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# On a special composition of tree automata

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In the theory of finite automata it is an interesting problem to describe such systems from which any automaton can be built under a given composition and isomorphic embedding as representation. Such systems are called isomorphically complete with respect to the considered composition. In particular, it is important to characterize those compositions for which there are finite isomorphically complete systems. In the works [1], [2] necessary conditions are given for the existence of finite isomorphically complete systems with respect to the classical automata and tree automata, respectively. In both cases it turned out that the existence of a finite isomorphically complete system yields the unboundedness of the feedback dependency of the composition. It is unknown yet whether this condition is sufficient. So it is interesting to investigate such compositions for which there are finite isomorphically complete systems. In [4] such a composition was introduced. Here we generalize this notion of composition to tree automata and give a necessary and sufficient condition of the isomorphic completeness. For this reason we recall some notions from [3] and [5].

By a set of *operational symbols* we mean a nonempty union $\Sigma = \Sigma_0 \bigcup \Sigma_1 \bigcup \ldots$, where $\Sigma_m$ $(m = 0, 1, \ldots)$ are pairwise disjoint sets of symbols. For any $m \geq 0$, the set $\Sigma_m$ is called the set of *m-ary operational symbols*. It is said that the *rank* or *arity* of a symbol $\sigma \in \Sigma$ is $m$ if $\sigma \in \Sigma_m$. Now let a set $\Sigma$ of operational symbols and a set $R$ of nonnegative integers be given. $R$ is called the *rank-type* of $\Sigma$ if for any integer $m \geq 0$, $\Sigma_m \neq \emptyset$ if and only if $m \in R$. Next we shall work under a fixed rank-type $R$.

Now let $\Sigma$ be a set of operational symbols with rank-type $R$. By a *$\Sigma$-algebra* $\mathcal{A}$ we mean a pair consisting of a nonempty set $A$ and a mapping that assigns to every operational symbol $\sigma \in \Sigma$ an $m$-ary operation $\sigma^{\mathcal{A}} : A^m \to A$, where the arity of $\sigma$ is $m$. The set $A$ is called the *set of elements of* $\mathcal{A}$ and $\sigma^{\mathcal{A}}$ is the *realization of* $\sigma$ *in* $\mathcal{A}$. The mapping $\sigma \to \sigma^{\mathcal{A}}$ will not be mentioned explicitly, but we write $\mathcal{A} = (A, \Sigma)$. It is said that a $\Sigma$-algebra $\mathcal{A}$ is *finite* if $A$ is finite, and it is of *finite type* if $\Sigma$ is finite. By *a tree automaton* we mean a finite algebra of finite type. Finally, it is said that the *rank-type of a tree automaton* $\mathcal{A} = (A, \Sigma)$ is $R$ if the rank-type of $\Sigma$ is $R$.

Now let us denote by $\mathcal{U}_R$ the class of all tree automata with rank-type $R$. A composition of tree automata from $\mathcal{U}_R$ can be represented as a network in which each vertex denotes a tree automaton and the actual operation of a tree automaton may depend only on those automata which have direct connection to the given one.

In order to define this notion of composition let $\mathcal{D}$ denote an arbitrary nonempty fixed set of finite directed graphs. Let $\mathcal{A} = (A, \Sigma) \in \mathcal{U}_R$ and $\mathcal{A}_j = (A_j, \Sigma^j) \in \mathcal{U}_R$ $(j = 1, \ldots, n)$. Moreover, take a family $\Psi$ of mappings

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$$\Psi_{mj} : (A_1 \times \ldots \times A_n)^m \times \Sigma_m \to \Sigma_m^j \qquad (m \in R,\ 1 \le j \le n)\,.$$

It is said that the tree automaton $\mathcal{A}$ is a *$\mathcal{D}$-product* of $\mathcal{A}_j$ $(j = 1, \ldots, n)$ with respect to $\Psi$ if the following conditions are satisfied:

$$\text{(i)} \quad A = \prod_{j=1}^{n} A_j\,,$$

(ii) there exists a graph $D = (\{1, \ldots, n\}, E)$ in $\mathcal{D}$ such that for any $m \in R$, $j \in \{1, \ldots, n\}$ and

$$((a_{11}, \ldots, a_{1n}), \ldots, (a_{m1}, \ldots, a_{mn})) \in A^m$$

the mapping $\Psi_{mj}$ is independent of the elements $a_{ts}$ $(t = 1, \ldots, m)$ if $(s, j) \notin E$,

