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The Self-organizing List and Processor Problems under Randomized Policies

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Abstract

We consider the self-organizing list problem in the case that only one item has a different request probability and show that transposition has a steady state cost stochastically smaller than any randomized policy that moves the requested item, found in position i, to position j with some probability $a_{ij}, i \geq j$. A random variable X is said to be stochastically smaller than another random variable Y, written $X \leq_{st} Y$ if $Pr\{X \geq k\} \leq Pr\{Y \geq k\}$, for any k. This is a stronger statement than $E[X] \leq E[Y]$. We also show that the steady state cost under the policy that moves the requested item i positions forward is stochastically increasing in i. Sufficient conditions are given for the steady state cost under a randomized policy **A** to be stochastically smaller than that under another randomized policy **B**. Similar results are obtained for the processor problem, where a list of processors is considered.

OPTIMAL LIST ORDER; MEMORY CONSTRAINTS; TRANSPOSITION RULE; RAMDOMIZATION

0 Introduction

A *self-organizing list problem* is characterized by a sequential list of n items subject to a reordering policy. At the beginning of each time period, an item is requested and the list is searched sequentially from the first position until the requested item is found. Each of these n items has an unknown probability of being requested. Let $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ be the request probability vector, where p_i is the request probability of item $i, i = 1, \ldots, n$, and $0 < p_i \leq 1$, $\sum_{i=1}^{n} p_i = 1$. At the end of each period, the items on the list are reordered according to the reordering policy. The cost of each period is taken to be the position where the requested item is found. We are interested in the steady state costs under various policies. A reordering policy is called *optimal* if it minimizes the expected steady state cost for any given request probability vector $\mathbf{p}$. The self-organizing list problem will now be called *the list problem* and the *policy* will mean the reordering policy.

Kan and Ross [6] define a *no-memory* policy as a reordering policy that depends only on the position of the requested item and the current ordering. Some of the most studied examples of the no-memory policies are the *transposition, move-to-front,* and *move-i-position* policies. Keeping the relative positions of all other

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items unchanged, the move-i-position policy moves the requested item i positions closer to the front if the requested item is found at position $j, j > i$, otherwise the requested item is moved to the first position. Transposition is just move-1-position and move-to-front is move-$(n-1)$-position for a problem of n items. Hendricks [3,4] gives the steady state probability distributions of states under move-to-front and transposition. See Hester and Hirschberg [5] for a recent survey of the list problem.

Anderson, Nash and Weber [1] show by counterexample that transposition is not optimal. However, their counterexample not only moves the requested item but also changes the positions of other items. So it is still an open question if transposition is optimal among policies that move only the requested item, leaving the relative ordering of the rest unchanged.

In the special case where only one item has a different request probability, Kan and Ross [6] and Phelps and Thomas [7] show that transposition is indeed optimal among policies that move only the requested item. We will show in Section 1.2 that transposition is optimal in a stronger sense. In particular, by extending the induction argument used by Phelps and Thomas, we can show that transposition has a steady state cost *stochastically smaller* than that of any *randomized policy*. Let $C(\mathrm{p};\mathbf{A})$ be the steady state cost of the list problem with request probability vector p under policy **A**. Then $C(\mathrm{p};\mathbf{A})$ is stochastically smaller than $C(\mathrm{p};\mathbf{B})$, written $C(\mathrm{p};\mathbf{A}) \leq_{st} C(\mathrm{p};\mathbf{B})$, if $Pr\{C(\mathrm{p};\mathbf{A}) \geq k\} \leq Pr\{C(\mathrm{p};\mathbf{B}) \geq k\}, k = 1, 2, \ldots, n$. It follows immediately that $\mathrm{E}[C(\mathrm{p};\mathbf{A})] \leq \mathrm{E}[C(\mathrm{p};\mathbf{B})]$. A randomized policy is a policy which, when an item is requested and found at position i, moves that item to position j with some probability $a_{ij}, \sum_{j=1}^{i} a_{ij} = 1$, leaving the relative ordering of others unchanged.

Section 1.1 defines the randomized policy and shows its properties. By the introduction of the randomized policy, we also show in Section 1.2 that move-i-position has a steady state cost stochastically increasing in i. This partially supports the conjecture of Gonnett, Munro, and Suwanda [2]. Their conjecture says that if **A** and **B** are two no-memory policies such that if the requested item is found at position i, it is moved forward $A(i)$ and $B(i)$ positions by the policies **A** and **B** respectively, and $A(i) \leq B(i), i = 1, \ldots, n$, then the expected steady state cost under **A** is smaller than or equal to that under **B**, but **B** converges to its asymptotic behavior more quickly than **A**. Furthermore, it also follows that if the cost is taken to be an increasing function of the position where the requested item is found, move-i-position will have an expected steady state cost increasing in i. A special case of this situation is found in the *paging problem* as also discussed by Phelps and Thomas [7] where for a fixed integer $m, 1 \leq m \leq n$, the cost is taken to be zero if the requested item is found in a postion less than m, and one otherwise.

