}.

Let $0 < m < n$ and $P \in \mathcal{P}_n$. Using the property P, we can define two (not necessarily non-trivial) monotone graph properties on $\mathcal{G}_m$. For this reason, divide the set of nodes, P is defined on, into disjoint sets V_1 and V_2 so that $|V_1| = m$ and $|V_2| = n - m$. Let $ind(P|m)$ and $red(P|m)$ denote the following m-vertex properties:

$G \in ind(P|m)$ iff adding all nodes in V_2 to G and keeping the original edge-set, we obtain a graph having property P.

$G \in red(P|m)$ iff adding all nodes in V_2 together with all possible edges incident to them to G, we get a graph having property P.

Obviously $C^R(ind(P|m)) \leq C^R(P)$ and $C^R(red(P|m)) \leq C^R(P)$.

We have to build up the same system of notions for the universe of labeled bipartite graphs with colour classes $V = \{1, 2, \ldots, n\}$ and $W = \{\overline{1}, \overline{2}, \ldots, \overline{m}\}$ denoted by $\mathcal{G}_{n,m}$. The set of all non-trivial, monotone bipartite graph properties on $\mathcal{G}_{n,m}$ is denoted by $\mathcal{P}_{n,m}$. We also use the other corresponding notions $C^R(P)$, $\min(P)$ and $E(G)$.

If $G \in \mathcal{G}_{n,m}$ and U is a subset of the vertices then let us denote by $d_{\max,U}(G)$ and $d_{av,U}(G)$ the maximal and average degree in the set U, and by $K_{n,m}$, $E_{n,m}$ the complete bipartite graph and the empty bipartite graph, respectively

Let $0 < r < n$ and $P \in \mathcal{P}_{n,m}$. Divide V into disjoint sets V_1 and V_2 so that $|V_1| = r$ and $|V_2| = n - r$. Let $ind_V(P|r)$ and $red_V(P|r)$ denote the following bipartite graph properties defined on $\mathcal{G}_{n,m}$:

$G \in ind_V(P|r)$ iff adding all nodes of V_2 to G, we obtain a bipartite graph having property P.

$G \in red_V(P|r)$ iff adding all nodes of V_2 together with all possible edges between V_2 and W to G, we get a bipartite graph having property P.

Obviously $C^R(ind_V(P|r)) \leq C^R(P)$ and $C^R(red_V(P|r)) \leq C^R(P)$.

Finally let

$$C^R(n, m) = \min\{C^R(P) | P \in \mathcal{P}_{n,m}\}.$$

In lower bound proofs for the complexity of monotone graph properties the following reduction to bipartite graph properties plays an important role.

Let $P \in \mathcal{P}_n$ and $0 < r < n$. Furthermore, let $bipart(P|r, n-r)$ be the following bipartite graph property defined on $\mathcal{G}_{r,n-r}$

$G \in bipart(P|r, n-r)$ iff adding all edges between nodes in W, we obtain a graph having property P.

Obviously $C^R(bipart(P|r, n-r)) \leq C^R(P)$ and so if $bipart(P|r, n-r)$ is non-trivial, then $C^R(r, n-r) \leq C^R(P)$.

A good survey of previous techniques can be found in [1]. We only mention those, we will apply.

Theorem 2.3 *(Basic Method [6]) (i) Let $P \in \mathcal{P}_n$ and $G \in \min(P)$ be any minimal graph for P. Then*

$$C^R(P) \geq |E(G)|.$$

(ii) Let $P \in \mathcal{P}_{n,m}$ and $G \in \min(P)$ be any minimal graph for P. Then

$$C^R(P) \geq |E(G)|.$$

Definition 2.4 *Let $\mathcal{L}$ be a list of graphs from $\mathcal{G}_{n,m}$. For each $G \in \mathcal{L}$ let us consider the sequence of degrees in colour clas V. Let $d_1 \geq d_2 \geq \ldots \geq d_n$ be the ordered list of degrees. If $(d_1, d_2, \ldots, d_n)$ is the lexicographically minimal sequence considering all the ordered lists then we refer to G as the V-*lexicographically first element *of $\mathcal{L}$*

Theorem 2.5 *(Yao's Method [7]) Let $P \in \mathcal{P}_{n,m}$ and G be the V-lexicographically first graph of $\min(P)$. Then*

$$C^R(P) = \Omega(\frac{d_{\max,V}(G)}{d_{av,V}(G)} \cdot |V|).$$

A very useful tool for proving lower bounds is dualiy. For every non-trivial, monotone boolean function f we can define the dual function f^D *as follows:*

$$f^D(\underline{x}) = \neg f(\neg \underline{x}).$$

It is easy to see that f^D is also non-trivial, monotone and $C^R(f^D) = C^R(f)$.

