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Initial and Final Congruences

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Abstract

The paper contains some results which establish connections between different types of algebraic structures which appeared in the process of algebrization of the theory of flowchart schemes [3,4,5,6,7]. Two basic constructions [2,4] are used, first separately, and afterwards combinations between them or their duals, in order to obtain the theorems.

1 Introduction

We recall the definitions of the algebraic structures which will be used subsequently in the paper.

Let B be a category whose objects form a monoid $(M,+,0)$ and such that for each $a,b,c,d \in M$ a sum operation is given

$$+ : B(a,b) \times B(c,d) \longrightarrow B(a+c,b+d)$$

B is called a *strict monoidal category* (smc for short) if Axioms 1-4 from Table 1 are satisfied. If B satisfies the weaker axioms 1, 2, 3, 4a and 4b of Table 1, then B is called a *nonpermutable strict monoidal category* (nsmc for short).

$1. f+(g+h)=(f+g)+h$
$2. f+I_0=f=I_0+f$
$3. I_a+I_b=I_{a+b}$

$4.(f+g)(u+v)=fu+gv$	$4a.(f+g)(I_b+v)=f+gv$ $4b.(f+I_d)(u+v)=fu+v$ $4c.(I_{c+d}+f)({}^cX^d+I_b)={}^cX^d+f$
$5.{}^aX^c(g+f){}^dX^b=f+g$ for $f:a\longrightarrow b$ and $g:c\longrightarrow d$	$5a.{}^aX^c(I_c+f){}^cX^b=f+I_c$ for $f:a\longrightarrow b$

$6.{}^aX^0=I_a$
$7.{}^aX^{b+c}=({}^aX^b+I_c)(I_b+{}^aX^c).$

Table 1: Axioms for ssmc and snsmc

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Suppose that for every $a, b \in M$ some distinguished morphisms ${}^aX^b \in B(a+b, b+a)$ are given. An smc is called *symmetric* (ssmc for short) if Axioms 5, 6, 7 of Table 1 are satisfied. An nsmc is called *symmetric* (*snsmc* for short) if Axioms 4c, 5a, 6 and 7 are satisfied. Obviously, every ssmc is an snsmc.

Let us notice that an snsmc which satisfies *either* Axiom 4 *or* Axiom 5 of Table 1 is an ssmc. Moreover, this remains true if we replace Axiom 4 (or Axiom 5) with the following weaker axiom:

$$f + g = (I_a + g)(f + I_d)$$

for every $f \in B(a, b)$ and $g \in B(c, d)$.

In an snsmc an *aα-morphism* is a composite of morphisms of the form $I_a + {}^bX^c + I_d$. Axiom 4 of Table 1 is satisfied in an snsmc for g or u $a\alpha$-morphisms. In an snsmc B we denote by B_a the subcategory of $a\alpha$-morphisms and we notice that B_a is an ssmc. Note also that every $a\alpha$-morphism is an isomorphism, its inverse being also an $a\alpha$-morphism.

The concept of an xy-ssmc depends on two parameters, $x \in \{a, b, c, d\}$ and $y \in \{\alpha, \beta, \gamma, \delta\}$. For every $e \in M$ we will use the distinguished morphisms $\top_e \in B(0, e)$, $\perp^e \in B(e, 0)$, $\vee_e \in B(e+e, e)$ and $\wedge^e \in B(e, e+e)$. Table 2 shows for every value of the parameters the distinguished morphisms which are involved. The axioms which are satisfied by the distinguished morphisms are chosen from Table 3 for each xy case according to the rule: select all those axioms (and only those) in which the distinguished morphisms of the xy case appear.

x	distinguished morphisms	y	distinguished morphisms
a	none	α	none
b	$\perp^e$	β	$\top_e$
c	$\wedge^e$	γ	$\vee_e$
d	$\perp^e$ and $\wedge^e$	δ	$\top_e$ and $\vee_e$

Table 2: Distinguished morphisms for xy-ssmc and xy-snsmc

Thus, an ssmc (snsmc) will be called an *xy-ssmc (xy-snsmc)* if every object $e \in M$ is endowed with the distinguished morphisms which appear in Table 2 corresponding to the xy case and if all axioms of Table 3 which contain only these distinguished morphisms are fulfilled.

Notice that in an xy-ssmc axioms $\mathbf{P}\top, \mathbf{P}\perp, \mathbf{P}\vee$, and $\mathbf{P}\wedge$ are automatically satisfied as a consequence of Axiom 4 in Table 1.

$A. (\vee_a + I_a)\vee_a = (I_a + \vee_a)\vee_a$ $\qquad$ $A^\circ. \wedge^a (\wedge^a + I_a) = \wedge^a(I_a + \wedge^a)$

$B. {}^aX^a\vee_a = \vee_a$ $\qquad$ $B^\circ. \wedge^a {}^aX^a = \wedge^a$

$C. (\top_a + I_a)\vee_a = I_a$ $\qquad$ $C^\circ. \wedge^a (\perp^a + I_a) = I_a$

$D. \vee_a \perp^a = \perp^a + \perp^a$ $\qquad$ $D^\circ. \top_a\wedge^a = \top_a + \top_a$

$E. \top_a\perp^a = I_0$

$F. \vee_a \wedge^a = (\wedge^a + \wedge^a)(I_a + {}^aX^a + I_a)(\vee_a + \vee_a)$

$G. \wedge^a \vee_a = I_a$

$SV1. \top_0 = I_0$ $\qquad$ $SV1^\circ. \perp^0 = I_0$

$SV2. \top_{a+b} = \top_a + \top_b$ $\qquad$ $SV2^\circ. \perp^{a+b} = \perp^a + \perp^b$

$SV3. \vee_0 = I_0$ $\qquad$ $SV3^\circ. \wedge^0 = I_0$

$SV4. \vee_{a+b} = (I_a + {}^bX^a + I_b)(\vee_a + \vee_b)$ $\qquad$ $SV4^\circ. \wedge^{a+b} = (\wedge^a + \wedge^b)(I_a + {}^aX^b + I_b)$

$P\top. g(\top_a + I_c) = \top_a + g$ $\qquad$ $P\perp. (I_a + g)(\perp^a + I_c) = \perp^a + g$

$P\vee. (I_{a+a} + g)(\vee_a + I_c) = \vee_a + g$ $\qquad$ $P\wedge. (I_a + g)(\wedge^a + I_c) = \wedge^a + g$

for $g : b \longrightarrow c$

Table 3: Axioms for xy-ssmc and xy-snsmc

$S\top. \top_a f = \top_b$ $\qquad$ $S\perp. f\perp^b = \perp^a$

$S\vee. (f + f)\vee_b = \vee_a f$ $\qquad$ $S\wedge. \wedge^a (f + f) = f\wedge^b$

for $f : a \longrightarrow b$

Table 4: Axioms for strong xy-ssmc

Let us consider the order $<_L$ on $\{a, b, c, d\}$ given by $a <_L b <_L d$ and $a <_L c <_L d$, and the same order $<_G$ on the corresponding Greek letters, so that, e.g. $\alpha <_G \beta <_G \delta$. For $x' \leq_L x$ and $y' \leq_G y$, we define an *$x'y'$-strong xy-ssmc* to be an xy-ssmc in which all the axioms in Table 4 corresponding to the $x'y'$ case hold. A *strong xy-ssmc* will be, by definition, an xy-strong xy-ssmc.