(iii) for any $m \in R$, $\sigma \in \Sigma_m$ and $((a_{11}, \ldots, a_{1n}), \ldots, (a_{m1}, \ldots, a_{mn})) \in A^m$,

$$\sigma^{\mathcal{A}}((a_{11}, \ldots, a_{1n}), \ldots, (a_{m1}, \ldots, a_{mn})) = (\sigma_1^{\mathcal{A}_1}(a_{11}, \ldots, a_{m1}), \ldots, \sigma_n^{\mathcal{A}_n}(a_{1n}, \ldots, a_{mn}))$$

where

$$\sigma_j = \Psi_{mj}((a_{11}, \ldots, a_{1n}), \ldots, (a_{m1}, \ldots, a_{mn}), \sigma) \qquad (j = 1, \ldots, n)\,.$$

We shall use the notation

$$\prod_{j=1}^{n} \mathcal{A}_j(\Sigma, \Psi, D)$$

for the product introduced above and sometimes we shall indicate only those variables of $\psi_{mj}$ on which it may depend.

Now let $\mathcal{B}$ be a system of tree automata from $\mathcal{U}_R$. It is said that $\mathcal{B}$ is *isomorphically complete for $\mathcal{U}_R$ with respect to the $\mathcal{D}$-product* if any tree automaton from $\mathcal{U}_R$ can be embedded isomorphically into a $\mathcal{D}$-product of tree automata from $\mathcal{B}$.

The first characterization of isomorphically complete systems of tree automata was given in [5] with respect to the Gluskov-type product, which can be defined considering the set of finite directed complete graphs as possible networks. Now taking the set of the $n$-dimensional hyper cubes $(n = 2, 3, \ldots)$ as possible networks, we prove that this cube-product is equivalent to the Gluskov-type product with respect to the isomorphic completeness. For this purpose we need some preparation.

Let $n \ge 2$ be an arbitrary integer. Let us consider the $n$-dimensional hyper cube. The set of the vertices of this hyper cube is $S_n = \{(s_1, \ldots, s_n) : s_i \in \{0, 1\}\ (i = 1, \ldots, n)\}$. Define the mapping $\lambda_n$ on the set $S_n$ as follows: for any vector $(s_1, \ldots, s_n)$

$$\lambda_n(s_1, \ldots, s_n) = 1 + \sum_{t=1}^{n} s_t \cdot 2^{n-t}\,.$$

Then $\lambda_n$ is a one-to-one mapping of $S_n$ onto the set $\{1, \ldots, 2^n\}$.

Let us form the directed graph $D_n^* = (\{1, \ldots, 2^n\}, V_n)$, where for any $1 \le i, j \le 2^n$, $(i, j) \in V_n$ if and only if $\lambda_n^{-1}(i)$ is adjacent to $\lambda_n^{-1}(j)$. For any $u \in \{1, \ldots, 2^n\}$

let us denote by $J_u^{(n)}$ the set of all ancestors of $u$ in $D_n^*$. It is obvious that $\lambda_n^{-1}(u) = (s_1, \ldots, s_n)$ is adjacent to a vertex $(r_1, \ldots, r_n)$ if and only if there exists an index $1 \leq i \leq n$ such that $r_i = 1 - s_i$ and $r_j = s_j$ if $1 \leq j \leq n$ and $i \neq j$. Therefore, $|J_u^{(n)}| = n$, i.e. each vertex of $D_n^*$ has exactly $n$ ancestors. On the other hand, it is easy to see that

if $1 \leq u \leq 2^{n-1}$, then $u$ has one ancestor in the set $\{2^{n-1}+1, \ldots, 2^n\}$ and $n-1$ ancestors in the set $\{1, \ldots, 2^{n-1}\}$,

if $2^{n-1} < u \leq 2^n$, then $u$ has one ancestor in the set $\{1, \ldots, 2^{n-1}\}$ and $n-1$ ancestors in the set $\{2^{n-1}+1, \ldots, 2^n\}$.

Now let us suppose that $n > 2$ and consider the graphs $D_n^*$ and $D_{n-1}^*$. Then using the above observation, one can prove the following equalities:

$$J_u^{(n-1)} = J_u^{(n)} \setminus \{u + 2^{n-1}\} \quad \text{if} \quad 1 \leq u \leq 2^{n-1} \quad \text{and} \tag{1}$$

$$J_{u-2^{n-1}}^{(n-1)} = \{v - 2^{n-1} : v \in (J_u^{(n)} \setminus \{u - 2^{n-1}\})\} \quad \text{if} \quad 2^{n-1} < u \leq 2^n \ . \tag{2}$$

Now we are ready to prove our statement.