Definition 2.6 *(i) Let $G, H \in \mathcal{G}_n$ with vertex sets V and V', respectively. A* packing *is an identification between V and V' such that no edge of G is identified with any edge of H.*

(ii) Let $G, H \in \mathcal{G}_{n,m}$ with colour classes V, W and V', W', respectively. A bipartite packing *is an identification between V and V' and between W and W' such that no edge of G is identified with any edge of H.*

Lemma 2.7 *(Yao [6]) (i) If $P \in \mathcal{P}_n, G \in \min(P)$ and $H \in \min(P^D)$ then G and H can't be packed. (ii) If $P \in \mathcal{P}_{n,m}, G \in \min(P)$ and $H \in \min(P^D)$ then G and H can't be packed as bipartite graphs.*

3 Results

By a *covering* of a graph G we mean a subset K of V such that any edge of G is adjacent to at least one vertex in K. A covering K is *minimal* if G has no covering K' with $|K'| < |K|$.

The *width* of a graph G denoted by width(G) is the size of a minimal covering of G. The *trace* of a graph G denoted by trace(G) is the minimal number of edges we have to remove from G in order to decrease its width.

Now we extend these notions to monotone graph properties. The *width of a monotone graph property* P is defined as follows:

$$width(P) = \min\{width(G) | G \in \min(P)\}$$

The *trace of a monotone graph property* P is defined by

$$trace(P) = \min\{trace(G) | G \in \min(P) \text{ and } width(P) = width(G)\}$$

The following assertions show some fundamental properties of these notions.

Lemma 3.1 *If $P \in \mathcal{P}_n$ and $1 \le r < n$ is a fixed integer then*
(i) $width(P) \ge r$ iff $K^*_{n+1-r} \in P^D$
(ii) If $width(P) > r$ then

$$red(P|n-r) \in \mathcal{P}_{n-r}, width(red(P|n-r)) = width(P)-r, trace(red(P|n-r)) = trace(P).$$

Lemma 3.2 *If $P \in calP_n$ and $width(P) = 1$ then for any $G \in P^D, G$ has at least $\frac{1}{2}n \cdot (n-$ $trace(P))$ edges.*

Proof. Since P^D is a non-trivial, monoton graph property, it is sufficient to prove the statement for $G \in \min(P^D)$. Indeed, let $G \in \min(P^D)$ be arbitrary and let $H \in \min(P)$ such a graph for which width(H) = width(P) and trace(H) = trace(P) holds. With other words, H is a star with trace(P) mary edges. According to Lemma 2.7. G and H can't be packed. This implies $\delta(G) \ge |V(G)| - trace(H) = n - trace(H)$, therefore $|E(G)| \ge \frac{1}{2}n \cdot (n - trace(H))$.

Lemma 3.3 *For any $P \in \mathcal{P}_n$ the following assertions hold:*
(i) $C^R(P) \geq width(P) \cdot trace(P)$.
(ii) $C^R(P) \geq \frac{1}{2}(n+1- width(P)) \cdot (n+1-(width\ (P)+ trace(P)))$.
(iii) For any $0 < \varepsilon < 1$, *if* $width(P) \leq (1-\varepsilon) \cdot n$ *then* $C^R(P) \geq \frac{\varepsilon^3}{2-\varepsilon} n \cdot width(P)$.

Proof. Assertion (i) is a straightforward consequence of Theorem 2.3. To prove (ii) choose in Lemma 3.1. r =width $(P)-1$ and apply Lemma 3.2. to the reduced property re $\alpha(P|n-(\text{width } (P)-1))$. Finally Theorem 2.3. yields the result. If trace$(P) \geq \frac{\varepsilon^2}{2-\varepsilon} \cdot n$, then assertion (iii) follows from (i), else it can be proved, using (ii) and assumption width$(P) \leq (1-\varepsilon) \cdot n$.