Suppose that in an snsmc B we are given, for each $a, b, c \in M$, an unary operation

$$\uparrow^a \_ : B(a + b, a + c) \longrightarrow B(b, c)$$

called *(left) feedback.*

A *biflow (flow)* is an ssmc (snsmc) endowed with a feedback which satisfies all the axioms in Table 5. As we will use sometimes the right feedback $\_ \uparrow^a$: $B(b + a, c + a) \longrightarrow B(b, c)$ we mention the connections between the two feedbacks:

$$f \uparrow^a = \uparrow^a ({}^aX^b f \, {}^cX^a) \qquad \text{for } f : b + a \longrightarrow c + a.$$

$$\uparrow^a f = ({}^bX^a f \, {}^aX^c) \uparrow^a \quad \text{for } f : a + b \longrightarrow a + c.$$

1. $f(\uparrow^a g)h = \uparrow^a [(I_a + f)g(I_a + h)]$
2. $\uparrow^a f + I_d = \uparrow^a (f + I_d)$
3. $\uparrow^{a+b} [f({}^bX^a + I_d)] = \uparrow^{b+a} [({}^bX^a + I_c)f]$
   for $f : a + b + c \longrightarrow b + a + d$
4. $\uparrow^a \uparrow^b f = \uparrow^{b+a} f$
5. $\uparrow^a I_a = I_0$
6. $\uparrow^a {}^aX^a = I_a$.

Table 5: Axioms for feedback

All the algebraic structures previously defined form categories whose morphisms are functors which are monoid morphisms on objects and which preserve the additional algebraic structure. Sometimes we will be interested in certain subcategories, namely those in which the monoid of objects, M, is kept fixed (called M-smc, M-nsmc,..., M-biflow), and where the morphisms are object-preserving functors (called M-smc morphism, ..., M-biflow morphisms). These subcategories are equational varieties in the sense of the many-sorted universal algebra. Examples of the above algebraic structures may be found in [4,5,6].

Some of the results in the rest of the paper provide the construction of left adjoints for the following forgetful functors:

a) for $x \in \{b, d\}$ and $y \in \{\alpha, \beta, \gamma, \delta\}$
$b\alpha$-strong xy-ssmc $\longrightarrow$ xy-ssmc
biflow over a $b\alpha$-strong xy-ssmc $\longrightarrow$ biflow over an xy-ssmc

b) for $x \in \{b, d\}$ and $y \in \{\beta, \delta\}$
$b\beta$-strong xy-ssmc $\longrightarrow$ xy-ssmc
biflow over a $b\beta$-strong xy-ssmc $\longrightarrow$ biflow over an xy-ssmc

c) $b\alpha$-ssmc $\longrightarrow$ ssmc
biflow over a $b\alpha$-ssmc $\longrightarrow$ biflow

d) for $y \in \{\alpha, \beta, \gamma, \delta\}$
$b\alpha$-strong by-ssmc $\longrightarrow$ ay-ssmc
strong by-ssmc $\longrightarrow$ strong ay-ssmc
biflow over a $b\alpha$-strong by-ssmc $\longrightarrow$ biflow over an ay-ssmc
biflow over a strong by-ssmc $\longrightarrow$ biflow over a strong ay-ssmc

e) $b\alpha$-ssmc and $a\beta$-ssmc $\longrightarrow$ ssmc
biflow over a $b\alpha$-ssmc and $a\beta$-ssmc $\longrightarrow$ biflow.

Even if the existence of left adjoints follows from general principles, our constructions are more effective than the general ones.

Table 6 represents the structure of the paper as a graph, where each section (vertex) depends only on the sections at higher levels with which it is linked.

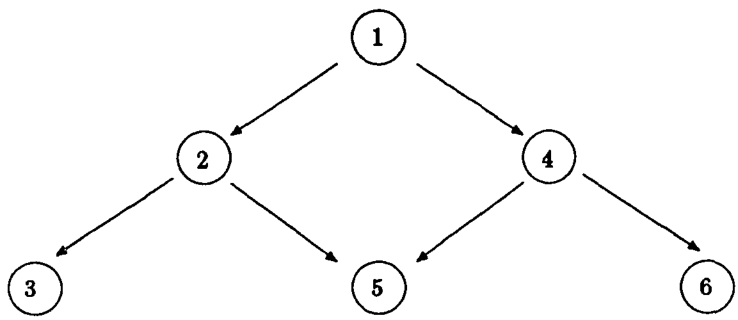

Table 6: The structure of the paper

2 Final congruences

In this section we will show that the congruence relation introduced by Bloom and Tindell [2] in an algebraic theory is useful also in the study of more general algebraic structures. We mention that, instead of the name "zero-congruence" used in [2], we will use the name "final congruence", one reason being that it can be dualized into "initial congruence".

In an smc B a congruence relation $\equiv$ is called *final* if $f \equiv g$ for every $f, g \in B(a, 0)$. Notice that if 0 is a weak final object in B, i.e. $B(a, 0) \neq \emptyset$ for every $a \in M$, then factorization with a final congruence makes 0 a final object.

Definition 2.1 [2] *For every $f, g \in B(a, b)$ we have $f\mathbf{P}g$ iff there exist: an object $x \in M$ and morphisms $h \in B(a, b + x)$ and $u, v \in B(x, 0)$, such that*

$$f = h(I_b + u) \text{ and } g = h(I_b + v).$$

Since a final congruence relation identifies any two morphisms u and v in $B(x, 0)$ it obviously includes the relation $\mathbf{P}$.

Note that for every $a, b \in M$ the relation $\mathbf{P}$ is reflexive and symmetric on $B(a, b)$.

The above defined relation was introduced in [2], and was used for different purposes in [1].

Lemma 2.2 *In an ssmc, relation $\mathbf{P}$ is compatible with composition and sum.*

Proof. With the notations of Definition 2.1, let $f\mathbf{P}g$.

a) Compatibility with composition. Suppose $p\mathbf{P}q$ in $B(b, c)$, that is there exist $j \in B(b, c + y)$ and $w, t \in B(y, 0)$ such that $p = j(I_c + w)$ and $q = j(I_c + t)$. Notice that

$$fp = h(I_b + u)j(I_c + w) = [h(j + I_x)](I_c + w + u)$$

and similarly,

$$gq = [h(j + I_x)](I_c + t + v).$$

b) Compatibility with sum. Suppose $p\mathbf{P}q$ in $B(c,d)$, i.e. there exist $j \in B(c, d+y)$ and $w, t \in B(y,0)$ such that $p = j(I_d + w)$ and $q = j(I_d + t)$. Notice that

$$f + p = (h + j)(I_b + u + I_d + w) = [(h + j)(I_b + {}^xX^d + I_y)](I_{b+d} + u + w)$$

and similarly,

$$g + q = [(h + j)(I_b + {}^xX^d + I_y)](I_{b+d} + v + t).$$

It is known that in a Σ-algebra the transitive closure of a reflexive and symmetric relation, compatible with the operations, is a congruence. This implies that $\mathbf{P}^+$, the transitive closure of $\mathbf{P}$, is an ssmc congruence.