Tenenbaum and Nemes [9] consider two spectra of policies. Assuming that only one item has a different request probability, the policies in each of the two spectra are ordered by the values of their expected steady state costs. Each spectrum has transposition at one end with the minimum expected steady state cost and move-to-front at the other with the maximum expected steady state cost. We will show in Section 1.2 that the steady state costs of these policies in each spectrum are stochastically smaller or larger than each other.

A problem related to the list problem is called *the processor problem* which was studied by, among others, Topkis [10]. In the processor problem, we consider a sequential list containing an ordering of the n processors. Each of these processors has an unknown probability that it will successfully process a given job. At the beginning of each time priod, there is an arrival of a job to be processed. The job is attempted by the processors successively according to the ordering in the list until either one of the processors succeeds or all of them fail. Then the job is dismissed. The cost in each period is taken to be the number of processors attempted until the job is processed, or, in the case that all n processors fail, it is taken to be n. At the end of each period, a reordering policy is applied in the same manner as in the list probelm. For example, we might move the successful processor to the beginning of the ordering (move-to-front), or we might just move it one position closer to the front (transposition).

Topkis [10] gives the steady state probabilities of the move-to-front and move-to-back policies and shows that move-to-front has a steady state cost stochastically smaller than move-to-back, which in turn, has a steady state cost stochastically smaller than the random policy where processors are equally likely to be in any of the $n!$ orderings.

Section 2.1 shows the properties of randomized policy when applied to the processor problem with only one processor having a different success probability. In this special case, Ross [8] shows that the expected steady state cost under transposition is smaller than or equal to that under move-to-front. In Section 2.2, we also use randomized policies to obtain results closely parallel to those of the list problem. That is, the steady state cost under transposition is stochastically smaller than that under any randomized policy. Furthermore, the steady state cost under move-i-position is stochastically smaller than that under move-$(i+1)$-position. The steady state costs under the policies in the two spectra proposed by Tenenbaum and Nemes [9] are also ordered such that the steady state cost of each policy is stochastically smaller or larger than its neighbors in the same spectrum.

1 The List Problem

When only item 1 has a different request probability, the expected steady state cost can be written in terms of the expected position of the item 1. That is, by conditioning on whether item 1 is being requested,

$$\begin{aligned} \mathrm{E}[C(\mathbf{p};\mathbf{A})] &= cp\mathrm{E}[Y_1(\mathbf{p};\mathbf{A})] + \frac{p(n-1)\mathrm{E}[1+2+\cdots+n-Y_1(\mathbf{p};\mathbf{A})]}{n-1} \\ &= p(c-1)\mathrm{E}[Y_1(\mathbf{p};\mathbf{A})] + \frac{pn(n+1)}{2}, \end{aligned}$$

where $Y_1(\mathbf{p};\mathbf{A})$ is the steady state position of item 1 of the list problem with request probability vector $\mathbf{p}$ under policy $\mathbf{A}, p_1 = cp, p_2 = p, \ldots, p_n = p$, and $c > 0$.

So when $c > 1$, we want to minimize $\mathrm{E}[Y_1(\mathbf{p};\mathbf{A})]$, and maximize it when $c < 1$. For the rest of the paper, we assume that $c > 1$. The results for $c < 1$ will be just the opposite.

1.1 Randomized Policy

A randomized policy is characterized by a matrix $\mathbf{A} = [A_{ij}]_{n\times n}$, where $A_{ij} = \sum_{k=1}^{j} a_{ik}$, and a_{ij} is the probability that given an item is requested and found at position i, it is moved to position j, where $\sum_{j=1}^{i} a_{ij} = 1$ for all i, and $0 \le a_{ij} \le 1$. So A_{ij} is the probability that given the requested item is found at position i, it is moved to a position less than or equal to j.