**Theorem 0.1** *Let $\mathcal{D}^* = \{D_n^* : n = 2, 3, \ldots\}$. A system $\mathcal{C} \subseteq \mathcal{U}_R$ of tree automata is isomorphically complete for $\mathcal{U}_R$ with respect to the $\mathcal{D}^*$-product if and only if $\mathcal{C}$ contains a tree automaton $\mathcal{A} = (A, \Sigma)$ which has two different states $a$, $b$ and for any $m \in R$, $(u_1, \ldots, u_m) \in \{a, b\}^m$, $u \in \{a, b\}$ there exists an $m$-ary operation $\sigma \in \Sigma$ with $\sigma^{\mathcal{A}}(u_1, \ldots, u_m) = u$.*

**Proof.** If $R = \{0\}$, then the validity of our statement can be proved easily. Now let us suppose that $R \neq \{0\}$. Then the necessity follows from the work [5].

In order to prove the sufficiency, first let us define the sequence of matrices $\mathbf{A}^{(1)}$, $\mathbf{A}^{(2)}$, ... as follows:

$$\mathbf{A}^{(1)} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \qquad \mathbf{A}^{(n+1)} = \begin{pmatrix} \mathbf{A}^{(n)} & \mathbf{A}^{(n)} \\ \mathbf{A}^{(n)} & \bar{\mathbf{A}}^{(n)} \end{pmatrix} \qquad (n = 1, 2, \ldots) ,$$

where $\bar{\mathbf{A}}^{(n)}$ is defined by $\bar{a}_{ij}^{(n)} = 1 - a_{ij}^{(n)}$ $(1 \leq i \leq 2^{n+1};\ 1 \leq j \leq 2^n)$ in the partitioned matrix.

We shall show that for any $n \geq 2$ and $1 \leq u \leq 2^n$ the $n$-tuples $(a_{ti_1}^{(n)}, \ldots, a_{ti_n}^{(n)})$ $(t = 1, \ldots, 2^n)$ are pairwise different, where $\{i_1, \ldots, i_n\} = J_u^{(n)}$.

We proceed by induction on $n$. The case $n = 2$ can be checked easily. Now let $n > 2$ and assume that the statement is valid for $n-1$. Let $1 \leq u \leq 2^n$ be arbitrary and $J_u^{(n)} = \{i_1, \ldots, i_n\}$. Let us suppose that $i_v < i_w$ if $v < w$. If the desired $n$-tuples are pairwise not different, then there are indices $j, k$ with $1 \leq j < k \leq 2^n$ such that

$$(a_{ji_1}^{(n)}, \ldots, a_{ji_n}^{(n)}) = (a_{ki_1}^{(n)}, \ldots, a_{ki_n}^{(n)}) \ . \tag{3}$$

Now we distinguish three cases.

**Case 1.** Let us suppose that $1 \le j < k \le 2^{n-1}$. If $1 \le u \le 2^{n-1}$, then $n-1$ ancestors of $u$ are in the set $\{1,\ldots,2^{n-1}\}$ and the $n$th ancestor is $u+2^{n-1}$. Therefore, by the ordering of $J_u^{(n)}$, $i_n = u+2^{n-1}$. Then, by (1), $J_u^{(n-1)} = \{i_1,\ldots,i_{n-1}\}$ and by the definition of $\mathbf{A}^{(n)}$,

$$(a_{ji_1}^{(n-1)},\ldots,a_{ji_{n-1}}^{(n-1)}) = (a_{ji_1}^{(n)},\ldots,a_{ji_{n-1}}^{(n)}) =$$

$$(a_{ki_1}^{(n)},\ldots,a_{ki_{n-1}}^{(n)}) = (a_{ki_1}^{(n-1)},\ldots,a_{ki_{n-1}}^{(n-1)})$$

which contradicts our induction assumption.

If $2^{n-1} < u \le 2^n$, then $n-1$ ancestors of $u$ are in the set $\{2^{n-1}+1,\ldots,2^n\}$ and the $n$th ancestor is $u-2^{n-1}$. Therefore, $i_1 = u-2^{n-1}$. Let $w_t = i_t - 2^{n-1}$ $(t = 2,\ldots,n)$. Then by (2), $J_{i_1}^{(n-1)} = \{w_2,\ldots,w_n\}$. But then using the equality (3) and the definition of $\mathbf{A}^{(n)}$, we obtain that

$$(a_{jw_2}^{(n-1)},\ldots,a_{jw_n}^{(n-1)}) = (a_{kw_2}^{(n-1)},\ldots,a_{kw_n}^{(n-1)})$$

which contradicts our induction assumption.