Proposition 2.3 *The congruence $\mathbf{P}^+$ is the least final congruence.*

To simplify the notation in the following, each time $\mathbf{R}$ is a reflexive and symmetric relation on A, we will denote by $A/\mathbf{R}$ the quotient of A by $\mathbf{R}^+$.

Proposition 2.4 *If B is a $b\alpha$-ssmc, then $B/\mathbf{P}$ is a strong $b\alpha$-ssmc. If C is a strong $b\alpha$-ssmc and $G : B \longrightarrow C$ is a $b\alpha$-ssmc morphism, then there exists a unique $b\alpha$-ssmc morphism $H : B/\mathbf{P} \longrightarrow C$ such that $G = F_B \bullet H$, where $F_B : B \longrightarrow B/\mathbf{P}$ is the canonical factorization morphism.*

Proof. It is sufficient to notice that $f\mathbf{P}g$ implies $G(f) = G(g)$.

The above proposition tells us that the forgetful functor from strong $b\alpha$-ssmc to $b\alpha$-ssmc has a left adjoint. The same construction can be used in other cases as well, giving us the left adjoints for the forgetful functors from the categories of $b\alpha$-strong xy-ssmc to the categories of xy-ssmc, for every $x \in \{b, d\}$ and every $y \in \{\alpha, \beta, \gamma, \delta\}$. If we note furthermore that in a biflow, relation $\mathbf{P}$ is compatible with the feedback, then the above result remains true for biflows over an xy-ssmc.

3 Congruences which are simultaneously initial and final

The concept of initial congruence is dual to that of final congruence, and the construction and results of the previous section can be readily dualized.

The purpose of this section is to show that the factorization from the previous section and its dual can be merged into a single factorization.

Definition 3.1 *In an ssmc B we define relation $\mathbf{R}$ by the following: $f\mathbf{R}g$ in $B(a,b)$ iff there exist objects $x, y \in M$ and morphism $h \in B(a+y, b+x), u, v \in B(x,0)$ and $p, q \in B(0,y)$ such that we have decompositions*

$$\begin{aligned} f &= (I_a + p)h(I_b + u) \\ g &= (I_a + q)h(I_b + v). \end{aligned}$$

Obviously, $\mathbf{R}$ is reflexive and symmetric. Since an initial and final congruence relation identifies any two morphisms u and v in $B(x,0)$ and any two morphisms p and q in $B(0,y)$ it obiously includes the relation $\mathbf{R}$.

Lemma 3.2 *The relation $\mathbf{R}$ is compatible with composition and sum.*

Proof. With the above notation let $f\mathbf{R}g$ in $B(a,b)$.

a) Compatibility with composition. Suppose we have $f'\mathbf{R}g'$ in $B(b,c)$, i.e. there exist $h' \in B(b+y', c+x')$, $u',v' \in B(x',0)$, $p',q' \in B(0,y')$ such that $f' = (I_b+p')h'(I_c+u')$ and $g' = (I_b+q')h'(I_c+v')$.

It follows that

$$\begin{aligned} ff' &= (I_a+p)h(I_b+u)(I_b+p')h'(I_c+u') \\ &= (I_a+p+p')(h+I_{y'})(I_b+u+I_{y'})h'(I_c+u') \\ &= [I_a+(p+p')][(h+I_{y'})(I_b+{}^{x}X^{y'})(h'+I_x)][I_c+(u'+u)], \end{aligned}$$

and similarly,

$$gg' = [I_a+(q+q')][(h+I_{y'})(I_b+{}^{x}X^{y'})(h'+I_x)][I_c+(v'+v)],$$

which imply $ff'\mathbf{R}gg'$.

b) Compatibility with sum. Suppose that $f'\mathbf{R}g'$ in $B(c,d)$, i.e. there exist $h' \in B(c+y', d+x'), u',v' \in B(x',0), p',q' \in B(0,y')$ such that

$$f' = (I_c+p')h'(I_d+u') \text{ and } g' = (I_c+q')h'(I_d+v').$$

It follows that

$$\begin{aligned} f+f' &= (I_a+p+I_c+p')(h+h')(I_b+u+I_d+u') \\ &= [I_{a+c}+(p+p')][(I_a+{}^{c}X^{y}+I_{y'})(h+h')(I_b+{}^{x}X^{d}+I_{x'})][I_{b+d}+(u+u')], \end{aligned}$$

and

$$g+g' = [I_{a+c}+(q+q')][(I_a+{}^{c}X^{y}+I_{y'})(h+h')(I_b+{}^{x}X^{d}+I_{x'})][I_{b+d}+(v+v')],$$

which imply $(f+f')\mathbf{R}(g+g')$.

Proposition 3.3 *The congruence* $\mathbf{R}^+$ *is the least initial and final congruence.*

Proposition 3.4 *If B is a $b\beta$-ssmc, then $B/\mathbf{R}$ is a strong $b\beta$-ssmc. If C is a strong $b\beta$-ssmc and $G : B \longrightarrow C$ is a $b\beta$-ssmc morphism, then there exists a unique $b\beta$-ssmc morphism $H : B/\mathbf{R} \longrightarrow C$ such that $G = F_B \bullet H$, where $F_B : B \longrightarrow B/\mathbf{R}$ is the canonical factorization morphism.*

The same factorisation gives us the left adjoints for the forgetful functors from the categories of $b\beta$-strong xy-ssmc to the categories of xy-ssmc, for every $x \in \{b,d\}$ and every $y \in \{\beta,\delta\}$. Since in any biflow, $\mathbf{R}$ is compatible with the feedback, the result holds also for biflows over the above xy-ssmc-ies.

4 The adjunction of $\perp$

We recall here briefly, following [4], the construction which associates to every ssmc a $b\alpha$-ssmc. We give some motivation. First, we mention the following identities in a $b\alpha$-ssmc:

a) $f(\perp^x + I_b)g(\perp^y + I_c) = [f(I_x + g)](\perp^{x+y} + I_c)$
for $f : a \longrightarrow x + b$ and $g : b \longrightarrow y + c$.

b) $f(\perp^x + I_b) + g(\perp^y + I_d) = [(f + g)(I_x + {}^bX^y + I_d)](\perp^{x+y} + I_{b+d})$
for $f : a \longrightarrow x + b$ and $g : c \longrightarrow y + d$.

c) $f = f(\perp^0 + I_b)$ for $f : a \longrightarrow b$.

d) $\perp^x = I_x(\perp^x + I_0)$.

