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A note on intersections of isotone clones

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Abstract

We show that for every $k > 3$ there exists two chains P_1, P_2 over a base set A, $|A| = k$ such that the only isotone functions P_1 and P_2 have in common are the constants and projections. This settles a question raised by Demetrovics, Miyakawa, Rosenberg, Simovici and Stojmenović. We prove a related result which generalizes the observation that two 3-element chains over the same ground set always admit a nontrivial common order preserving operation.

1 Introduction

Let A be a nonempty finite set. An *n-ary operation over A* is a function from A^n to A. $O_n(A)$ denotes the set of all n-ary operations over A and we put $O(A) = \cup_{n\geq 0} O_n(A)$. A set of operations $C \subseteq O(A)$ is a *clone* over A if it contains the projections and is closed under arbitrary superpositions (cf. Jablonskii [J58], Pöschel, Kaluznin [PK79], Szendrei [SZ86]). The set of all clones over A is denoted by $L(A)$. $L(A)$ is a partially ordered set with respect to inclusion and is closed under intersection. Clearly the set K_A of all projections and constant operations form a clone over A.

Let $P = < A, \leq >$ be a partial order (a poset for short) on A. We say that an operation $f \in O_n(A)$ *preserves* P if $x_1 \leq y_1,\ x_2 \leq y_2, \ldots, x_n \leq y_n$ implies that $f(x_1, x_2, \ldots, x_n) \leq f(y_1, y_2, \ldots, y_n)$, for every $x_i, y_i \in A$. In this case f is called an *isotone function* (with respect to P). It is easy to see that

$$Pol(P) = \{f \in O(A);\ f \textit{ preserves } P\ \}$$

is a clone over A and $Pol(P) \supseteq K_A$. In [DMRSS90] Demetrovics, Miyakawa, Rosenberg, Simovici and Stojmenović studied intersections of clones of the form $Pol(P)$. In the context of semirigid relations they proved that if $|A| > 7$ or $|A| = 6$ then there exists two posets P_1, P_2 over A for which we have $Pol(P_1) \cap Pol(P_2) = K_A$. Also, they constructed four chains Q_1, Q_2, Q_3, Q_4 over A for which the clones $Pol(Q_i)$ intersect in K_A. The objective of this note is to improve the latter result. For $|A| > 3$ we exhibit two chains P_1, P_2 over A with the property $Pol(P_1) \cap Pol(P_2) = K_A$ (Theorem A). It is easy to see that any two chains over a 3-element set admit a common order preserving function. This observation is generalized in Theorem B. We show for a large class of posets P that any two isomorphic copies of P over the same ground set have a common order preserving operation. This class, besides the 3-element chain, includes the diamond and the pentagon. The note is concluded with a problem for further research.

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2 The results

Recall that a pair of elements $a < b$ of a poset P forms a *cover* if there is no $c \in P$ such that $a < c < b$. In this case we say also that b is an *upper cover* of a and a is a *lower cover* of b. A poset P is *bounded* if there exist $x, y \in P$ such that for every $z \in P$ we have $x \leq z \leq y$. In the sequel we shall use the following result (cf. [LP84], [P84]).

Lemma 1 *Let* $|A| > 2$ *and* C *be a clone over* A. *Then* $C = K_A$ *if and only if* $C \cap O_1(A) = K_A \cap O_1(A)$.

In simple terms, Lemma 1 states that a clone C is K_A exactly when the unary fuctions in C are the constants and the identity function. For $k > 0$ let A_k denote the set $\{0, 1, \ldots k-1\}$.

Theorem A. *For every integer* k Ú 3 *there exists two chains* P_1, P_2 *on* A_k *such that* $Pol(P_1) \cap Pol(P_2) = K_{A_k}$.

