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A criterion for the simplicity of finite Moore automata

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Abstract

A Moore automaton $\mathbf{A} = (A, X, Y, \delta, \lambda)$ can be obtained in two steps: first we consider the triplet (A, X, δ) – called a semiautomaton and denoted by $\mathbf{S}$ – and then we add the components Y and λ which concern the output functioning. Our approach is: $\mathbf{S}$ is supposed to be fixed, we vary λ in any possible way, and – among the resulting automata – we want to separate the simple and the nonsimple ones from each other. This task is treated by combinatorial methods. Concerning the efficiency of the procedure, we note that it uses a semiautomaton having $|A|(|A|+1)/2$ states.

1 Introduction and terminology

§ 1.

The question, when a Moore automaton is simple, has already been the subject of a series of previous papers.[1] Let some earlier results be outlined.[2] If, particularly, only autonomous automata are considered (i.e., $|X| = 1$ is required), the question has been solved in [4] as a consequence of the theory describing all congruences of autonomous automata. Without the restriction to autonomousness, a result of certain theoretical importance has been obtained in [2]; this statement does not seem to be worthy practically, because its algorithmic complexity depends on $|A|$ exponentially. Investigations of recursive character are contained in [5] and [6], the general problem of simplicity was there reduced to the question, when a strongly connected automaton (i.e., an automaton having no proper subautomata) is simple.

In the present considerations the problem of simplicity is dealt with for the eintirety of automata, we rely on the result achieved in [2]. We choose the way that first a semiautomaton $\mathbf{S} = (A, X, \delta)$ is thought to be fixed, and we form

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[1]The researcher of this problem feels his situation to be similar to that of a mountain-climber who besieges a difficultly reachable peak from various sides, since he does not know in advance where he must turn back because of a too steep rise.

[2]Out of the three results to be mentioned now, the first and second ones are restated in this paper as Propositions D and A (in § 11 and § 3, respectively).

then several automata $\mathbb{A}_\lambda = (A, X, Y, \delta, \lambda)$ so that the output components Y, λ are prescribed in every (essentially different) manner. We obtain a necessary and sufficient condition that separates the simple $\mathbb{A}_\lambda'$s from the nonsimple ones. We use combinatorial tools, and our considerations are in connection with the articles [7], [8] where partial results were gained in common with I. Babcsányi and F. Wettl.

Sketching the content of this paper, let it be mentioned first that the terminology concerning automata is introduced in §§ 2, 3; together with restating some former results. A glance is thrown at the graph theory in § 4.

The construction, elaborated in § 5, and the Theorem, exposed at the end of § 6, are the principal purport of the article. § 7 contains the proof of the Theorem and of two cognate propositions.

The condition for the simplicity of automata, asserted in the Theorem, allows sometimes a useful further analysis by logical methods; an insight into this possibility is explained in § 8. In § 9 examples are treated on how the Theorem can be applied in practice.

The paper terminates with touching some questions on combinatorial complexity, arising if the method is applied. These considerations do not set up a claim for completeness at all, they are of intuitive nature. The possibilities of future contributions to this topics are specified as open problems in § 11. It is probable that a genuine expert of the combinatorial complexity theory may conceive essential further thoughts in addition to the ideas formulated in §§ 10, 11.

§ 2.

As usual, we understand by a *finite Moore automaton* a quintuple $\mathbb{A} = (A, X, Y, \delta, \lambda)$ where A, X, Y are finite sets (called the set of states, set of input symbols, set of output symbols, respectively), δ (the transition function) is a mapping of $A \times X$ into A and λ (the output function) is a mapping of A onto Y.

The finite sequences (of arbitrary nonnegative length) consisting of elements of X are called input words. The set of all input words is denoted by $F(X)$. The meaning of $\delta(a,p)$ is the customary where p is an input word.

Let a and b be two states of an automaton. We say that b is *accessible* from a if there exists an input word p such that $\delta(a,p) = b$. The accessibility is a reflexive and transitive relation. If any of a, b is accessible from the other, then it is said that a and b are *mutually accessible*. The mutual accessibility is an equivalence relation in A, the equivalence classes are called the *strongly connected blocks* – or, for the sake of brevity, the blocks – of A. If there is only one block, we say that A is a *strongly connected* automaton.

Let π be an arbitrary equivalence relation in A. π is called a *congruence* (of A) if $a \equiv b \pmod{\pi}$ implies the formulae $\delta(a,x) \equiv \delta(b,x) \pmod{\pi}$ and $\lambda(a) = \lambda(b)$ whenever $a \in A, b \in A, x \in X$. The minimal partition of A is the *trivial congurence* of $\mathbb{A}$. It is said that $\mathbb{A}$ is *simple* (or *reduced*) if $\mathbb{A}$ has no nontrivial congruence.

If we do not take into consideration the third component Y and the fifth component λ of a Moore automaton $\mathbb{A} = (A, X, Y, \delta, \lambda)$, then the resulting structure $\mathbb{S} = (A, X, \delta)$ is called a *semiautomaton*.[3] We say then that $\mathbb{S}$ is the *scheme* (or *projection*) of $\mathbb{A}$ and, reciprocally, that $\mathbb{A}$ is an *automaton completion* (or a-completion) of $\mathbb{S}$. Of course, a semiautomaton $\mathbb{S}$ has many a-completions, depending on how λ is chosen. We shall use the notation $\mathbb{A}_\lambda$ sometimes when the a-completion of a semiautomaton with the output function λ is regarded.