To obtain a $b\alpha$-ssmc from an ssmc B we have to add for every object x a new morphism $\perp^x : x \longrightarrow 0$. Consequently, it will be necessary to add for every morphism $f : a \longrightarrow x+b$ the morphism $f(\perp^x + I_b)$. The above identities imply that this suffices for our purpose. We represent the newly added morphism $f(\perp^x + I_b)$ as a pair $(f, x) : a \longrightarrow b$. Since in any ssmc $j\perp^x = \perp^y$ for every $a\alpha$-morphism $j : y \longrightarrow x$, we have $g(\perp^y + I_b) = g(j + I_b)(\perp^x + I_b)$ for every $g : a \longrightarrow y + b$. We deduce that the pairs (g, y) and $(g(j + I_b), x)$ represent the same morphism. Therefore we shall need a factorization which identifies them in order to accomplish our construction.

Let B be an ssmc with $(M, +, 0)$ the monoid of its objects. Consider category $K(B)$, having the same objects as B, with morphism defined for every $a, b \in M$ by

$$K(B)(a, b) := \{(f, x) | x \in M, f \in B(a, x + b)\}$$

and with composition defined by

$$(f, x)(g, y) := (f(I_x + g), x + y).$$

Note that the identity morphism of $a \in M$ in $K(B)$ is $(I_a, 0)$.

$K(B)$ becomes an snsmc defining the sum of $(f, x) : a \longrightarrow b$ and $(g, y) : c \longrightarrow d$ to be

$$(f, x) + (g, y) := ((f + g)(I_x + {}^bX^y + I_d), x + y)$$

and the distinguished morphism ${}^aX^b$ as $({}^aX^b, 0)$.

In $K(B)$ the distinguished morphisms $\perp^a := (I_a, a) \in K(B)(a, 0)$ have the following properties: $\perp^0 = I_0$ and $\perp^{a+b} = \perp^a + \perp^b$.

We define $I_B : B \longrightarrow K(B)$ as being the identity on objects and mapping every morphism f of B to $I_B(f) = (f, 0)$. I_B will be an M-snsmc morphism, and for every $(f, x) \in K(B)(a, b)$ we will have $(f, x) = I_B(f)(\perp^x + I_b)$, which is called the canonical decomposition of (f, x).

Definition 4.1 [4] *For every $a, b \in M$ we define a relation $\sim$ on $K(B)(a, b)$ in the following manner : $(f, x) \sim (g, y)$ in $K(B)(a, b)$ iff there exists an $a\alpha$-morphism $j \in B_a(y, x)$ such that $f = g(j + I_b)$.*

Note that $\sim$ is a congruence and that $K(B)/\sim$ is a $b\alpha$-ssmc.

The ssmc morphism $K_B : B \longrightarrow K(B)/\sim$ is by definition the composite of I_B with the canonical factorization morphism.

Proposition 4.2 *For every $b\alpha$-ssmc C and every ssmc morphism $F : B \longrightarrow C$ there exists a unique $b\alpha$-ssmc morphism $H : K(B)/\sim \longrightarrow C$ such that $F = K_B \bullet H$.*

Suppose now B is a biflow. Defining in $K(B)$ the right feedback of (f,x) : $a+c \longrightarrow b+c$ to be

$$(f,x)\uparrow^c = (f\uparrow^c, x)$$

we note that $K(B)$ becomes a flow. Because congruence $\sim$ is a flow congruence, $K(B)/\sim$ becomes a biflow over a $b\alpha$-ssmc and K_B becomes an M-biflow morphism. The next proposition is a version of Proposition 4.2.

Proposition 4.3 *For every biflow over a $b\alpha$-ssmc C and every biflow morphism $F : B \longrightarrow C$ there exists a unique biflow and $b\alpha$-ssmc morphism $H : K(B)/\sim \longrightarrow C$ such that $F = K_B \bullet H$.*

5 The strong adjunction of $\perp$

The results of Sections 2 and 4 imply that the passage from an ssmc B to a strong $b\alpha$-ssmc is a three step contruction: we construct first $K(B)$ and then we factor succesively through $\sim$ and $\mathbf{P}^+$. In this section we show that the two succesive factorzations can be replaced by a single one.

We give some motivation for the next definition. Since in any strong $b\alpha$-ssmc we have the identity $g(q+I_b)(\perp^z + I_b) = g(\perp^y + I_b)$ for every $g : a \longrightarrow y+b$ and $q : y \longrightarrow z$, we deduce that the pairs (g,y) and $(g(q+I_b),z)$ from $K(B)$ represent the same morphism. Analogously, for $f : a \longrightarrow x+b$ and $p : x \longrightarrow z$, the pairs (f,x) and $(f(p+I_b),z)$ represent the same morphism. Therefore, the equality $f(p+I_b) = g(q+I_b)$ is a sufficient condition to identify the pairs (f,x) and (g,y).

Definition 5.1 *For $(f,x),(g,y) \in K(B)(a,b)$ we say that $(f,x)\mathbf{Q}(g,y)$ iff there exist an object $z \in M$ and morphisms $p \in B(x,z)$ and $q \in B(y,z)$ such that $f(p+I_b) = g(q+I_b)$.*

Notice that $\mathbf{Q}$ is a reflexive and symmetric relation.

Lemma 5.2 *Relation $\mathbf{Q}$ is compatible with composition and sum.*

Proof. With the above notation, suppose $(f,x)\mathbf{Q}(g,y)$.

a) Compatibility with composition. Let $(f',x')\mathbf{Q}(g',y')$ in $K(B)(b,c)$, i.e. there exist an object $z' \in M$ and morphisms $p' \in B(x',z')$ and $q' \in B(y',z')$ such that $f'(p'+I_c) = g'(q'+I_c)$. We note that

$$\begin{aligned}[(f(I_x+f')][(p+p')+I_c] &= f(p+I_b)(I_x+f'(p'+I_c))\\ &= g(q+I_b)(I_x+g'(q'+I_c))\\ &= [g(I_y+g')][(q+q')+I_c],\end{aligned}$$

which implies that $[(f,x)(f',x')]\mathbf{Q}[(g,y)(g',y')]$.

b) Compatibility with sum. Suppose $(f',x')\mathbf{Q}(g',y')$ in $K(B)(c,d)$, i.e. there exist an object $z' \in M$ and morphisms $p' \in B(x',z')$ and $q' \in B(y',z')$ such that $f'(p'+I_d) = g'(q'+I_d)$. We note that

$$\begin{aligned}&[(f+f')(I_x + {}^bX^{x'} + I_d)][(p+p')+I_{b+d}]\\ &= [f(p+I_b) + f'(p'+I_d)](I_z + {}^bX^{z'} + I_d)\\ &= [[g(q+I_b) + g'(q'+I_d)](I_z + {}^bX^{z'} + I_d)\\ &= [(g+g')(I_y + {}^bX^{y'} + I_d)][(q+q')+I_{b+d}],\end{aligned}$$

which implies $[(f,x)+(f',x')]\mathbf{Q}[(g,y)+(g',y')]$.