Proof. We give first the definitions of P_1 and P_2 by specifying the covers in the respective orders:

$$P_1: \quad 0 < 1 < 2 < \ldots < k-2 < k-1.$$

In the definition of P_2, we distinguish two cases, corresponding to the parity of k. If $k = 2m$ then we put

$$P_2: \quad 2m-2 < 2m-4 < \ldots < 2 < 0 < 2m-1 < 2m-3 < \ldots < 3 < 1.$$

If $k = 2m+1$ then we set

$$P_2: \quad 2m-2 < 2m-4 < \ldots < 2 < 2m < 0 < 2m-1 < 2m-3 < \ldots < 3 < 1.$$

In other words, P_1 is the standard ordering of A_k, while in P_2 we have first the *even* integers from the interval $[0, k-1]$ in a decreasing order (with respect to the standard ordering) followed by the odd numbers listed decreasingly again, provided that k is even. If k is odd then a little perturbation is introduced: $k-1$ is placed between 2 and 0 rather than to the beginning of the sequence. This is possible because $k > 3$ and therefore $2 \neq 2m$.

As for the proof, let $f \in Pol(P_1) \cap Pol(P_2)$ be a nontrivial unary function (i.e. f is not constant and not the identical function on A_k). Chains have no nontrivial automorphisms, therefore there exists $a \neq b \in A_k$ such that $f(a) = f(b)$. Using that $f \in Pol(P_1)$, we can assume that $b = a+1$, hence a and b have different parities. Now from $f \in Pol(P_2)$ we infer that $f(0) = f(k-1)$ if k is even and $f(0) = f(k-2)$ if k is odd. Switching back to P_1 we obtain that $f(0) = f(1) = \cdots = f(k-1)$ for k even. In this case the proof is finished. For k odd the same argument gives that $f(0) = f(1) = f(2) = \cdots = f(k-2)$. From the relations $2 < 2m < 0$ in P_2 we infer $f(0) = f(2m) = f(2)$ and conclude that f is a constant. The proof is complete. □

The unary functions over A_2 are the identity function and the constants. If P_1 and P_2 are chains over A_3 then an easy argument shows that $Pol(P_1) \cap Pol(P_2)$ is nontrivial. Next we prove a generalization of this observation. A finite bounded poset has the *cover property* if every element, except possibly the least and the greatest elements, has either a unique lower cover or a unique upper cover. We argue that there are many posets having the cover property. In fact, if P is an arbitrary bounded poset then if we replace every $z \in P$ (except possibliy the greatest and the least elements of P) by a two-element chain then the resulting poset will have the cover property.

Theorem B. *Let P be a bounded poset on the finite base set A. Let $0, 1 \in A$ denote the least and the greatest elements of P. Suppose that there is an element $a \in P$ such that $0 < a$ and $a < 1$ are covers and that the poset $P \setminus \{a\}$ has the cover property. Let Q be an other poset on the base set A isomorphic to P. Then $Pol(P)$ and $Pol(Q)$ have a nontrivial intersection, i.e. $Pol(P_1) \cap Pol(P_2) \supset K_A$.*

Proof. Let $\phi : A \to A$ denote the map establishing an isomorphism $phi : P \to Q$ and put $b = \phi(a)$. Observe first that an arbitary map $f : P \to P$ which is the identical map on $P \setminus \{a\}$ is actually an order preserving map of P. For this reason if $b = a$ then for the map $g : A \to A$ defined as $g(a) = 1$ and $g(y) = y$ if $y \neq a$ we have $g \in Pol(P) \cap Pol(Q)$. We can henceforth assume that $a \neq b$. If $b \notin \{0, 1\}$ then we can easily construct a nontivial function $h \in Pol(P) \cap Pol(Q)$ as follows. As $P \setminus \{a\}$ has the cover property, $b \in P$ has either a unique upper cover in P or a unique lower cover in P. We shall assume that $c \in P$ is a unique upper cover of b in P (the other case can be treated in exactly the same way). Now set $h(b) = c$ and $h(z) = z$ if $z \in A \setminus \{b\}$. From the fact that c is a unique upper cover of b in $P \setminus \{a\}$ an therefore in P, we obtain that $h \in Pol(P)$. By our first observation we have $h \in Pol(Q)$ as well.