Given a policy $\mathbf{A}$ defined in a system of n items, define a related policy $\mathbf{A}^d$ in a system of $n-1$ items as follows.

$$\mathbf{A}^d = [A^d_{ij}]_{(n-1)\times(n-1)},$$

where $A^d_{ij} = \sum_{k=1}^{j} a^d_{ik}$, and

$$a^d_{ij} = \begin{cases} a_{i+1,1} + a_{i+1,2} & , j = 1 \\ a_{i+1,j+1} & , j \geq 2. \end{cases} \tag{1.1}$$

Let π^A_i be the steady state probability that item 1 is at position i under policy **A**. That is, $\pi^A_i = \Pr\{Y_1(\mathbf{p};\mathbf{A}) = i\}$. Alternatively, we can say $Y_1(\mathbf{p};\mathbf{A}) \leq_{st} Y_1(\mathbf{p};\mathbf{B})$ by using the notation $\{\pi^A_i\} \leq_{st} \{\pi^B_i\}$. Define $K^A_i = \pi^A_i/\pi^A_n$. Lemma 1.1 to Lemma 1.4 below show the relationships between $\{\pi^A_i\}$ and $\{\pi^{dA}_i\}$ under the assumption that $(p^d_1, p^d_2, \ldots p^d_{n-1}) = (cp^d, p^d, \ldots, p^d)$. Lemma 1.1 and Lemma 1.2 are also obtained by Phelps and Thomas [7], where they consider only policies that move the requested item, found at position i, to a fixed position $\tau(i), \tau(i) \leq i$.

Lemma 1.1 *Under policy* **A**, *for* $i = 2, \ldots, n$,

$$\pi^A_i/\pi^A_n = K^A_i = \pi^{dA}_{i-1}/\pi^{dA}_{n-1}.$$

Proof. The transition matrix, showing only columns $1, r+1$ and n can be written as

$$\begin{bmatrix} cp + p\sum_{i=2}^{n}\sum_{j=2}^{i} a_{ij} & \cdots & 0 & \cdots & 0 \\ cpa_{21} & \cdots & \vdots & \cdots & \vdots \\ cpa_{31} & \cdots & 0 & \cdots & \\ \vdots & \cdots & p\sum_{i=r+1}^{n}\sum_{j=1}^{r} a_{ij} & \cdots & \\ & \cdots & cpa_{r+1,r+1} + p\theta & \cdots & \vdots \\ & \cdots & cpa_{r+2,r+1} & \cdots & 0 \\ \vdots & \cdots & \vdots & \cdots & p(1-a_{nn}) \\ cpa_{n1} & \cdots & cpa_{n,r+1} & \cdots & cpa_{nn} + p(n-1) \end{bmatrix} \tag{1.2}$$

where $\theta = r + \sum_{i=r+2}^{n}\sum_{j=r+2}^{i} a_{ij}$

Except the first column, column $r+1$ contains zeros from row 1 to row r. Using the $(r+1)^{st}$ column of the transition matrix and suppressing the superscript **A**, we have

$$\begin{aligned} \pi_{r+1} &= \pi_r p\left(\sum_{i=r+1}^{n}\sum_{j=1}^{r} a_{ij}\right) + \pi_{r+1}\left[cpa_{r+1,r+1} + p\theta\right] \\ &+ \sum_{i=r+2}^{n} cpa_{i,r+1}\pi_i, \end{aligned}$$

for $r = 1, \ldots, n-1$.

Since $p = \frac{1}{c+n-1}$, the above equation becomes

$$\begin{aligned}
&\pi_r\left(\sum_{i=r+1}^{n}\sum_{j=1}^{r}a_{ij}\right)\\
&= \pi_{r+1}\left[c+n-1-ca_{r+1,r+1}-\left(r+\sum_{i=r+2}^{n}\sum_{j=r+2}^{i}a_{ij}\right)\right]-\sum_{i=r+2}^{n}ca_{i,r+1}\pi_i\\
&= \pi_{r+1}\left[n+c(1-a_{r+1,r+1})-(r+1)-\sum_{i=r+2}^{n}\sum_{j=r+2}^{i}a_{ij}\right]-c\sum_{i=r+2}^{n}a_{i,r+1}\pi_i,
\end{aligned} \tag{1.3}$$

where $r = 1,\ldots,n-1$.

For policy $\mathbf{A}^d$, using the r^{th} column of the transition matrix of $n-1$ items and noting that $p^d = \frac{1}{c+n-2}$, we have in the same manner as (1.3) above

$$\begin{aligned}
&\pi^d_{r-1}\left(\sum_{i=r}^{n-1}\sum_{j=1}^{r-1}a^d_{ij}\right)\\
&= \pi^d_r\left[c+n-2-ca^d_{rr}-\left(r-1+\sum_{i=r+1}^{n-1}\sum_{j=r+1}^{i}a^d_{ij}\right)\right]-\sum_{i=r+1}^{n-1}ca^d_{ir}\pi^d_i\\
&= \pi^d_r\left[n+c(1-a^d_{rr})-(r+1)-\sum_{i=r+1}^{n-1}\sum_{j=r+1}^{i}a^d_{ij}\right]-c\sum_{i=r+1}^{n-1}a^d_{ir}\pi^d_i,
\end{aligned} \tag{1.4}$$