It follows that the transitive closure $\mathbf{Q}^+$ of $\mathbf{Q}$ is a congruence. We denote by Q_B the composite of the morphism I_B, defined in Section 4, with the canonical factorization morphism $K(B) \longrightarrow K(B)/\mathbf{Q}$.

Proposition 5.3 *$K(B)/\mathbf{Q}$ is a strong bα-ssmc. For every strong bα-ssmc C and for every ssmc morphism $F : B \longrightarrow C$, there exists a unique bα-ssmc morphism $H : K(B)/\mathbf{Q} \longrightarrow C$, such that $F = Q_B \bullet H$.*

Proof. We show first that $K(B)/\mathbf{Q}$ is an ssmc. For $(f,x) \in K(B)(a,b)$ and $(g,y) \in K(B)(c,d)$, taking into account that

$$[I_a + (g,y)][(f,x)+I_d] = ((f+g)({}^{x+b}X^y + I_d), y+x),$$

we note that there exists ${}^yX^x \in B(y+x, x+y)$ such that

$$(f+g)({}^{x+b}X^y + I_d)({}^yX^x + I_{b+d}) = (f+g)(I_x + {}^bX^y + I_d),$$

and thus $[I_a + (g,y)][(f,x)+I_d]\mathbf{Q}[(f,x)+(g,y)]$.

The existence of the distinguished morphisms $\perp^a = (I_a, a) \in K(B)(a,0)$ and their properties make $K(B)/\mathbf{Q}$ a bα-ssmc. To prove that it is strong, let $(f,x) \in K(B)(a,0)$. The equality $f(I_x + I_0) = I_a(f + I_0)$ implies $(f,x)\mathbf{Q}\perp^a$.

Let $F : B \longrightarrow C$ be an ssmc morphism, with C a strong bα-ssmc. We define $G : K(B) \longrightarrow C$ in the following way:

$$\begin{aligned} G(a) &:= F(a), \text{ for every object } a \in M,\\ G(f,x) &:= F(f)(\perp^{F(x)} + I_{F(b)}), \text{ for every } (f,x) \in K(B)(a,b). \end{aligned}$$

It follows that G is the unique snsmc morphism such that $F = I_B \bullet G$ and $G(\perp^a) = \perp^{G(a)}$ for every $a \in M$.

We now prove that $(f,x)\mathbf{Q}(g,y)$ in $K(B)(a,b)$ implies $G(f,x) = G(g,y)$. Let $p \in B(x,z)$ and $q \in B(y,z)$ be such that $f(p+I_b) = g(q+I_b)$. Applying F and then composing on the right with $\perp^{F(z)} + I_{F(b)}$ we deduce that

$$F(f)(\perp^{F(x)} + I_{F(b)}) = F(g)(\perp^{F(y)} + I_{F(b)}).$$

So, there exists $H : K(B)/\mathbf{Q} \longrightarrow C$ a unique bα-ssmc morphism such that $F = Q_B \bullet H$.

Corollary 5.4 *$(K(B)/\sim)/\mathbf{P}$ is isomorphic to $K(B)/\mathbf{Q}$.*

Proposition 5.5 *If B is an ay-ssmc, where $y \in \{\alpha,\beta,\gamma,\delta\}$, then $K(B)/\mathbf{Q}$ is a bα-strong by-ssmc and $Q_B : B \longrightarrow K(B)/\mathbf{Q}$ is an ay-ssmc morphism. If C is a bα-strong by-ssmc and $F : B \longrightarrow C$ is an ay-ssmc morphism, then the unique bα-ssmc morphism $H : K(B)/\mathbf{Q} \longrightarrow C$, such that $F = Q_B \bullet H$, given by Proposition 5.3, is a by-ssmc morphism.*

Proof. Case $y = \alpha$ is precisely Proposition 5.3. For the remainder of the cases, from the distinguished morphisms of B, $\top_a \in B(0,a)$ or $\vee_a \in B(a+a,a)$ we obtain the distinguished morphisms of $K(B)/\mathbf{Q}$, $Q_B(\top_a)$ or $Q_B(\vee_a)$.

The axioms fulfilled by $\top_a$ and/or $\vee_a$ in B, will be also fulfilled in $K(B)/\mathbf{Q}$. The only axioms which remain to be verified are those relating $\perp^a$ with $Q_B(\top_a)$ and $Q_B(\vee_a)$, that is

$$Q_B(\top_a)\perp^a = I_0 \text{ and } Q_B(\vee_a)\perp^a = \perp^a + \perp^a.$$

Their validity is a consequence of the fact that $K(B)/\mathbf{Q}$ is $b\alpha$-strong.

Proposition 5.6 *If B is a strong ay-ssmc, where $y \in \{\alpha,\beta,\gamma,\delta\}$, then $K(B)/\mathbf{Q}$ is a strong by-ssmc. If C is a strong by-ssmc and $F : B \longrightarrow C$ is an ay-ssmc morphism, then there exists a unique by-ssmc morphism $H : K(B)/\mathbf{Q} \longrightarrow C$ such that $F = Q_B \bullet H$.*

Proof. We prove only the first assertion. Assume $(f,x) \in K(B)(a,b)$.

For the distinguished morphism $\top_a$ of B, using the equalities

$$I_B(\top_a)(f,x) = (\top_a,0)(f,x) = (\top_a(I_0+f),x) = (\top_a f,x) = (\top_{x+b},x)$$

and noting that

$$\top_{x+b}(I_x+I_b) = \top_b(\top_x+I_b),$$

it follows that $[I_B(\top_a)(f,x)]\mathbf{Q}I_B(\top_b)$.

For the distinguished morphisms $\vee_a$ of B, using the equalitites

$$I_B(\vee_a)(f,x) = (\vee_a f,x)$$

$$[(f,x)+(f,x)]I_B(\vee_b) = ((f+f)(I_x+{}^bX^x+I_b)(I_{x+x}+\vee_b),x+x)$$

and noting that

$$[(f+f)(I_x+{}^bX^x+I_b)(I_{x+x}+\vee_b)](\vee_x+I_b) = \vee_a f(I_x+I_b),$$

we deduce that $[((f,x)+(f,x))I_B(\vee_b)]\mathbf{Q}[I_B(\vee_a)(f,x)]$.

Suppose now that B is a biflow. Because the relation $\mathbf{Q}$ is compatible with the feedback, it follows that $K(B)/\mathbf{Q}$ is a biflow over a strong $b\alpha$-ssmc, and we obtain the corresponding version of Propositions 5.5 and 5.6.