A pair (a, b) is called a *proper* pair if $a \neq b$.

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[3]The present use of the word "semiautomaton" differs from the terminology of [10].

§ 3.

Throughout this section let (a, b) be a (proper or nonproper) unordered pair of states of a Moore automaton.

We denote by $H_{a,b}$ the set of all input words p satisfying $\delta(a,p) \neq \delta(b,p)$. It is said that

(a, b) is a pair of *first type* if $|H_{a,b}| < \infty$,
(a, b) is a pair of *second type* if $H_{a,b} = F(X)$,
(a, b) is a pair of *third type* if $H_{a,b} \subset F(X)$ and $|H_{a,b}| = \infty$.

The difference set $F(X) - H_{a,b}$ is obviously either empty of infinite. The pairs of second and third type are necessarily proper.

We say that (a, b) is a *distinguishable* pair if there exists an input word p such that $\lambda(\delta(a,p)) \neq \lambda(\delta(b,p))$. In the contrary case (a, b) is *indistinguishable*. The relation of indistinguishability, to be denoted by $\pi_{\max}$, is clearly an equivalence relation. The following fact establishes a connection between simplicity and distinguishability (see [2], § 5; [5], § 4):

Proposition A. *Consider $\pi_{\max}$ in a Moore automaton* **A**. *The relation $\pi_{\max}$ is a congruence of* **A** *and each congruence of* **A** *is a refinement of $\pi_{\max}$.* **A** *is simple if and only if $\pi_{\max}$ equals the minimal partition of A (or, equivalently, if each proper pair (a, b) is distinguishable where $a \in A, b \in A$).*

If a proper pair (a, b) of states is indistinguishable and of first type, then we say that a and b are *weakly indistinguishable*. If a pair (a, b) is indistinguishable and of second type, then we say that a and b are *strongly indistinguishable*. If (a, b) is indistinguishable and of third type, then we say that a and b are *compoundly indistinguishable*.

The three kinds of indistinguishability introduced above are pairwise excluding. The subsequent assertion follows from [8], Proposition 2:

Proposition B. *The weak indistinguishability is a transitive relation.*

The analogous statement does not hold for the other two indistinguishability types (cf. [8], Chapter III).

Let (a, b) be a state pair. If $\lambda(a) = \lambda(b)$ holds and $\delta(a, x) = \delta(b, x)$ is valid for every $x (\in X)$, then we say that (a, b) is an *associated* pair. The relation of being associated is an equivalence in A.

It is clear that any associated proper pair is weakly indistinguishable. The converse of this fact does not hold (in general), but we have the following sentence (see [7], Proposition 2):

Proposition C. *Consider the state pairs in a Moore automaton. There is a proper associated pair if and only if there is a weakly indistinguishable pair.*

§ 4.

It is not superfluous to say here a few words on graph theory, because we shall construct a nondirected graph at the end of § 5, and our automaton-theoretical considerations use sometimes certain ideas that originate from the theory of directed graphs. Let [11], [13] be mentioned as reference books of the two main branches of graph theory.

The graph got in § 5 is simple in the sense that each vertex pair [or, in another terminology, point pair] is joined by at most one edge [line], and each edge joins two different vertices. We use the notation $[ab]$ for the edge joining a and b.

The notions of accessibility (introduced in § 2) correspond precisely to the analogous concepts in directed graph theory (for the latter, see e.g. Chapter 3 of [13]). One can show easily that we get a cycle-free directed graph if we form the condensation of the strongly connected blocks [strong components] in a directed graph ([13], Theorem 3.6). Keeping this fact in mind, the reader may perhaps understand better Steps 2-4 of the Construction of § 5.

2 Results

§ 5.

Let $\mathbf{S} = (A, X, \delta)$ be a semiautomaton where $|A| \geq 2$. $\mathbf{S}$ is regarded to be fixed in Chapter 2. We denote $|A|$ by v.

If the output function $\lambda : A \to Y$ is varied, we can get several automaton completions $\mathbf{A}_\lambda = (A, X, Y, \delta, \lambda)$ from $\mathbf{S}$. Our aim is to examine the question: when is a simple $\mathbf{A}_\lambda$ obtained (depending on the choice of λ). Among the automata $\mathbf{A}_\lambda$, it is yielded always a simple one (if $|Y| = v$ and λ is bijective), and also a nonsimple one (if $|Y| = 1$).

In the next construction, we are going to establish a pair (G, ρ) where G is a nondirected graph (whose vertex set equals A) and ρ is a partition of the edges of G.

CONSTRUCTION. The procedure consists of five steps.