where $r = 2,\ldots,n-1$. By the definition of a^d_{ij} given in (1.1), (1.4) becomes

$$\begin{aligned}
&\pi^d_{r-1}\left(\sum_{i=r+1}^{n}\sum_{j=1}^{r}a_{ij}\right)\\
&= \pi^d_r\left[n+c(1-a_{r+1,r+1})-(r+1)-\sum_{i=r+2}^{n}\sum_{j=r+2}^{i}a_{ij}\right]-c\sum_{i=r+2}^{n}a_{i,r+1}\pi^d_{i-1},
\end{aligned} \tag{1.5}$$

where $r = 2,\ldots,n-1$. From (1.3) and (1.5), $(\pi_2,\ldots,\pi_n)$ and $(\pi^d_1,\ldots,\pi^d_{n-1})$ satisfy the same set of equations. We will use this fact to show that $K_i = K^d_{i-1}, i = 2,\ldots,n$, and this proves the Lemma. Since $K_n = K^d_{n-1} = 1$ by definition, we use the induction hypothesis that $K_i = K^d_{i-1}, i = r+1,\ldots,n$. We will show that it is also true for $i = r$. But this follows immediately by dividing both sides of (1.3) and (1.5) by π_n and π^d_{n-1} respectively. □

If we know π^A_i, $i = 1,\ldots,n$, then we know π^{dA}_i, $i = 1,\ldots,n-1$. The exact relationship is given in Lemma 1.2.

Lemma 1.2 *Under policy* **A**, *for* $i = 2, 3, \ldots, n$,

$$\pi_i^A = (1 - \pi_1^A)\pi_{i-1}^{dA}.$$

Proof. From Lemma 1.1, we need to show that $\pi_n^A = (1-\pi_1^A)\pi_{n-1}^{dA}$. By suppressing superscript **A**,

$$1 = \sum_{i=1}^{n-1} \pi_i^d = \sum_{i=1}^{n-1} K_i^d \pi_{n-1}^d = \pi_{n-1}^d \sum_{i=2}^{n} K_i = \pi_{n-1}^d (1 - \pi_1)/\pi_n.$$

□

Conversely, given π_i^{dA}, $i = 1, \ldots, n-1$, we can compute π_i^A, $i = 1, \ldots, n$, using Lemma 1.2 and the following Lemma 1.3.

Lemma 1.3 *Under policy* **A**,

$$\pi_1^A = \frac{c(a_{21}\pi_2^A + a_{31}\pi_3^A + \cdots + a_{n1}\pi_n^A)}{a_{21} + a_{31} + \cdots + a_{n1}}.$$

Proof. The Lemma is proved by using the first column of the transition matrix (1.2) and noting that $p = \frac{1}{c+n-1}$. □

From Lemma 1.1 and Lemma 1.3, Lemma 1.4 below says that we can write K_j in terms of $K_{j+1}, K_{j+2}, \ldots, K_n$. Note that $A_{i1} = a_{i1}, i = 2, \ldots, n$. So Lemma 1.3 and Lemma 1.4 are equivalent when $j = 1$.

Lemma 1.4 *Under policy* **A**, *for* $j = 1, 2, \ldots, n-1$,

$$K_j^A = \frac{c(A_{j+1,j}K_{j+1}^A + A_{j+2,j}K_{j+2}^A + \cdots + A_{nj}K_n^A)}{A_{j+1,j} + A_{j+2,j} + \cdots + A_{nj}}.$$

Proof. From Lemma 1.1, $K_2 = K_1^d, K_3 = K_2^d = K_1^{d^2}, \ldots, K_j = K_1^{d^{j-1}}$. By exactly the same argument, we have $K_k^{d^{j-1}} = K_{k+1}^{d^{j-2}} = \cdots = K_{j+k-1}, k = 2, \ldots, n-j+1$. From Lemma 1.3,

$$K_1^{d^{j-1}} = \frac{c(a_{21}^{d^{j-1}} K_2^{d^{j-1}} + a_{31}^{d^{j-1}} K_3^{d^{j-1}} + \cdots + a_{n-j+1,1}^{d^{j-1}} K_{n-j+1}^{d^{j-1}})}{a_{21}^{d^{j-1}} + a_{31}^{d^{j-1}} + \cdots + a_{n-j+1,1}^{d^{j-1}}}.$$

Now, by definiton (1.1),

$$\begin{aligned} a_{21}^{d^{j-1}} &= a_{31}^{d^{j-2}} + a_{32}^{d^{j-2}} \\ &= a_{41}^{d^{j-3}} + a_{42}^{d^{j-3}} + a_{43}^{d^{j-3}} \\ &\vdots \\ &= a_{j+1,1} + a_{j+1,2} + \ldots + a_{j+1,j} \\ &= A_{j+1,j}. \end{aligned}$$

Similarly, $a_{k1}^{d^{j-1}} = A_{j+k-1,j}, k = 3, \ldots, n-j+1$. So follows the Lemma. □

1.2 Comparison of the Steady State Costs and Probabilities of Two Lists under Two Different Policies

Let S be the set of policies that the resulting probability distribution $\{\pi_i\}$ is decreasing in i when $p_1 > p$ and increasing in i otherwise. The question of how to determine if a policy is in S will be addressed later. We are now ready to prove the following Theorem that compares $\{\pi_i\}$ of two different policies.