Proposition 5.7 *If B is a biflow over an ay-ssmc, where $y \in \{\alpha,\beta,\gamma,\delta\}$, then $K(B)/\mathbf{Q}$ is a biflow over a $b\alpha$-strong by-ssmc and Q_B is a biflow and ay-ssmc morphism. If C is a biflow over a $b\alpha$-strong by-ssmc and $F : B \longrightarrow C$ is a biflow and ay-ssmc morphism, then there exists a unique biflow and by-ssmc morphism $H : K(B)/\mathbf{Q} \longrightarrow C$, such that $F = Q_B \bullet H$.*

Proposition 5.8 *If B is a biflow over a strong ay-ssmc, where $y \in \{\alpha,\beta,\gamma,\delta\}$, then $K(B)/\mathbf{Q}$ is a biflow over a strong by-ssmc. If C is a biflow over a strong by-ssmc and $F : B \longrightarrow C$ is a biflow and ay-ssmc morphism, then there exists a unique biflow and by-ssmc morphism $H : K(B)/\mathbf{Q} \longrightarrow C$ such that $F = Q_B \bullet H$.*

6 The simultaneous adjunction of $\top$ and $\bot$

In this section we show that the construction of Section 4, together with its dual, can be merged into a single one.

To understand the construction which follows we mention that for every $f : a + y \longrightarrow x + b$ the triple (y, f, x) represents the morphism $(I_a + \top_y)f(\bot^x + I_b)$, where $\bot^x : x \longrightarrow 0$ and $\top_y : 0 \longrightarrow y$ are the newly added morphisms. The definitions for composition and sum are based on the following identities which hold in any ssmc which is simultaneously an $a\beta$-ssmc and a $b\alpha$-ssmc:

$$a)\quad (I_a + \top_y)f(\bot^x + I_b)(I_b + \top_z)g(\bot^w + I_c) = (I_a + \top_{y+z})[(f + I_z)(I_x + g)](\bot^{x+w} + I_c)$$

where $f : a + y \longrightarrow x + b$ and $g : b + z \longrightarrow w + c$,

$$b)\quad (I_a + \top_y)f(\bot^x + I_b) + (I_c + \top_z)g(\bot^w + I_d) = (I_{a+c} + \top_{y+z})[(I_a + {}^cX^y + I_z)(f + g)(I_x + {}^bX^w + I_d)](\bot^{x+w} + I_{b+d})$$

where $f : a + y \longrightarrow x + b$ and $g : c + z \longrightarrow w + d$.

Let B be an ssmc, with a fixed monoid of objects $(M, +, 0)$. Consider the category $J(B)$, having the same objects as B, and as morphisms, for each $a, b \in M$,

$$J(B)(a, b) := \{(y, f, x) | y, x \in M, f \in B(a + y, x + b)\}$$

with composition defined by

$$(y, f, x)(z, g, w) := (y + z, (f + I_z)(I_x + g), x + w).$$

Notice that the identiy morphism of $a \in M$ in $J(B)$ is $(0, I_a, 0)$, and it will be subsequently denoted by I_a.

In $J(B)$ define the sum of $(y, f, x) \in J(B)(a, b)$ and $(z, g, w) \in J(B)(c, d)$ as

$$(y, f, x) + (z, g, w) := (y + z, (I_a + {}^cX^y + I_z)(f + g)(I_x + {}^bX^w + I_d), x + w).$$

We prove next that $J(B)$ is an nsmc.

Let $(u, h, v) \in J(B)(p, q)$. We have

$$[(y, f, x) + (z, g, w)] + (u, h, v) =$$

$$= (y + z + u, (I_{a+c} + {}^pX^{y+z} + I_u)[(I_a + {}^cX^y + I_z)(f + g)(I_x + {}^bX^w + I_d) + h]\bullet$$
$$\bullet(I_{x+w} + {}^{b+d}X^v + I_q), x + w + v) =$$
$$= (y + z + u, (I_{a+c} + {}^pX^{y+z} + I_u)(I_a + {}^cX^y + I_{z+p+u})(f + g + h)\bullet$$
$$\bullet(I_x + {}^bX^w + I_{d+v+q})(I_{x+w} + {}^{b+d}X^v + I_q), x + w + v) =$$
$$= (y + z + u, (I_a + {}^{c+p}X^y + I_{z+u})(I_{a+y+c} + {}^pX^z + I_u)\bullet$$
$$\bullet(f + g + h)(I_{x+b+w} + {}^dX^v + I_q)(I_x + {}^bX^{w+v} + I_{d+q}), x + w + v) =$$
$$= (y + z + u, (I_a + {}^{c+p}X^y + I_{z+u})\bullet$$
$$\bullet[f + (I_c + {}^pX^z + I_u)(g + h)(I_w + {}^dX^v + I_q)](I_x + {}^bX^{w+v} + I_{d+q}), x + w + v) =$$

$$= (y, f, x) + [(z, g, w) + (u, h, v)].$$

It is easy to prove that $(y, f, x) + I_0 = (y, f, x) = I_0 + (y, f, x)$ and that $I_a + I_b = I_{a+b}$.

For Axiom 4a, we take $(y, f, x) : a \longrightarrow b$, $(z, g, w) : c \longrightarrow d$ and $(u, h, v) : d \longrightarrow e$, and we have

$$[(y, f, x) + (z, g, w)][I_b + (u, h, v)] =$$

$$= (y+z+u, [(I_a + {}^cX^y + I_z)(f+g)(I_x + {}^bX^w + I_d) + I_u][I_{x+w} + (I_b + h)({}^bX^v + I_e)], x+w+v) =$$

$$= (y + z + u, (I_a + {}^cX^y + I_{z+u})[f + (g + I_u)(I_w + h)](I_x + {}^bX^{w+v} + I_e), x + w + v) =$$

$$= (y, f, x) + (z, g, w)(u, h, v).$$

The proof of Axiom 4b is analogous.

Define now the distinguished symmetry morphisms of $J(B)$ to be ${}^aX^b := (0, {}^aX^b, 0)$. It can be proven that $J(B)$ is now a snsmc. We prove here only the validity of Axiom 5a.

$${}^aX^c(I_c + (y, f, x))^cX^b =$$

$$= (y, ({}^aX^c + I_y)(I_c + f)({}^cX^x + I_b)(I_x + {}^cX^b), x) =$$

$$= (y, ({}^aX^c + I_y)(I_c + f)^cX^{x+b}, x) =$$

$$= (y, ({}^aX^c + I_y)^cX^{a+y}(f + I_c), x) =$$

$$= (y, (I_a + {}^cX^y)(f + I_c), x) =$$

$$= (y, f, x) + I_c.$$

Define the canonical M-snsmc morphism $I_B : B \longrightarrow J(B)$ to be, for every $f \in B(a, b)$,

$$I_B(f) := (0, f, 0).$$

In $J(B)$ we define the distinguished morphisms

$$\perp^a := (0, I_a, a) \text{ and } \top_a := (a, I_a, 0)$$

and we notice that

$$\perp^0 = I_0 = \top_0$$

$$\perp^a + \perp^b = \perp^{a+b} \text{ and } \top_a + \top_b = \top_{a+b}.$$

For every $(y, f, x) \in J(B)(a, b)$ the following identity holds:

$$(y, f, x) = (I_a + \top_y)I_B(f)(\perp^x + I_b).$$

Definition 6.1 *For every $a, b \in M$, we define relation $\equiv$ in $J(B)(a, b)$ in the following manner: $(y, f, x) \equiv (z, g, w)$ iff there exist morphisms $k \in B_a(y, z)$ and $j \in B_a(w, x)$ such that $f = (I_a + k)g(j + I_b)$.*

Lemma 6.2 *The relation $\equiv$ is a congruence.*

Proof. Because every $a\alpha$-morphism is invertible, its inverse being also an $a\alpha$-morphism, it follows easily that $\equiv$ is an equivalence.