Step 1. Let a semiautomaton $\mathbf{R} = (C, X, \delta_R)$ be introduced in the following manner: let C be the set of all (proper and nonproper) unordered pairs (a, b) where $a \in A, b \in A$, define δ_R by the rule

$$\delta_R((a,b),x) = (\delta(a,x), \delta(b,x)). \tag{1}$$

Comments to Step 1. The right-hand side of (1) is meant as an unordered pair. Clearly $|C| = v(v+1)/2$. If (a,b) is a nonproper pair, then the values $\delta_R((a,b),x)$ are again nonproper pairs, hence $\mathbf{R}$ has a subsemiautomaton isomorphic to $\mathbf{S}$. In the terminology of products of automata, we can say that the factor semiautomaton $(\mathbf{S} \otimes \mathbf{S})/\sigma$ is denoted by $\mathbf{R}$ where $\otimes$ is the sign of direct product and σ is the congruence of $\mathbf{S} \otimes \mathbf{S}$ defined by the rule that $(a,b) \equiv (c,d) \pmod{\sigma}$ exactly if either $a = c$, $b = d$ are true or $a = d, b = c$ hold.

Step 2. Denote by ε the equivalence relation of mutual accessibility in C.

Comment to Step 2. If K is an equivalence class modulo ε, then either each element of K is a proper pair or each element of K is a nonproper pair.

Step 3. Consider the equivalence classes K modulo ε (in C) satisfying the conditions (a) and (b):

(a) K consists of proper pairs,

(b) whenever $(a, b) \in K$ and $x \in X$, then

$$\delta_R((a, b), x) \in K. \tag{2}$$

Denote the number of these classes by j and themselves the classes by $K_1, K_2, \ldots, K_j$.

Step 4. Consider the equivalence classes K modulo ε (in C) such that K does not satisfy (b), it fulfils (a) and the following condition (c):

(c) whenever $(a, b) \in K$ and $x \in X$, then either $\delta_R((a, b), x)$ is a nonproper pair or (2) is true.

Denote the number of these classes by k and themselves the classes by $K_{j+1}, K_{j+2}, \ldots, K_{j+k}$.

Comments to Steps 3, 4. Condition (b) can be expressed by saying that K determines a subsemiautomaton of $\mathbf{R}$. The ordering of the classes $K_1, \ldots, K_j$ is arbitrary and the same holds for $K_{j+1}, \ldots, K_{j+k}$. The $j + k$ classes are pairwise disjoint because they have arisen as different classes of an equivalence relation. The number $j + k$ is positive by the finiteness of C.

Step 5. Denote by G the nondirected graph whose vertex set is A and in which two vertices a, b are joined by an edge $[ab]$ precisely when

$$(a, b) \in K_1 \cup K_2 \cup \ldots \cup K_{j+k}.$$

Moreover, let the edge $[a, b]$ belong to the i-th class (modulo ρ), L_i, exactly when $(a, b) \in K_i$ (where $1 \leq i \leq j + k$).

§ 6.

We state two propositions and a theorem on an arbitrary a-completion $\mathbf{A}_\lambda = (A, X, Y, \delta, \lambda)$ of $\mathbf{S}$ and on the partitioned graph (G, g). The verification of the results will be done in the next section.

Proposition 1 *The following two assertions are equivalent:*
(α) *There is a strongly indistinguishable state pair in* $\mathbf{A}_\lambda$.
(β) *There exists a number i, fulfilling $1 \leq i \leq j$, such that, whenever $[ab] \in L_i$, then $\lambda(a) = \lambda(b)$.*

Proposition 2 *If there is a weakly indistinguishable state pair in $\mathbf{A}_\lambda$, then there exists a number i such that the subsequent assertions are true:*

$$j + 1 \leq i \leq j + k,$$

$$|L_i| = 1, and$$

we have $\lambda(a) = \lambda(b)$ for the single element $[ab]$ of L_i.

We have arrived to the main result of the paper.

Theorem 1 *Let an output function $\lambda : A \to Y$ be added to* **S**. *The following two conditions are equivalent for the resulting automaton* $\mathbf{A}_\lambda$*:*

(I) $\mathbf{A}_\lambda$ *is simple.*

(II) *In any class L_i (where $1 \le i \le j+k$) there exists at least one edge $[a_i b_i]$ such that $\lambda(a_i) \neq \lambda(b_i)$.*

§ 7.

Proof of Proposition 1. $(\alpha) \Rightarrow (\beta)$. To any (unordered) proper state pair (a,b) let us denote by $Q(a,b)$ the set of the (proper and nonproper) state pairs (c,d) which satisfy

$$(c,d) = (\delta(a,p), \delta(b,p))$$

with some $p(\in F(X))$. First we mention immediate consequences of this definition. We have $(a,b) \in Q(a,b)$. The pair (c,d) is accessible from (a,b) if and only if $Q(c,d) \subseteq Q(a,b)$.

$$(a,b) \equiv (c,d) (\mathrm{mod}\ \varepsilon)$$

if and only if $Q(a,b) = Q(c,d)$. If (a,b) is a strongly indistinguishable pair and $(c,d) \in Q(a,b)$, then also (c,d) is strongly indistinguishable.

Consider now a strongly indistinguishable state pair (a,b) in $\mathbf{A}_\lambda$. We can choose a pair (c_0, d_0), belonging to $Q(a,b)$, in such a manner that the strict inclusion

$$Q(c,d) \subset Q(c_0, d_0) \tag{3}$$

is false for any $(c,d)(\in Q(a,b))$. (This choice is possible by the finiteness of **R**.)