Before we state our main results we apply this method to some special graph properties. For this reason let us denote the property that an n-vertex graph contains a Hamiltonian cycle by PH_n and the property that an n-vertex graph has a vertex colouring with k colours by $PC_{k,n}$.

Theorem 3.4

$$C^R(PH_n) \geq \frac{1}{8} \cdot (n^2 - 1)$$

$$C^R(PC_{n,k}) \geq \binom{n+1-k}{2}.$$

Proof. We have only to determine the width and trace of the given properties. The required values are:

$$width(PH_n) = \lceil \frac{n}{2} \rceil,$$

$$trace(PH_n) = \begin{cases} 1, & \text{if } n \text{ is odd} \\ 2, & \text{if } n \text{ is even} \end{cases}$$

Since $PC_{k,n}$ itself is not monotone, we consider instead of $PC_{k,n}$ the property $\neg PC_{k,n}$ which is non-trivial, monotone and obviously, $C^R(\neg PC_{k,n}) = C^R(PC_{k,n})$. It can be seen that the corresponding values are:

$$width(\neg PC_{k,n}) = k$$

$$trace(\neg PC_{k,n}) = 1.$$

The following theorem improves the known reductions of non-trivial, monotone graph properties to bipartite graph properties. Although King [3] has already stated a similar result, the new approach can help to prove better uniform lower bounds, since the reduction is to colour classes both of size $\Omega(n)$

Theorem 3.5 *The randomized decision tree complexity of any non-trivial, monotone graph property $P \in \mathcal{P}_n$ is*

$$C^R(P) \geq \min\{\frac{1}{40} \cdot n^{3/2}, C^R(\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil)\}.$$

Proof. We have only to consider the case that the property P can't be reduced to a non-trivial bipartite graph property bipart$P||\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil)$. This implies $K^*_{\lfloor \frac{n}{2} \rfloor} \in P$ or $K^*_{\lceil \frac{n}{2} \rceil} \in P^D$. Therefor, it remains only to prove, that for any $P \in \mathcal{P}_n$, if $K^*_{\lceil \frac{n}{2} \rceil} \in P$ then $C^R(P) \geq \frac{1}{40} \cdot n^{3/2}$ holds.

Let us suppose that we found a property $P \in \mathcal{P}_n$ with $K^*_{\lceil \frac{n}{2} \rceil} \in P$ and $C^R(P) < \frac{1}{40} \cdot n^{3/2}$. Let us construct the following sequence of induced graph properties

$$\{P_i | 0 \leq i \leq \lfloor \frac{1}{2} n^{1/2} \rfloor\}, P_i = ind(P|\lceil \frac{3}{4} n + \frac{1}{2} i \cdot n^{1/2} \rceil).$$

Since $K^*_{\lceil \frac{n}{2} \rceil} \in P$ and for any i P_i is an induced property of P on at least $\lceil \frac{3}{4} n \rceil$ vertices, P_i is non-trivial and

$$C^R(P_i) \leq C^R(P) < \frac{1}{40} n^{3/2} \tag{1}$$

$K^*_{\lceil \frac{n}{2} \rceil} \in P_0$ implies width$(P_0) \leq \lceil \frac{n}{2} \rceil - 1 \leq \frac{1}{2} n$. Assertion (iii) of Lemma 3.3. yields $C^R(P_0) \geq \frac{1}{20} n \cdot$ width(P_0). Hence

$$width(P_0) \leq \lfloor \frac{1}{2} n^{1/2} \rfloor. \tag{2}$$

Obviously $G \in P_i$ implies $G \in P_{i+1}$. Therefore

$$width(P_{i+1}) \leq width(P_i), i \geq 0. \tag{3}$$

Let us suppose, that for some $i \geq 0$ width(P_{i+1}) = width(P_i) holds. Then trace$(P_{i+1}) \leq$ trace(P_i). Now Lemma 3.3. yields

$$\begin{aligned}
C^R(P_{i+1}) &\geq \frac{1}{2}(\lceil \frac{3}{4} n + \frac{1}{2}(i+1) \cdot n^{1/2} \rceil + 1 - width(P_{i+1})) \cdot \\
&\quad (\lceil \frac{3}{4} n + \frac{1}{2}(i+1) \cdot n^{1/2} \rceil + 1 - (width(P_{i+1}) + trace(P_{i+1}))) \\
&\geq \frac{1}{2} \cdot (\frac{3}{4} n + \frac{1}{2} n^{1/2} - width(P_0)) \cdot \\
&\quad (\frac{3}{4} n + \frac{1}{2} i \cdot n^{1/2} + 1 - (width(P_i) + trace(P_i)) + \frac{1}{2} n^{1/2}) \\
&\geq \frac{3}{16} n^{3/2} > \frac{1}{40} n^{3/2},
\end{aligned}$$

which contradicts (1).