Theorem 1.5 *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be two policies such that, for* $j = 1, 2, \ldots, n-1, k = j+1, \ldots, n$,

$$\frac{A_{j+1,j} + A_{j+2,j} + \cdots + A_{kj}}{A_{j+1,j} + A_{j+2,j} + \cdots + A_{nj}} \geq \frac{B_{j+1,j} + B_{j+2,j} + \cdots + B_{kj}}{B_{j+1,j} + B_{j+2,j} + \cdots + B_{nj}}, \tag{1.6}$$

and at least one of these two conditions holds:

(a) $\mathbf{A} \in S$ *and* B_{ij} *is decreasing in* i *for all* $j = 1, \ldots, n$,
(b) $\mathbf{B} \in S$ *and* A_{ij} *is decreasing in* i *for all* $j = 1, \ldots, n$.

Then $\{\pi_i^A\} \leq_{st} \{\pi_i^B\}$ *for any* $\mathbf{p} = (cp, p, \ldots, p), c > 1$.

Proof. We will prove this Theorem by induction. It is easily checked that the Theorem is true for $n = 2$. Assume that it is true for the problem of $n-1$ items. Now given such policies $\mathbf{A}$ and $\mathbf{B}$, their corresponding policies $\mathbf{A}^d$ and $\mathbf{B}^d$ also satisfy all the conditions above. We can check this by first noting by that by (1.1)

$$A_{ij}^d = a_{i1}^d + a_{i2}^d + \cdots + a_{ij}^d = a_{i+1,1} + a_{i+1,2} + a_{i+1,3} + \cdots + a_{i+1,j+1} = A_{i+1,j+1}.$$

Therefore, A_{ij}^d is also decreasing in i, and

$$\begin{aligned}\frac{A_{j+1,j} + A_{j+2,j} + \cdots + A_{kj}}{A_{j+1,j} + A_{j+2,j} + \cdots + A_{nj}} &= \frac{A_{j,j-1}^d + A_{j+1,j-1}^d + \cdots + A_{k-1,j-1}^d}{A_{j,j-1}^d + A_{j+1,j-1}^d + \cdots + A_{n-1,j-1}^d} \\ &\geq \frac{B_{j,j-1}^d + B_{j+1,j-1}^d + \cdots + B_{k-1,j-1}^d}{B_{j,j-1}^d + B_{j+1,j-1}^d + \cdots + B_{n-1,j-1}^d}.\end{aligned}$$

Secondly, since $\mathbf{A} \in S, \pi_1^A \geq \pi_2^A \geq \cdots \geq \pi_n^A$. But from Lemma 1.2, $\pi_i^{dA} = \frac{\pi_{i+1}^A}{1-\pi_1^A}$, so $\pi_1^{dA} \geq \pi_2^{dA} \geq \cdots \geq \pi_{n-1}^{dA}$. This means $\mathbf{A}^d \in S$. So we have the induction hypothesis that

$$\{\pi_i^{dA}\} \leq_{st} \{\pi_i^{dB}\}.$$

From Lemma 1.2, $\pi_i^A + \pi_{i+1}^A + \cdots + \pi_n^A = (1-\pi_1^A)(\pi_{i-1}^{dA} + \pi_i^{dA} + \cdots + \pi_{n-1}^{dA})$. All we need to show is that $\pi_1^A \geq \pi_1^B$. From Lemma 1.2 and Lemma 1.3,

$$\pi_1^A = c(1-\pi_1^A)\frac{A_{21}\pi_1^{dA} + A_{31}\pi_2^{dA} + \cdots + A_{n1}\pi_{n-1}^{dA}}{A_{21} + A_{31} + \cdots + A_{n1}}.$$