**Case 2.** Assume that $2^{n-1} < j < k \le 2^n$.

Let $r = j-2^{n-1}$, $s = k-2^{n-1}$. Then $1 \le r < s \le 2^{n-1}$. On the other hand, by the construction of $\mathbf{A}^{(n-1)}$, from (3) it follows that $(a_{ri_1}^{(n)},\ldots,a_{ri_n}^{(n)}) = (a_{si_1}^{(n)},\ldots,a_{si_n}^{(n)})$ which yields a contradiction in the same way as in Case 1.

**Case 3.** Let us suppose that $1 \le j \le 2^{n-1} < k \le 2^n$.

If $1 \le u \le 2^{n-2}$, then by (1), $i_n = u+2^{n-1}$, $i_{n-1} = u+2^{n-2}$ and $J_u^{(n-2)} = \{i_1,\ldots,i_{n-2}\} \subseteq \{1,\ldots,2^{n-2}\}$. Since $i_n = u+2^{n-1}$, by the definition of $\mathbf{A}^{(n)}$ and (3), we obtain $a_{ju}^{(n)} = a_{ji_n}^{(n)} = a_{ki_n}^{(n)} = a_{ku}^{(n)}$. By (3), $a_{ji_{n-1}}^{(n)} = a_{ki_{n-1}}^{(n)}$, which results that $k \ne j+2^{n-1}$. Now let $r = k-2^{n-1}$. Then $1 \le r \le 2^{n-1}$. Since $1 \le u \le 2^{n-2}$ and $2^{n-2} < i_{n-1} \le 2^{n-1}$, by the construction of $\mathbf{A}^{(n-1)}$, we obtain $a_{ku}^{(n)} = a_{ru}^{(n)}$, $a_{ki_{n-1}}^{(n)} = 1-a_{ri_{n-1}}^{(n)}$. But then $a_{ji_{n-1}}^{(n)} = 1-a_{ri_{n-1}}^{(n)}$. On the other hand, $1 \le u \le 2^{n-2}$, $u+2^{n-2} = i_{n-1}$, $a_{ju}^{(n)} = a_{ru}^{(n)}$, $1 \le j,r \le 2^{n-1}$ yield that $a_{ji_{n-1}}^{(n)} = a_{ri_{n-1}}^{(n)}$ which is a contradiction.

If $2^{n-2} < u \le 2^{n-1}$, then on the bases of (1) and (2), $i_n = u+2^{n-1}$, $i_1 = u-2^{n-2}$ and $\{i_2,\ldots,i_{n-1}\} \subseteq \{2^{n-2}+1,\ldots,2^{n-1}\}$. But then, by (3) and the definition of $\mathbf{A}^{(n)}$, $a_{ju}^{(n)} = a_{ku}^{(n)}$, which yields $k \ne j+2^{(n-1)}$. Let $r = k-2^{n-1}$. By the construction of $\mathbf{A}^{(n-1)}$, $a_{ru}^{(n)} = 1-a_{ku}^{(n)}$, and so, $a_{ru}^{(n)} = 1-a_{ju}^{(n)}$. On the other hand, by (3), $a_{ki_1}^{(n)} = a_{ji_1}^{(n)}$, and so, by the construction of $\mathbf{A}^{(n-1)}$, $a_{ri_1}^{(n)} = a_{ji_1}^{(n)}$. Since $i_1+2^{n-2} = u$ and $1 \le j,r \le 2^{n-1}$, by the construction of $\mathbf{A}^{(n-1)}$, we obtain that the last equality yields $a_{ru}^{(n)} = a_{ju}^{(n)}$ which is a contradiction.

If $2^{n-1} < u \le 2^n$, then $i_1 = u-2^{n-1}$. Let $w_{t-1} = i_t - 2^{n-1}$ $(t = 2,\ldots,n)$ and $w_n = i_1+2^{n-1} = u$. Then by (1) and (2), $J_{i_1}^{(n)} = \{w_1,\ldots,w_n\}$. On the other hand, by (3) and the definition of $\mathbf{A}^{(n)}$, we obtain the equality $(a^{(n)}_{jw_1},\ldots,a^{(n)}_{jw_n}) = (a^{(n)}_{kw_1},\ldots,a^{(n)}_{kw_n})$. Since $1 \le i_1 \le 2^{n-1}$, we have traced back the considered case to the above treated ones.