The sequence of positive integers $\{$width $(P_i) | 0 \leq i \leq \lfloor \frac{1}{2} n^{1/2} \rfloor\}$ therefore decreases strictly monotone, and so

$$width(P_0) \geq \lfloor \frac{1}{2} n^{1/2} \rfloor + 1,$$

which contradicts (2).

Since our assumption $C^R(P) < \frac{1}{40} n^{3/2}$ led to a contradiction we have completed the proof.

A straightforward consequence of the improved reduction is the following result.

Theorem 3.6 *For the randomized decision tree complexity of any subgraph isomorphism property $P_G \in \mathcal{P}_n$*

$$C^R(P_G) = \Omega(n^{3/2}).$$

Proof. According to Theorem 3.5., we have only to settle the case that $\text{bipart}(P_G|\lfloor\frac{n}{2}\rfloor, \lceil\frac{n}{2}\rceil)$ is nontrivial. Depending on $\text{width}(P_G)$ and $\text{trace}(P_G)$ we shall distinguish three cases.

Case 1. Assume that $\text{width}(P_G) \geq \frac{1}{4}n$. Since $\text{bipart}(P_G|\lfloor\frac{n}{2}\rfloor, \lceil\frac{n}{2}\rceil)$ is non-trivial, we get $\text{width}(P_G) \leq \frac{1}{2}n$ and assertion (iii) of Lemma 3.3. implies a lower bound of $\Omega(n^2)$.

Case 2. If $\text{width}(P_G) < \frac{1}{4}n$ and $\text{trace}(P_G) < \frac{2}{3}n$, then we can apply assertion (ii) of Lemma 2.5. and get also a lower bound of $\Omega(n^2)$.

Case 3. Suppose that $\text{width}(P_G) < \frac{1}{4}n$ and $\text{trace}(P_G) \geq \frac{2}{3}n$. Since $\text{trace}(P_G) \geq \frac{2}{3}n$ the corresponding bipartite graph property $\text{bipart}(P_G|\lfloor\frac{n}{2}\rfloor, \lceil\frac{n}{2}\rceil)$ has only such minimal graphs H for that $D_V(H) \geq \frac{1}{6}n$ holds. If $\text{bipart}(P_G|\lfloor\frac{n}{2}\rfloor, \lceil\frac{n}{2}\rceil)$ has a minimal graph with at least $n^{3/2}$ edges we can apply Theorem 2.3. Otherwise we can apply Theorem 2.5. In both cases we get a lower bound of $\Omega(n^{3//2})$ which completes the proof.

Before we prove the sharper version of Theorem 2.6. we consider some special bipartite graph properties. Let us denote by $S_{n,m}$ the graph which has one vertex of positive degree in V and m edges.

Lemma 3.7 *Let $P_S \in \mathcal{P}_{n,m}$ denote the property of containing a subgraph isomorph to $S_{n,m}$. Then*

$$C^R(P_S) \geq \frac{1}{2}m \cdot n.$$

Proof. (analogue to Yao [7]). We consider the dual property P_S^D, which is easy to see to contain exactly those graphs, that have no isolated nodes in colour class W. According to Lemma 2.2. wee choose as a "hard" input distribution the uniform distribution over all minimal graphs. Let be A an optimal deterministic decision tree, that computes our P_S^D. We denote by $X_i(G)$ the number of edges incident to w_i asked by A. Then

$$\begin{aligned} C^R(P_S^D) &\geq E(\sum_{i=1}^{m} X_i(G)) \\ &= \sum_{i=1}^{m} E(X_i(G)) \end{aligned}$$

Since for any value of i we have to find one edge out of n edges, we get

$$E(X_i(G)) \geq \frac{1}{2}n$$

and finally

$$C^R(P_S) \geq \frac{1}{2}m \cdot n.$$

Lemma 3.8 *Let $P \in \mathcal{P}_{n,m}$ such a property, that every $G \in \min(P)$ has exactly $k \leq \frac{1}{2}n$ vertices of positive degree in colour class V. Then*

$$C^R(P) \geq \frac{1}{6}m \cdot n.$$

Proof. We consider the reduced graph property $P' = red_V(P|n+1-k)$. Obviously, P' is non-trivial and $\min(P')$ contains up to isomorphy exactly one graph. This graph has exactly one vertex with positive degree (d) in the colour class V'. We distinguish two cases.