Since $\pi_1^A \geq \pi_1^B$ if and only if $\frac{\pi_1^A}{1-\pi_1^A} \geq \frac{\pi_1^B}{1-\pi_1^B}$, we need to show that

$$\frac{A_{21}\pi_1^{dA} + A_{31}\pi_2^{dA} + \cdots + A_{n1}\pi_{n-1}^{dA}}{A_{21} + A_{31} + \cdots + A_{n1}} \geq \frac{B_{21}\pi_1^{dB} + B_{31}\pi_2^{dB} + \cdots + B_{n1}\pi_{n-1}^{dB}}{B_{21} + B_{31} + \cdots + B_{n1}}.$$

Assume first that (a) holds. Then, by (1.6) with $j = 1$ and, because $\mathbf{A}^d \in S$, $\pi_1^{dA} \geq \pi_2^{dA} \geq \cdots \geq \pi_{n-1}^{dA}$,

$$\begin{aligned}\frac{A_{21}\pi_1^{dA} + A_{31}\pi_2^{dA} + \cdots + A_{n1}\pi_{n-1}^{dA}}{A_{21} + A_{31} + \cdots + A_{n1}} &\geq \frac{B_{21}\pi_1^{dA} + B_{31}\pi_2^{dA} + \cdots + B_{n1}\pi_{n-1}^{dA}}{B_{21} + B_{31} + \cdots + B_{n1}} \\ &\geq \frac{B_{21}\pi_1^{dB} + B_{31}\pi_2^{dB} + \cdots + B_{n1}\pi_{n-1}^{dB}}{B_{21} + B_{31} + \cdots + B_{n1}}.\end{aligned}$$

The second inequality follows from the assumption that B_{i1} is decreasing in i and from the induction hypothesis that $\{\pi_i^{dA}\} \leq_{st} \{\pi_i^{dB}\}$.

Similarly, if (b) holds,

$$\begin{aligned}\frac{A_{21}\pi_1^{dA} + A_{31}\pi_2^{dA} + \cdots + A_{n1}\pi_{n-1}^{dA}}{A_{21} + A_{31} + \cdots + A_{n1}} &\geq \frac{A_{21}\pi_1^{dB} + A_{31}\pi_2^{dB} + \cdots + A_{n1}\pi_{n-1}^{dB}}{A_{21} + A_{31} + \cdots + A_{n1}} \\ &\geq \frac{B_{21}\pi_1^{dB} + B_{31}\pi_2^{dB} + \cdots + B_{n1}\pi_{n-1}^{dB}}{B_{21} + B_{31} + \cdots + B_{n1}}\end{aligned}$$

□

A consequence of this Theorem is that the steady state cost under policy **A** is stochastically smaller than the steady state cost under policy **B**.

Corollary 1.6 *Under the conditions of Theorem 1.5,* $C(\mathbf{p};\mathbf{A}) \leq_{st} C(\mathbf{p};\mathbf{B})$.

Proof. Conditioning on whether item 1 is at the first position, for $k = 2, \ldots, n$,

$$\begin{aligned}&\Pr\Big\{C(\mathbf{p};\mathbf{A} \geq k\Big\} \\ =\ &\pi_1^A \Pr\Big\{C(\mathbf{p};\mathbf{A}) \geq k|Y_1(\mathbf{p};\mathbf{A}) = 1\Big\} + (1-\pi_1^A)\Pr\Big\{C(\mathbf{p};\mathbf{A}) \geq k|Y_1(\mathbf{p};\mathbf{A}) \neq 1\Big\} \\ =\ &\pi_1^A \Pr\Big\{C(\mathbf{p};\mathbf{B}) \geq k|Y_1(\mathbf{p};\mathbf{B}) = 1\Big\} + (1-\pi_1^A)\Pr\Big\{C(\mathbf{p};\mathbf{A}) \geq k|Y_1(\mathbf{p};\mathbf{A}) \neq 1\Big\}.\end{aligned}$$

Now given that item 1 is not at position 1, the probability that it will be at position i, $2 \leq i \leq n$, is $\frac{\pi_i^A}{1-\pi_1^A}$, which is exactly π_{i-1}^{dA} by Lemma 1.2. That is, given item 1 is not at position 1, its probability distribution over $\{2, 3, \ldots, n\}$ is the same as the probability distribution of $\{\pi_i^{dA}\}$ over $\{1, 2, \ldots n-1\}$. Using the induction hypothesis that the Corollary is true for the list of size $n-1$, we have

$$\begin{aligned}\Pr\Big\{C(\mathbf{p};\mathbf{A}) \geq k|Y_1(\mathbf{p};\mathbf{A}) \neq 1\Big\} &= (1-p)\Pr\Big\{C(\mathbf{p}^d;\mathbf{A}^d) \geq k-1\Big\} \\ &\leq (1-p)\Pr\Big\{C(\mathbf{p}^d;\mathbf{B}^d) \geq k-1\Big\} \\ &= \Pr\Big\{C(\mathbf{p};\mathbf{B}) \geq k|Y_1(\mathbf{p};\mathbf{B}) \neq 1\Big\}.\end{aligned}$$