Denote $Q(c_0, d_0)$ by K. The condition (b) in § 5 is obviously valid for K and K consists of strongly indistinguishable pairs only, furthermore our condition on the falsity of (3) implies that K is just a complete class modulo ε. Consequently, K equals one of the classes $K_1, K_2, \ldots, K_j$ (introduced in Step 3 of the Construction), thus the validity of (β) is clear.

$(\beta) \Rightarrow (\alpha)$. Suppose (β) for a number i, consider an arbitrary edge $[ab]$ in L_i. We can see easily that $Q(a,b) = K_i$, hence (a,b) is a strongly indistinguishable pair.

Proof of Proposition 2. Assume the existence of a weakly indistinguishable pair. Then there is (by Proposition C) a proper associated state pair (a,b).$\lambda(a) = \lambda(b)$ is clear. The one-element set $\{(a,b)\}$ is evidently a class $K_i (\mathrm{mod}\ \varepsilon)$ and i fulfils $j < i \le j+k$.

Proof of the Theorem.

First we show that the falsity of (I) implies the falsity of (II). Denote the set of indistingushable state pairs of $\mathbf{A}_\lambda$ by J. If $\mathbf{A}_\lambda$ is not simple, then $\pi_{\max}$ differs from the minimal partition of A (by Proposition A in § 3), therefore $J \neq \emptyset$. We separate three cases (the first and second ones can overlap each other).

Case 1: J contains a strongly indistinguishable pair. Proposition 1 applies, the truth of (β) shows that (II) does not hold.

Case 2: J contains a weakly indistinguishable pair. We get now by Proposition 2 that (II) is not fulfilled.

Case 3: any element of J is compoundly indistinguishable. Recall the notation $Q(a,b)$ (where $(a,b) \in J$). Define $Q'(a,b)$ as the difference set $Q(a,b) - P$ where P is the set of nonproper state pairs. We have always $(a,b) \in Q'(a,b) \subseteq$

J. Analogously to the first proof in § 7, we start with an arbitrary $(a,b)(\in J)$ and we choose a $(c_0,d_0)(\in J)$ such that

$$Q'(c,d) \subset Q'(c_0,d_0)$$

is false when (c,d) is an arbitrary element of $Q'(c_0,d_0)$. It is obtained that $Q'(c_0,d_0)$ is one of the classes $K_1, K_2, \ldots, K_{j+k}$, say, K_i. (II) is not satisfied with this i because $\lambda(a_i) = \lambda(b_i)$ whenever $(a_i,b_i) \in Q'(c_0,d_0)$.

Conversely, let us assume that (II) is not fulfilled. There is an i such that $[ab] \in L_i$ (that is, $(a,b) \in K_i$) implies $\lambda(a) = \lambda(b)$. Remember how K_i has been constructed in § 5. Choose an arbitrary element (a_0,b_0) of K_i. Whenever $(c,d) \in Q(a_0,b_0)$, then either $(c,d) \in K_i$ or $c = d$; we get $\lambda(c) = \lambda(d)$ in both cases. We have shown that (a_0,b_0) is an indistinguishable proper pair. Thus $\pi_{\max}$ is not the minimal partition of A, hence (by Proposition A in § 3) $\mathbf{A}_\lambda$ is not simple.

3 Discussion and examples

§ 8.

Suppose that we consider some semiautomaton **S** and we want to use the Theorem for getting an overview of the simple automata among all the automata obtained as $\mathbf{A}_\lambda$.

There is no difficulty if the graph G and its edge-partition ρ are enough perspicuous. In the contrary case (i.e. when (G,ρ) is involved), it is possible to utilize logical methods (see e.g. [1] for the occurring logical notions).

We regard that the elements of A are denoted by $a_1, a_2, \ldots, a_v$ (where $v = |A|$). The condition, stated in the Theorem, can be formulated as a conjunctive normal form $\mathfrak{N}$, expressing a truth function f. This function has $\binom{v}{2}$ variables $\mathfrak{w}_{rs}$ (where $1 \le r < s \le v$) such that $\mathfrak{w}_{rs}$ is true or false according as $\lambda(a_r) \ne \lambda(a_s)$ or $\lambda(a_r) = \lambda(a_s)$, respectively. We form, to any class L_i, the disjunction of the variables $\mathfrak{w}_{rs}$ such that the edge $[a_r a_s]$ (exists in G and) belongs to L_i. We get $j+k$ elementary disjunctions (of nonnegated variables) in this manner; f is obtained by the formula $\mathfrak{N}$ which is the conjunction of these $j+k$ disjunctions.

It is known that a disjunctive normal form is often a more treatable representation of a truth function, than a conjunctive one. Therefore, if we continue the study of f, it may be useful to transform $\mathfrak{N}$ into a disjunctive normal form. Some methods for performing this are described in Chapter 3 of [1].