To prove compatibility with composition, using the same notation as in Definition 6.1, let $(y,f,x) \equiv (z,g,w)$ and let $(y',f',x') \equiv (z',g',w')$ in $J(B)(b,c)$, i.e. there exist $k' \in B_a(y',z')$ and $j' \in B_a(w',x')$ such that $f' = (I_b + k')g'(j' + I_c)$. We have

$$(f + I_{y'})(I_x + f') =$$

$$= (I_a + k + I_{y'})(g + I_{y'})(j + I_{b+y'})(I_{x+b} + k')(I_x + g')(I_x + j' + I_c) =$$

$$= [I_a + (k + k')][(g + I_{z'})(I_w + g')][(j + j') + I_c],$$

which implies that $(y,f,x)(y',f',x') \equiv (z,g,w)(z',g',w')$.

To prove compatibility with sum, let $(y,f,x) \equiv (z,g,w)$ again as in Definition 6.1, and $(y',f',x') \equiv (z',g',w')$ in $J(B)(c,d)$, i.e. there exist $k' \in B_a(y',z')$ and $j' \in B_a(w',x')$ such that $f' = (I_c + k')g'(j' + I_d)$. It follows that

$$(I_a + {}^cX^y + I_{y'})(f + f')(I_x + {}^bX^{x'} + I_d) =$$

$$= (I_a + {}^cX^y + I_{y'})(I_a + k + I_c + k')(g + g')(j + I_b + j' + I_d)(I_x + {}^bX^{x'} + I_d) =$$

$$= [I_{a+c} + (k + k')][(I_a + {}^cX^z + I_{z'})(g + g')(I_w + {}^bX^{w'} + I_d)][(j + j') + I_{b+d}],$$

which implies that $(y,f,x) + (y',f',x') \equiv (z,g,w) + (z',g',w')$.

Proposition 6.3 *$J(B)/\equiv$ is an $a\beta$-ssmc and a $b\alpha$-ssmc.*

Proof. We will prove only the permutability of $J(B)/\equiv$, that is, we will show, for every $(y,f,x) \in J(B)(a,b)$ and $(z,g,w) \in J(B)(c,d)$, that

$$(y,f,x) + (z,g,w) \equiv (I_a + (z,g,w))((y,f,x) + I_d).$$

This follows easily from the following calculations:

$$(I_a + (z,g,w))((y,f,x) + I_d) =$$

$$= (z, (I_a + g)({}^aX^w + I_d), w) \bullet (y, (I_a + {}^dX^y)(f + I_d), x) =$$

$$= (z + y, (I_a + g + I_y)({}^aX^w + I_{d+y})(I_{w+a} + {}^dX^y)(I_w + f + I_d), w + x) =$$

$$= (z + y, (I_a + g + I_y)(I_a + {}^{w+d}X^y)({}^{a+y}X^w + I_d)(I_w + f + I_d), w + x) =$$

$$= (z + y, (I_a + {}^{c+z}X^y)(I_{a+y} + g)(f + I_{w+d})({}^{x+b}X^w + I_d), w + x) =$$

$$= (x + y, (I_{a+c} + {}^zX^y)[(I_a + {}^cX^y + I_z)(f + g)(I_x + {}^bX^w + I_d)]({}^xX^w + I_{b+d}), w + x).$$

Let $J_B : B \longrightarrow J(B)/\equiv$ be the ssmc morphism obtained by compsoing I_B with the canonical factorization morphism.

Proposition 6.4 *If C is an $a\beta$-ssmc and a $b\alpha$-ssmc and $F : B \longrightarrow C$ is an ssmc morphism, then there exists a unique $a\beta$-ssmc and $b\alpha$-ssmc morphism $H : J(B)/\equiv \longrightarrow C$ such that $F = J_B \bullet H$.*

Proof. We will prove that there exists a unique snsmc morphism $G : J(B) \longrightarrow C$ such that $F = I_B \bullet G, G(\bot^a) = \bot^{G(a)}$ and $G(\top_a) = \top_{G(a)}$. Define G as follows:

$$G(a) := F(a), \text{ for every object } a \in M,$$

$$G(y, f, x) := (I_{F(a)} + \top_{F(y)}) \bullet F(f) \bullet (\bot^{F(x)} + I_{F(b)},) \text{ for every } (y, f, x) \in J(B)(a, b).$$

For $f \in B(a, b)$, notice that

$$G(I_B(f)) = G(0, f, 0) = F(f),$$

from which follows also that

$$G(I_a) = I_{G(a)} \text{ and } G({}^aX^b) = {}^{G(a)}X^{G(b)}.$$

For $(y, f, x) \in J(B)(a, b)$ and $(z, g, w) \in J(B)(b, c)$ we have:

$$G((y, f, x)(z, g, w)) = (I_{F(a)} + \top_{F(y+z)})F((f + I_z)(I_x + g))(\bot^{F(x+w)} + I_{F(c)}) =$$
$$= (I_{F(a)} + \top_{F(y)})(F(f) + \top_{F(z)})(\bot^{F(x)} + F(g))(\bot^{F(w)} + I_{F(c)}) =$$
$$= (I_{F(a)} + \top_{F(y)})F(f)(\bot^{F(x)} + I_{F(b)})(I_{F(b)} + \top_{F(z)})F(g)(\bot^{F(w)} + I_{F(c)}) =$$
$$= G(y, f, x) \bullet G(z, g, w).$$

For $(y, f, x) \in J(B)(a, b)$ and $(z, g, w) \in J(B)(c, d)$ we have:

$$G((y, f, x) + (z, g, w)) =$$
$$= (I_{F(a+c)} + \top_{F(y+z)})F((I_a + {}^cX^y + I_z)(f + g)(I_x + {}^bX^w + I_d))(\bot^{F(x+w)} + I_{F(b+d)}) =$$
$$= (I_{F(a)} + \top_{F(y)} + I_{F(c)} + \top_{F(z)})F(f + g)(\bot^{F(x)} + I_{F(b)} + \bot^{F(w)} + I_{F(d)}) =$$
$$= G(y, f, x) + G(z, g, w).$$