Therefore,

$$
\begin{aligned}
& \Pr\Big\{C(\mathbf{p};\mathbf{A}) \geq k\Big\} \\
\leq \;& \pi_1^A \Pr\Big\{C(\mathbf{p};\mathbf{B}) \geq k | Y_1(\mathbf{p};\mathbf{B}) = 1\Big\} + (1-\pi_1^A) \Pr\Big\{C(\mathbf{p};\mathbf{B}) \geq k | Y_1(\mathbf{p};\mathbf{B}) \neq 1\Big\} \\
\leq \;& \pi_1^B \Pr\Big\{C(\mathbf{p};\mathbf{B}) \geq k | Y_1(\mathbf{p};\mathbf{B}) = 1\Big\} + (1-\pi_1^B) \Pr\Big\{C(\mathbf{p};\mathbf{B}) \geq k | Y_1(\mathbf{p};\mathbf{B}) \neq 1\Big\} \\
= \;& \Pr\Big\{(\mathbf{p};\mathbf{B}) \geq k\Big\}.
\end{aligned}
$$

The second inequality follows from the fact that $\pi_1^A \geq \pi_1^B$ and, when $p_1 > p$, $\Pr\Big\{C(\mathbf{p};\mathbf{B}) \geq k | Y_1(\mathbf{p};\mathbf{B}) = 1\Big\} \leq \Pr\Big\{C(\mathbf{p};\mathbf{B}) \geq k | Y_1(\mathbf{p};\mathbf{B}) \neq 1\Big\}$. □

By Lemma 1.2 and Corollary 1.6, transposition is optimal in the sense that it has a steady state cost stochastically smaller than any randomized policy. Let T denote the transposition policy.

Corollary 1.7 *For any policy* $\mathbf{A}, C(\mathbf{p};T) \leq_{st} C(\mathbf{p};\mathbf{A})$.

Proof. Given $c > 1$, Phelps and Thomas [7] show that $\pi_1^T \geq \pi_1^Z$ for any policy $\mathbf{Z}$ that moves the requested item strictly forward by using the fact that $\pi_i^Z = (1-\pi_1^Z)\pi_{i-1}^{dZ}$. Since this fact also holds for any randomized policy $\mathbf{A}$ as shown in Lemma 1.2, so $\pi_1^T \geq \pi_1^A$ and thus $\{\pi_i^T\} \leq_{st} \{\pi_i^A\}$ by the same induction argument in Theorem 1.5. The Corollary then follows by Corollary 1.6. □

The next question is how we know if $\mathbf{A} \in S$. The counterexample below shows that not every policy $\mathbf{A}$ is in S even with A_{ij} nonincreasing in i for all j.

A counterexample:

Let A be a policy characterized by the following matrix.

$$
\mathbf{A} = \begin{bmatrix}
1 & 0 & 0 & & & 0 & 0 & 0 \\
\varepsilon & 1 & 0 & & & \vdots & \vdots & \vdots \\
\varepsilon & 1-\varepsilon & 1 & \ddots & & & & \\
\vdots & \varepsilon & 1-\varepsilon & \ddots & & & & \\
 & \vdots & \varepsilon & \ddots & \ddots & & & \\
 & & \vdots & \ddots & \ddots & 0 & & \\
 & & & & \ddots & 1 & 0 & \vdots \\
\vdots & \vdots & \vdots & & \ddots & 1-\varepsilon & 1 & 0 \\
\varepsilon & \varepsilon & \varepsilon & & & \varepsilon & 1-\varepsilon & 1
\end{bmatrix}
$$

Let ε be some small number. The policy $\mathbf{A}$ almost always moves the requested item one position closer unless the requested item is founded at position 2 where it stays put with probability $1-\varepsilon$ and moves to position 1 with probability ε. By selecting small enough ε, we can get the values of K_i, as given by Lemma 1.4, to approach c^{n-i} arbitrarily close for $i \geq 2$. The value of K_1, as also given by Lemma

1.4, is

$$\begin{aligned} K_1 &= \frac{c(\varepsilon c^{n-2} + \varepsilon c^{n-3} + \cdots + \varepsilon c + \varepsilon)}{\varepsilon + \varepsilon + \cdots + \varepsilon} \\ &= \frac{c(c^{n-1} - 1)}{(n-1)(c-1)}. \end{aligned}$$

With $c = 3$ and $n = 6$, $K_1 = 72.6$ while $K_2 = 3^4 = 81$. So here K_i is not decreasing in i when $c > 1$. Thus not every policy has $\{\pi_i\}$ decreasing in i when $c > 1$. ■

The following Proposition gives a sufficient condition for $\mathbf{A} \in S$. This sufficient condition turns out to be true for any policy $\mathbf{A}$ under which the distribution of the number of positions to move the requested item is independent of the position where it is found. In other words, there is only one distribution for all positions. Call these policies *position independent*. One can interpret a position independent policy as one that uses a mixture of move-i-position, $i = 1, \ldots, n-1$.