If a function f is analyzed, sometimes we may gain advantage from the idea that the variables $\mathfrak{w}_{rs}$ are not independent of each other. Indeed, the equality is transitive, thus the formula

$$(\overline{\mathfrak{w}}_{rs} \& \overline{\mathfrak{w}}_{st}) \longrightarrow \overline{\mathfrak{w}}_{rt}$$

– or, equivalently, the formula

$$\mathfrak{w}_{rt} \longrightarrow (\mathfrak{w}_{rs} \vee \mathfrak{w}_{st})$$

– must be true for any choice of the subscripts r,s,t (where $\overline{\mathfrak{w}}_{rs}$ denotes the negation of $\mathfrak{w}_{rs}$).

§ 9.

In this section some examples will be studied. The semiautomata, analyzed in what follows, are mostly schemes of some automata occurring in previous articles.[4]

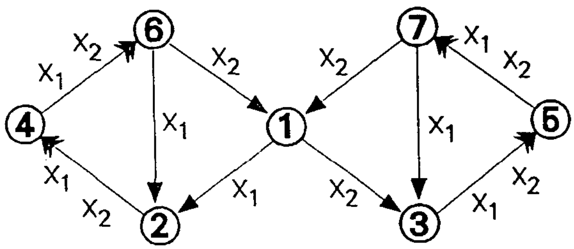

Fig. 1.

Example 1 *Put $A = \{1, 2, \ldots, 7\}$ and $X = \{x_1, x_2\}$, let δ be defined by Table 1 (see Fig. 1). Applying the first step of the Construction for this semiautomaton* $\mathbf{S}$, *we get the semiautomaton $\mathbf{R} = (C, X, \delta_R)$ seen in Fig. 2. (We write e.g. simply 2 instead of (2,2) in this figure.)* $\mathbf{R}$ *has 28 states, there are 21 proper pairs among the elements of C. There are four classes modulo ε, one class consists of the nonproper pairs. The proper pairs are distributed into three classes. One of these three classes is $\{(1,2)\}$, another class is*

$$\{(2,3),\ (4,5),\ (6,7)\}, \tag{4}$$

and the remaining 17 proper pairs belong to the third class. No class fulfils the conditions posed in Step 3 of the Construction. There is one class - namely (4) - which satisfies the conditions posed in Step 4. These facts mean that we have $j = 0, k = 1$ and

$$K_1 = \{(2,3),\ (4,5),\ (6,7)\}$$

in the present example.

[4]Compare the present Examples 1-3 with Example 3 in [4], Example 7 in [7], Example 6 in [8], respectively.