We also note that

$$G(\bot^a) = G(0, I_a, a) = (I_{F(a)} + \top_{F(0)})F(I_a)(\bot^{F(a)} + I_{F(0)}) = \bot^{G(a)}$$

and, similarly,

$$G(\top_a) = G(a, I_a, 0) = (I_{F(0)} + \top_{F(a)})F(I_a)(\bot^{F(0)} + I_{F(a)}) = \top_{G(a)}.$$

If $(y, f, x) \equiv (z, g, w)$, using the same notations as in Definition 6.1, we have

$$G(y, f, x) = (I_{F(a)} + \top_{F(y)})F(f)(\bot^{F(x)} + I_{F(b)}) =$$
$$= (I_{F(a)} + \top_{F(y)})(I_{F(a)} + F(k))F(g)(F(j) + I_{F(b)})(\bot^{F(x)} + I_{F(b)}) =$$
$$= (I_{F(a)} + \top_{F(z)})F(g)(\bot^{F(w)} + I_{F(b)}) = G(z, g, w),$$

from which we deduce the existence of the morphism H with the required properties.

Suppose for the rest of this section that B is a biflow.

For $(y,f,x) \in J(B)(c+a,c+b)$ we define the left feedback

$$\uparrow^c (y,f,x) := (y, \uparrow^c [f({}^xX^c + I_b)], x)$$

and we prove that $J(B)$ becomes a flow.

For $(z,g,w) : d \longrightarrow a$, we have:

$$\begin{aligned}
&\uparrow^c [(I_c + (z,g,w))(y,f,x)] = \\
&= \uparrow^c (z+y, (I_c + g + I_y)({}^cX^w + I_{a+y})(I_w + f), w+x) = \\
&= (z+y, \uparrow^c [(I_c + g + I_y)({}^cX^w + I_{a+y})(I_w + f)({}^{w+x}X^c + I_b)], w+x) = \\
&= (z+y, (g+I_y)(I_w + \uparrow^c (f({}^xX^c + I_b))), w+x) = \\
&= (z,g,w) \bullet \uparrow^c (y,f,x).
\end{aligned}$$

For $(z,g,w) : b \longrightarrow d$ we have

$$\begin{aligned}
&\uparrow^c [(y,f,x)(I_c + (z,g,w))] = \\
&= \uparrow^c [(y,f,x)(z,(I_c+g)({}^cX^w + I_d), w)] = \\
&= \uparrow^c (y+z, (f+I_x)(I_{x+c}+g)(I_x + {}^cX^w + I_d), x+w) = \\
&= (y+z, \uparrow^c [(f+I_x)(I_{x+c}+g)({}^xX^c + I_{w+d})], x+w) = \\
&= (y+z, \uparrow^c [(f({}^xX^c + I_b) + I_x)(I_c + I_x + g)], x+w) = \\
&= (y+z, (\uparrow^c [f({}^xX^c + I_b) + I_x])(I_x + g), x+w) = \\
&= (y+z, (\uparrow^c (f({}^xX^c + I_b)) + I_x)(I_x + g), x+w) = \\
&= \uparrow^c (y,f,x) \bullet (z,g,w).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\uparrow^c [(y,f,x) + I_d] = \uparrow^c (y, (I_{c+a} + {}^dX^y)(f + I_d), x) = \\
&= (y, \uparrow^c [(I_{c+a} + {}^dX^y)(f+I_d)({}^xX^c + I_{b+d}], x) = \\
&= (y, (I_a + {}^dX^y)[\uparrow^c (f({}^xX^c + I_b)) + I_d], x) = \uparrow^c (y,f,x) + I_d.
\end{aligned}$$

For $(y,f,x) : c+d+a \longrightarrow c+d+b$ we have:

$$\begin{aligned}
&\uparrow^{d+c} [({}^dX^c + I_a)(y,f,x)({}^cX^d + I_b)] = \\
&= \uparrow^{d+c} (y, ({}^dX^c + I_{a+y})f(I_x + {}^cX^d + I_b), x) = \\
&= (y, \uparrow^{d+c} [({}^dX^c + I_{a+y})f(I_x + {}^cX^d + I_b)({}^xX^{d+c} + I_b)], x) = \\
&= (y, \uparrow^{d+c} [({}^dX^c + I_{a+y})f({}^xX^{c+d} + I_b)({}^cX^d + I_{x+b})], x) = \\
&= (y, \uparrow^{c+d} [f({}^xX^{c+d} + I_b)], x) = \uparrow^{c+d} (y,f,x).
\end{aligned}$$

Noticing that for $f \in B(c+a,c+b)$ we have

$$\uparrow^c I_B(f) = \uparrow^c (0,f,0) = (0, \uparrow^c f, 0) = I_B(\uparrow^c f),$$

it follows easily that $\uparrow^c I_c = I_0$ and $\uparrow^c {}^cX^c = I_c$.

We prove now that $\equiv$ is a flow congruence. Suppose $(y, f, x) \equiv (z, g, w)$ in $J(B)(c+a, c+b)$, i.e. there exist $a\alpha$-morphisms $k \in B_a(y, z)$ and $j \in B_a(w, x)$ such that $f = (I_{c+a} + k)g(j + I_{c+b})$, and we notice that

$$\uparrow^c (y, f, x) = (y, \uparrow^c [(I_{c+a} + k)g(j + I_{c+b})({}^xX^c + I_b)], x) =$$
$$= (y, (I_a + k)(\uparrow^c [g({}^wX^c + I_b)])(j + I_b), x) \equiv \uparrow^c (z, g, w).$$

It follows that $J(B)/\equiv$ is a biflow and $J_B : B \longrightarrow J(B)/\equiv$ is a biflow morphism.

Proposition 6.5 *If C is a biflow over an $a\beta$-ssmc and a $b\alpha$-ssmc and if $F : B \longrightarrow C$ is a biflow morphism, then there exists a unique biflow, $a\beta$-ssmc and $b\alpha$-ssmc morphism $H : J(B)/\equiv\longrightarrow C$, such that $F = J_B \bullet H$.*

Proof. Using the proof of proposition 6.4 and keeping the same notation, it is sufficient to prove that G is a flow morphism.

Let $(y, f, x) \in J(B)(c+a, c+b)$. Then we have:

$$\uparrow^{G(c)} G(y, f, x) = \uparrow^{F(c)} [(I_{F(c+a)} + \top_{F(y)})F(f)(\perp^{F(x)} + I_{F(c+b)})] =$$
$$= (I_{F(a)} + \top_{F(y)}) \uparrow^{F(c)} [F(f)({}^{F(x)}X^{F(c)} + I_{F(b)})](\perp^{F(x)} + I_{F(b)}) =$$
$$= (I_{F(a)} + \top_{F(y)})F(\uparrow^c [f({}^xX^c + I_b)])(\perp^{F(x)} + I_{F(b)}) =$$
$$= G(\uparrow^c (y, f, x)).$$

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