Now let us suppose that $\mathcal{C}$ contains a tree automaton $\mathcal{A} = (A,\Sigma)$ satisfying the conditions of our Theorem with the elements $a$, $b$. Without loss of generality we may assume that $a = 0$ and $b = 1$. Furthermore, for any $m \in R$, $(u_1,\ldots,u_m) \in \{0,1\}^m$, $u \in \{0,1\}$ let us denote by $\sigma_{u_1,\ldots,u_m,u}$ an operational symbol from $\Sigma_m$ for which $\sigma^{\mathcal{A}}_{u_1,\ldots,u_m,u}(u_1,\ldots,u_m) = u$ holds.

Now let $\mathcal{B} = (\{b_1,\ldots,b_w\},\Sigma')$ be an arbitrary tree automaton. Choose an integer $n \ge 2$ such that $w \le 2^n$. Let $\mu$ be a one to one mapping of $\{b_1,\ldots,b_w\}$ onto the first $w$ rows of the matrix $\mathbf{A}^{(n)}$ defined by $\mu(b_k) = (a^{(n)}_{k1},\ldots,a^{(n)}_{k2^n})$ $(k = 1,\ldots,w)$. Denote by $S$ the set $\{\mu(b_k) : k = 1,\ldots,w\}$. Let $1 \le u \le 2^n$ be arbitrary. We know that the $n$-tuples $(a^{(n)}_{ti_1},\ldots,a^{(n)}_{ti_n})$ $(t = 1,\ldots,2^n)$ are pairwise different, where $\{i_1,\ldots,i_n\} = J^{(n)}_u$. But then there is a one to one mapping $\tau_u$ for which $\tau_u(a^{(n)}_{ti_1},\ldots,a^{(n)}_{ti_n}) = b_t$ $(t = 1,\ldots,w)$. Let us consider these mappings $\tau_u$ for any $1 \le u \le 2^n$.

Take the $\mathcal{D}^*$-product $\tilde{\mathcal{A}} = \prod_{j=1}^{2^n} \mathcal{A}(\Sigma',\Psi,D^*_n)$, where the family $\Psi$ of mappings is defined as follows:

For any $0 \ne m \in R$, $\sigma \in \Sigma'_m$, $1 \le u \le 2^n$ and $\mathbf{s}_t = (s_{t1},\ldots,s_{t2^n}) \in S$ $(t = 1,\ldots,m)$,

$$\Psi_{mu}(\mathbf{s}_1,\ldots,\mathbf{s}_m,\sigma) = \sigma_{s_{1i_n},\ldots,s_{mi_n},a^{(n)}_{ku}}$$

where $\sigma^{\mathcal{B}}(\tau_u(s_{1i_1},\ldots,s_{1i_n}),\ldots,\tau_u(s_{mi_1},\ldots,s_{mi_n})) = b_k$.

If $0 \in R$, $\sigma \in \Sigma'_0$ and $\mu(\sigma^{\mathcal{B}}) = (a^{(n)}_{k1},\ldots,a^{(n)}_{k2^n})$, then

$$\Psi_{mu}(\sigma) = \sigma^{\mathcal{A}}_{a^{(n)}_{ku}}$$

For any $m \in R$, $\sigma \in \Sigma'$, $1 \le u \le 2^n$ and $((u_{11},\ldots,u_{12^n}),\ldots(u_{m1},\ldots,u_{m2^n})) \in \{A^{2^n}\}^m \setminus S^m$, $\Psi_{mu}$ is defined arbitrarily in accordance with the definition of the $\mathcal{D}^*$-product.

It is easy to see that the mappings $\Psi_{mu}$ are well-defined, and so, we obtain a $\mathcal{D}^*$-product. On the other hand, one can prove that $\mu$ is an isomorphism of $\mathcal{B}$ into $\tilde{\mathcal{A}}$. Therefore, $\{\mathcal{A}\}$ is an isomorphically complete system for $\mathcal{U}_R$ with respect to the $\mathcal{D}^*$-product, which completes our proof.

**Remark.** Characterization of the isomorphically complete systems with respect to the Gluskov-type product (see [5]) is the same as the characterization in our Theorem. So this two kind of products are equivalent with respect to the isomorphically complete systems.

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