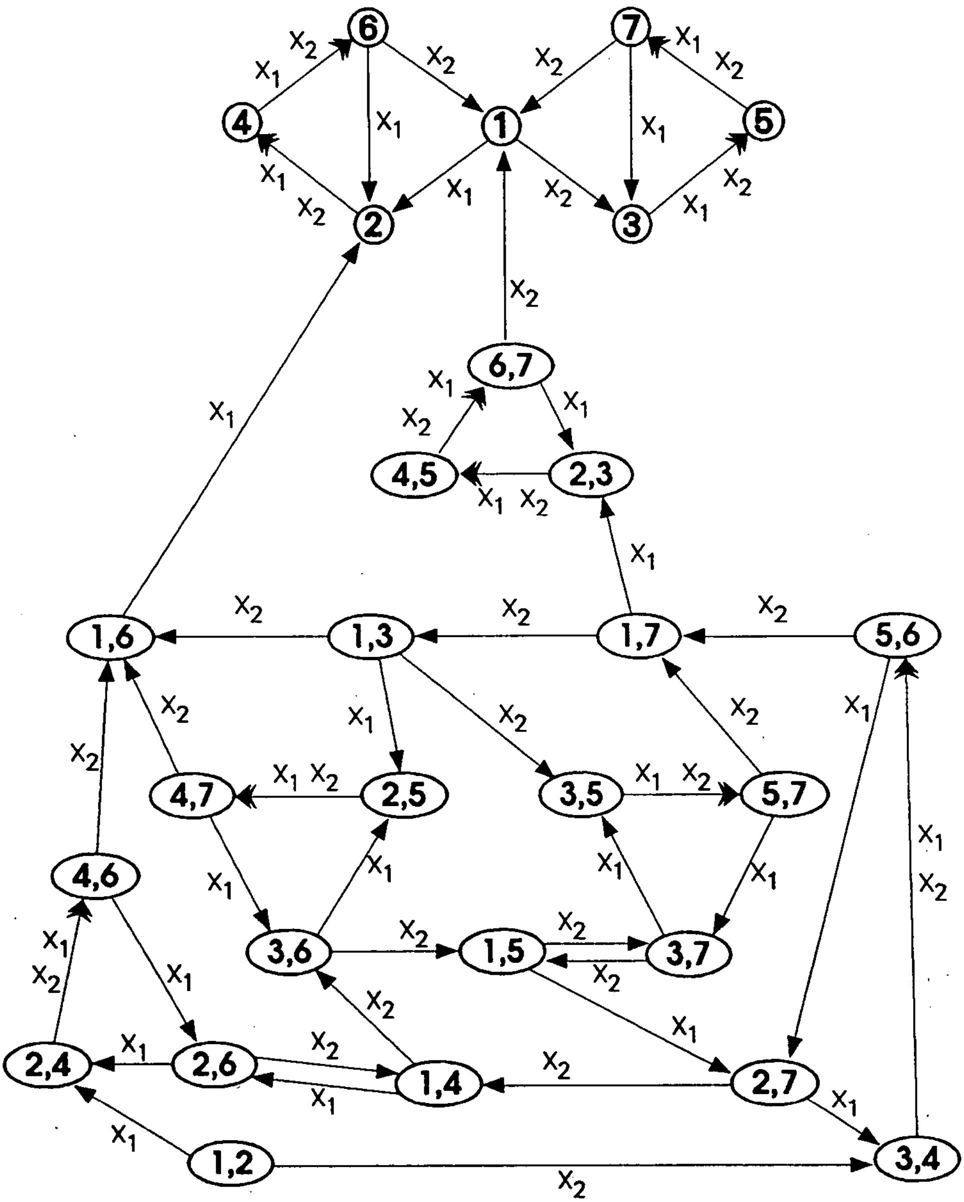

Fig. 2.

Fortunately, our discussion leads to a very simple situtation. The examination of the semiautomaton **S** terminates with constructing the graph G – seen in Fig. 3 – in which all the three edges belong to the same class L_1. Thus the criterion of the simplicity of an a-completion $\mathbf{A}_\lambda$ of **S** is

$$\lambda(2) \neq \lambda(3) \vee \lambda(4) \neq \lambda(5) \vee \lambda(6) \neq \lambda(7).$$

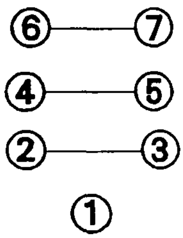

Fig. 3.

a	$\delta(a, x_1)$	$\delta(a, x_2)$
1	2	3
2	4	4
3	5	5
4	6	6
5	7	7
6	2	1
7	3	1

Table 1.

Example 2 *Put* $A = \{1, 2, \ldots, 5\}$ *and* $X = \{x_1, x_2\}$, *let* δ *be defined by Table 2 (see Fig. 4). In analogy to the preceding example, let* **R** *be constructed from this semiautomaton* $\mathbf{S} = (A, X, \delta)$. *(Some details can be left to the reader.) Among the 15 states of* **R** *there are 10 proper pairs. The number of classes mod* ϵ, *consisting of proper pairs, is three. Two of these classes fulfil the conditions of Step 3 of the Construction:*

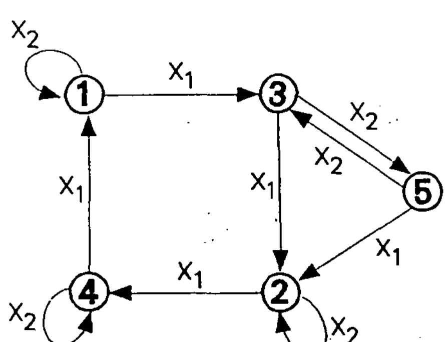

Fig. 4.

a	$\delta(a, x_1)$	$\delta(a, x_2)$
1	3	1
2	4	2
3	2	5
4	1	4
5	2	3

Table 2.

$$K_1 = \{(1,2),\ (3,4),\ (4,5)\},$$

$$K_2 = \{(1,3),\ (1,4),\ (1,5),\ (2,3),\ (2,4)\ (2,5)\}$$

(hence $j = 2$), and the third class

$$K_3 = \{(3,5)\}$$

satisfies the conditions of Step 4 (thus $k = 1$). The graph G has as many edges as possible, it is drawn in Fig. 5.

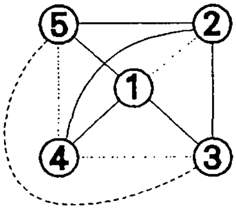

Fig 5.

Using the logical formalism considered in § 8, the criterion of the simplicity of an a-completion $\mathbf{A}_\lambda$ of $\mathbf{S}$ can be expressed by the conjunctive normal form

$$(\mathfrak{w}_{12} \vee \mathfrak{w}_{34} \vee \mathfrak{w}_{45})\&$$

$$\&(\mathfrak{w}_{13} \vee \mathfrak{w}_{14} \vee \mathfrak{w}_{15} \vee \mathfrak{w}_{23} \vee \mathfrak{w}_{24} \vee \mathfrak{w}_{25})\&\mathfrak{w}_{35}. \tag{5}$$

Observe that $\mathfrak{w}_{35} \rightarrow (\mathfrak{w}_{13} \vee \mathfrak{w}_{15})$ and $\mathfrak{w}_{35} \rightarrow (\mathfrak{w}_{34} \vee \mathfrak{w}_{45})$ are identically true formulae (cf. the end of § 8). We can infer that the formula (5) is equivalent to $\mathfrak{w}_{35}$, consequently $\mathbf{A}_\lambda$ is simple if and only if $\lambda(3) \neq \lambda(5)$.

Although (5) was enough complicated, we were in the advantageous situation that we could obtain a remarkable simplification of (5) by utilizing the transitivity of the equality relation.

By analyzing Example 2, we see that the conclusion of Proposition 2 may hold for some a-completions $\mathbf{A}_\lambda$ of $\mathbf{S}$, but the supposition of Proposition 2 is false for each choice of λ. Hence the converse of Proposition 2 does not hold in general.