Case 1. Assume that $d \leq \frac{2}{3}m$. Since the minimal graphs of P' and P'^D can't be packed as bipartite graphs, any $G \in \min(P'^D)$ has at least $(n+1-k)\cdot(m+1-d) \geq \frac{1}{6}m \cdot n$ edges. Hence Theorem 2.1. implies the required lower bound.

Case 2. If $d > \frac{2}{3}m$ then let us consider the induced property $ind_W(P'|d)$ on colour classes of size $n+1-k$ and d, respectively. Since $ind_W(P'|d) = S_{n+1-k,d}$, Lemma 3.7. yields the statement.

Lemma 3.9 *The randomized decision tree complexity of any subgraph isomorphism property $P_G \in \mathcal{P}_n$ with width at most $\frac{2}{3}n$ fulfilles*

$$C^R(P_G) \geq \frac{1}{24}(n^2 - 1).$$

Proof. Depending on width(P_G) and trace(P_G) we distinguish six cases.

Case 1. If $\frac{1}{2}n \leq width(P_G) \leq \frac{2}{3}n$ and trace$(P_G) \geq \frac{1}{12}n$, then assertion (i) of Lemma 3.3. implies the lower bound.

Case 2. If $\frac{1}{2}n \leq$ width$(P_G) \leq \frac{2}{3}n$ and trace$(P_G) < \frac{1}{12}n$, then assertion (ii) of Lemma 3.3. implies the lower bound.

Case 3. If $K^*_{\lfloor n/2 \rfloor} \in P_G$, then width$(P_G)+$ trace$(P_G) \leq \frac{1}{n}$. Therefore, by assertion (ii) of Lemma 3.3., we obtain $C^R(P_G) \geq \frac{1}{8}n^2 \geq \frac{1}{24}n^2$.

So far we have considered all the cases, when P_G can't be reduced to a non-trivial bipartite graph property bipart$(P_G)|\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil)$.

Case 4. If $\frac{n}{2} \leq$ width$(P_G) < \frac{n}{2}$, then we can apply assertion (iii) of Lemma 3.3. for $\varepsilon = \frac{1}{2}$ and get the required lower bound.

Case 5. If width$(P_G) < \frac{n}{4}$ and trace$(P_G) \geq \frac{5}{6}n$, then G contains width(P_G) vertices with degree at least $\frac{5}{9}n$. In our reduction to the bipartite graph property bipart$P_G||\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil)$ we have to put them all into V. On the other hand, these vertices build a covering of the graph G. Hence G contains no edge independent of this vertex set. Therefore any minimal graph of the property bipart$(P_G||\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil)$ has exactly width(P_G) vertices of positive degree in V and Lemma 3.8. implies

$$C^R(P_G) \geq (bipart(P_G||\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil)) \geq \frac{1}{24} \cdot (n^2 - 1)$$

Case 6. If width$(P_G) < \frac{n}{4}$ and trace$(P_G) < \frac{5}{9}n$, then by assertion (ii) of Lemma 3.3., we get that

$$C^R(P_G) \geq \frac{1}{2} \cdot \frac{3}{4}n \cdot (\frac{3}{4}n - \frac{5}{9}n) \geq \frac{1}{24}n^2,$$

which completes the proof.

The following statement is an immediate consequence of this theorem and generalizes the results of Yao [6].

Assertion 3.10 *For every $\varepsilon > 0$ we can find a $\lambda > 0$ which depends only on ε, such that the randomized decision tree complexity of any subgraph isomorphism property $P_G \in \mathcal{P}_n$ with $\overline{d}(G) \leq \varepsilon$ fulfilles:*

$$C^R(P_G) \geq \lambda(\varepsilon) \cdot n^2.$$

After finishing this manuscript the author has learnt that M. Karpinski et al [2] independently proved Theorem 3.5.

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Received April 25, 1989.