We are left with four cases to consider: $a \neq b$, $b \in \{0, 1\}$ and (by symmetry) $a \in \{\phi(0), \phi(1)\}$. In each case we shall define a nontrivial unary function $h \in Pol(P) \cap Pol(Q)$

(i) If $b = 0$ and $a = \phi(0)$ then we set $h(a) = h(b) = a$ and $h(y) = 1$ if $y \notin \{a, b\}$.
(ii) Analogously, if $b = 1$ and $a = \phi(1)$ then we set $h(a) = h(b) = a$ and $h(y) = 0$ if $y \notin \{a, b\}$.
(iii) If $b = 0$ and $a = \phi(1)$ then we set $h(a) = h(b) = a$ and $h(y) = 1$ if $y \notin \{a, b\}$.
(iv) Analogously, if $b = 1$ and $a = \phi(0)$ then we put $h(a) = h(b) = a$ and $h(y) = 0$ if $y \notin \{a, b\}$.

In all cases we have $|Im(h)| = 2$ therefore h neither is constant nor is the identity function on A. The easy verification of the fact that h is an isotone function with respect to both P and Q is left to the reader.

□

Corollary C. *Let P and Q be two posets on A_5 isomorphic to the pentagon (i.e. the poset on A_5 defined by the covers $0 < 1 < 2 < 3$ and $0 < 4 < 3$). Then $Pol(P)$ and $Pol(Q)$ have a nontrivial intersection.*

□

Example. In contrast to Corollary C, consider the posets R and S over the base set A_6 defined by covers as follows:

$$R: \quad 0 < 1 < 2 < 3 \text{ and } 0 < 4 < 5 < 3.$$

$$S: \quad 1 < 3 < 0 < 5 \text{ and } 1 < 4 < 2 < 5.$$

Note that R is obtained from the pentagon by inserting a new element between 4 and 3. Clearly R and S are isomorphic posets. We show that $Pol(R)$ and $Pol(S)$ have a trivial intersection, i.e. $Pol(R) \cap Pol(S) = K_{A_6}$.

To this end, let $f \in Pol(R) \cap Pol(S)$ be a unary function. We consider first the case when $f(0) \neq 0$ or $f(3) \neq 3$. We claim that in this case $|Im(f)| \leq 2$. Indeed, $f \in Pol(R)$ implies then that $Im(f)$ is bounded in R and is consequently a subset of one of the following four sets: $\{0,1,2\}$, $\{0,4,5\}$, $\{1,2,3\}$ and $\{3,4,5\}$. On the other hand, $Im(f)$ is a bounded poset with respect to S as well. As neither of the above four subsets of A_6 form a bounded subposet of S, the claim follows. If f is not a constant then we have $|Im(f)| = 2$ and $f(0) \not\equiv f(3)$. Now an inspection of S reveals that $f(1) = f(3)$ and $f(5) = f(0)$. Using again that $f \in Pol(R)$ we obtain that $f(2) = f(3)$ and $f(4) = f(5)$. The latter implies in S that $f(2) = f(5)$, showing that f is a constant, a contradiction.

From now on we can assume that $f(0) = 0$ and $f(3) = 3$. Now $f \in Pol(S)$ implies that $f(5) \in \{0,5\}$ and $f(1) \in \{1,3\}$. But $f(5) = 0$ would imply in R that $f(4) = 0$ which in S leads to $f(2) = 0$. The latter in R implies $f(1) = 0$ which in S leads to the contradictory $f(3) = 0$. A similar argument switching back and forth between R and S shows that $f(1) = 1$. At this point we have $f(i) = i$ for $i \in \{0,1,3,5\}$ and (from R) $f(4) \in \{0,4,5\}$. Here $f(4) \in \{0,5\}$ would give (in S) that $f(2) \in \{0,5\}$, which contradicts the relation

$$(*) \qquad f(2) \in \{1,2,3\}$$

obtained from R. We infer that $f(4) = 4$ and this gives in S that $f(2) \in \{2,4,5\}$. This together with (*) implies that $f(2) = 2$, i.e. f is the identity function of A_6. This proves the statement.

Motivated by our considerations we propose the follwing open research problem.

Problem. Find a characterization of the (bounded) posets $P = < A, \leq_P >$ for which there exists a poset $Q = < A, \leq_Q >$ such that P and Q are isomorphic and $Pol(P) \cap Pol(Q) = K_A$.

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