Example 3 *Put $A = \{1,2,\ldots,10\}$, $X = \{x_1,x_2,x_3\}$, let δ be defined by Table 3 (see Fig. 5 in [8]). Starting with this semiautomaton* **S**, *let* **R** *and the equivalence relation ε be constructed. Consider the proper pairs*

a	$\delta(a,x_1)$	$\delta(a,x_2)$	$\delta(a,x_3)$
1	2	5	6
2	3	3	2
3	2	1	3
4	5	1	4
5	4	4	5
6	7	10	1
7	8	8	7
8	7	6	8
9	10	6	9
10	9	9	10

Table 3.

in C only, then the number of elements in the 11 classes mod ε are: 24, 5, seven times 2, two times 1. By a further analysis we get that $j=1, k=2$ and the classes K_1, K_2, K_3 are:

$$\begin{aligned} K_1 &= \{(1,6),(2,7),(3,8),(4,9),(5,10)\}, \\ K_2 &= \{(2,5),(3,4)\}, \\ K_3 &= \{(7,10),(8,9)\}. \end{aligned}$$

Thus the necessary and sufficient condition for the simplicity of an a-completion $\mathbf{A}_\lambda$ of **S** is the fulfilment of the logical formula

$$(\mathfrak{w}_{16} \vee \mathfrak{w}_{27} \vee \mathfrak{w}_{38} \vee \mathfrak{w}_{49} \vee \mathfrak{w}_{5,10}) \& (\mathfrak{w}_{25} \vee \mathfrak{w}_{34}) \& (\mathfrak{w}_{7,10} \vee \mathfrak{w}_{8,9}).$$

For the sake of completeness, let also the other classes of C mod ε be listed. They are:

$$\{(1,2),(2,6)\}, \{(1,7),(6,7)\}, \{(2,8),(3,7)\},$$
$$\{(2,10),(3,9)\}, \{(4,8),(5,7)\}, \{(2,3)\}, \{(7,8)\};$$

moreover, a class consisting of the remaining 24 proper pairs and a class to which the 10 nonproper pairs belong.

The section will be finished with two sequences of semiautomata. All the semiautomata **S**, to be introduced in the sequel, have the property that, whenever a satate (i_1,i_2) of **R** is a proper pair, then $\{(i_1,i_2)\}$ is a separate class modulo ε.

Example 4 [5] *Choose a number $v (\geq 2)$. Put $A = \{1,2,\ldots,v\}, X = \{x_1,x_2\}$ and let δ be defined in the following manner:*

$$\begin{aligned} \delta(1,x_1) &= 2, \\ \delta(i,x_1) &= 1 \quad \text{if} \quad 2 \leq i \leq v, \\ \delta(i,x_2) &= i+1 \quad \text{if} \quad 1 \leq i \leq v-1, \\ \delta(v,x_2) &= v. \end{aligned}$$

[5] This example is due to A. Nagy (personal communication).

We can observe that the proper pairs are indeed pairwise incongruent mod ε, furthermore, $j = 0, k = 1$ and $K_1 = \{(v-1, v)\}$. Thus the criterion of simplicity is $\lambda(v-1) \neq \lambda(v)$.

Example 5 [6] *Choose a number* $v(\geq 3)$. *Put* $A = \{1, 2, \ldots, v\}, X = \{x_1, x_2, \ldots, x_v\}$ *and let* δ *be defined in the following manner:*

$$\begin{aligned} \delta(1, x_h) &= h \quad \text{if} \quad 1 \leq h \leq v, \\ \delta(i, x_1) &= 1 \quad \text{if} \quad 2 \leq i \leq v, \\ \delta(i, x_h) &= i \quad \text{if} \quad 2 \leq i \leq v, \text{ and } 2 \leq h \leq v. \end{aligned}$$

We find that $j = 0$ and – because for a proper pair (i_1, i_2) the set $\{(i_1, i_2)\}$ satisfies the conditions of Step 4 of the Construction precisely if $2 \leq i_1 \leq v, 2 \leq i_2 \leq v$ are valid – we have $k = \binom{v-1}{2}$ and the criterion of simplicity is

$$|\{\lambda(2), \lambda(3), \lambda(4), \ldots, \lambda(v)\}| = v - 1.$$

For the reader who is interested in this subject, it can be recommended to study also the schemes of other automata occurring as examples in [7] and [8].

§ 10.

A semiautomaton $\mathbf{S} = (A, X, \delta)$ can be considered to be an object of complexity vn (where $n = |X|$ and – as earlier – $v = |A|$), since it can be characterized by a table having vn entries. The product vn is also a good (lower) estimate for the complexity of an a-completion of $\mathbf{S}$. From the view point of practical applications, that (semi-)automata are of primary interest for which n is remarkably smaller than v.

Start with a semiautomaton S and effectuate a construction of another procedure concerning S. If the number of steps of the procedure is proportional to vn, then the procedure may be viewed economical as far as it is expectable. Such an optimal situation, however, is likely very infrequent. If the number of steps of a procedure is proportional to vn^β (with some exponent $\beta(> 1)$), then its complexity can be considered still as quite satisfactory. The procedures whose complexity is of order of magnitude $v^\alpha n^\beta$ (where $\alpha > 1$) are already worse ones, their profitableness decreases with the growth of α. At the other end of the scale, a procedure is not advantageous at all if its complexity cannot be estimated better than by an expression in which v occurs as an exponent.

Recall Proposition A, and consider the task that we are going to check whether or not the states of an automaton are pairwise indistinguishable. It is known[7] that two states a, b are distinguishable (if and) only if there is an input word p, fulfilling $\lambda(\delta(a, p)) \neq \lambda(\delta(b, p))$, such that the length of p does not exceed $v - 2$. The number of input words whose length is at most $v - 2$ equals

$$\frac{n^{v-1} - 1}{n - 1} (= 1 + n + n^2 + \cdots + n^{v-2}).$$

If we want to decide the simplicity of an automaton by using these ideas, we arrive at the following job:

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[6]This example is due to F. Wettl (personal communication).

[7]See e.g. § 5 and § 12 in [3].

we draft a matrix of size $v \times ((n^{v-1}-1)/(n-1))$,

we fill the matrix with the output signs $\lambda(\delta(a,p))$ as its entries, and

we examine the existence of two rows of the matrix that are from place to place coinciding.

The complexity of this process depends exponentially on v, consequently, it is not in the least economical.

The method, based in §§ 5-6 of this paper, is such an improvement of the "rough" application of Proposition A that its complexity remains already under polynomial bounds. The order magnitude of the semiautomaton $\mathbf{R}$ is $v(v+1)n/2$, this quantity is approximately proportional to v^2n. Although the number 2 (as exponent of v) is not quite reassuring, the author is afraid that it cannot be diminished notably (unless we restrict our attention to one or another particular class of semiautomata).

A known algorithm due to Tarjan (see [15]) shows that the classes of the mutual accessibility relation in directed graphs can be determined so that the complexity depends linearly on the number of vertices (if the ratio of the edge number and the vertex number is bounded); consequently, the computational complexity of our Construction is not increased in Steps 2-5 (in comparison to the complexity of Step 1).

§ 11.

In this final section further comments will be done concerning the Construction (in § 5), the Theorem (at the end of § 6) and the handling of the question by logical tools (see § 8).

It is not quite hopeless that the method (elaborated in §§ 5-8) can be refined into a more economical process under certain particular conditions. This subject will be concerned in the first three problems to be raised at once (they are rather heuristical than exact ones). The study of these problems is desirable primarily within the class of strongly connected semiautomata, because a reduction of the general question of the simplicity of automata to the strongly connected case is already known (see [5], [6]).

Problem 1. Find semiautomaton classes such that, for the elements of a class, the graph (G,ρ) can be obtained by some remarkably easier way, than through constructing the semiautomaton $\mathbf{R}$.

Problem 2. Study circumstances under which the truth function f - assigned to the graph (G,ρ) - admits an easy discussion. (f is, of course, easily treatable if it is got by a short formula. Beside this case, Problem 2 concerns whether the following methods can be utlized adavantageously: conversion of a conjunctive normal form into a disjunctive one, and/or use of the consequences of the transitivity of the equality relation.)

Consider again the partitioned graph (G,ρ) obtained in Step 5 of the Construction. Denote the number of the non-adjacent proper vertex pairs, i.e. the quantity

$$\binom{v}{2}-(|L_1|+|L_2|+\cdots+|L_{j+k}|),$$

by $\eta(G)$. The quotient

$$\frac{\max(|L_1|, |L_2|, \ldots, |L_{j+k}|, \eta(G))}{\binom{v}{2}} \tag{6}$$

can be viewed as a measure of in what degree (G, ρ) is perspicuous. The value (6) is clearly between $1/(j+k+1)$ and 1.

Problem 3. Find semiautomaton classes such that, for the elements of a class, the value of the expression (6) is near to one.

The last problem will be devoted to the connection between the general criterion of simplicity, asserted as the Theorem, and the known criterion for the simplicity of *autonomous* Moore automata, having been stated in [4]. The latter result can be formulated as follows:

Proposition D. ([4], Proposition 6). *Let* **S** *be an autonomous Moore semiautomaton. An a-completion* $\mathbf{A}_\lambda$ *of* **S** *is simple exactly if* λ *fulfils the following conditions:*

(i) each cycle is primitive,[8]

(ii) the cycles are pairwise non-isomorphic,

(iii) whenever $\delta(a, x) = \delta(b, x)$ *for a proper state pair* (a, b) *then* $\lambda(a) \neq \lambda(b)$.

Let condition (II) of the Theorem be applied for an autonomous semiautomaton. It is then almost obvious to see that condition (iii) is necessary for the simplicity of an $\mathbf{A}_\lambda$. In the other respects, however, it appears no immediate possibility for deriving Proposition D from the Theorem.

Problem 4. Show that the necessity of the conditions (i), (ii) and the sufficiency of (i) & (ii) & (iii) are consequences of the Theorem when (particularly) an autonomous semiautomaton is considered.

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[8]See [4], pp. 261-262 for the definition of primitivity.

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