Proposition 1.8 *A policy* $\mathbf{A} \in S$ *if, for* $j = 1, \ldots, n-1$,

$$\frac{A_{j+1,j} + \cdots + A_{nj}}{A_{j+2,j+1} + \cdots + A_{n,j+1}} \le \frac{A_{j+1,j} + \cdots + A_{n-1,j}}{A_{j+2,j+1} + \cdots + A_{n-1,j+1}} \le \cdots \le \frac{A_{j+1,j} + A_{j+2,j}}{A_{j+2,j+1}}. \tag{1.7}$$

Proof. Since $A^d_{ij} = A_{i+1,j+1}$, a condition similar to (1.7) holds for $\mathbf{A}^d$. By the induction hypothesis, $\mathbf{A}^d \in S$ and $\pi_1^{dA} \ge \pi_2^{dA} \ge \cdots \ge \pi_{n-1}^{dA}$. So by using Lemma 1.1 we have $\pi_2^A \ge \pi_3^A \ge \cdots \ge \pi_n^A$ and $K_2^A \ge K_3^A \ge \cdots \ge K_n^A$. Thus it remains to show that $\pi_1^A \ge \pi_2^A$. By Lemma 1.3, this means we have to show

$$\frac{A_{21}K_2^A + A_{31}K_3^A + \cdots + A_{n1}K_n^A}{A_{32}K_3^A + A_{42}K_4^A + \cdots + A_{n2}K_n^A} \ge \frac{A_{21} + A_{31} + \cdots + A_{n1}}{A_{32} + A_{42} + \cdots + A_{n2}}.$$

Rewrite the nominator on the left hand side of the above inequality as follows.

$$\begin{aligned} A_{21}K_2^A + A_{31}K_3^A + \cdots + A_{n1}K_n^A &= K_n^A(A_{21} + A_{31} + \cdots + A_{n1}) \\ &+ (K_{n-1}^A - K_n^A)(A_{21} + A_{31} + \cdots + A_{n-1,1}) + \\ &\cdots + (K_3^A - K_4^A)(A_{21} + A_{31}) + (K_2^A - K_3^A)A_{21} \end{aligned}$$

The left hand side of the last inequality becomes

$$\frac{K_n^A(A_{21} + A_{31} + \cdots + A_{n1}) + (K_{n-1}^A - K_n^A)(A_{21} + A_{31} + \cdots + A_{n-1,1}) + \cdots}{K_n^A(A_{32} + A_{42} + \cdots + A_{n2}) + \cdots}$$

$$\frac{\cdots + (K_3^A - K_4^A)(A_{21} + A_{31}) + (K_2^A - K_3^A)A_{21}}{\cdots + (K_{n-1}^A - K_n^A)(A_{32} + A_{42} + \cdots + A_{n-1,2}) + \cdots + (K_3^A - K_4^A)A_{32}},$$

and because $K_i^A - K_{i+1}^A \ge 0$, $i = 2, \ldots, n-1$, it is greater than the right hand side if

$$\frac{A_{21} + A_{31} + \cdots + A_{n1}}{A_{32} + A_{42} + \cdots + A_{n2}} \le \frac{A_{21} + A_{31} + \cdots + A_{n-1,1}}{A_{32} + A_{42} + \cdots + A_{n-1,2}} \le \cdots \le \frac{A_{21} + A_{31}}{A_{32}},$$

which is just (1.7) with $j = 1$. This follows from the fact that, $\frac{a}{b} \le \frac{a+c}{b+d}$ if $\frac{a}{b} \le \frac{c}{d}$, where a, b, c and d are positive. □

We will show next that (1.7) holds for any position independent policy that moves, with probability $a_i, \sum_{i=0}^{n-1} a_i = 1$, the requested item i positions forward if it is found at a position greater than or equal to $i+1$. Otherwise the policy moves the requested item to the first position. Thus, $a_{ij} = a_{i-j}, j > 1$, and $a_{i1} = a_{i-1} + a_i + \cdots + a_{n-1}$. Let $\overline{A}_i = \sum_{k=i}^{n-1} a_k$ be the probability that the requested item is moved more than or equal to i positions. Thus,

$$A_{ij} = a_{i1} + a_{i2} + \cdots + a_{ij} = (a_{i-1} + \cdots + a_{n-1}) + a_{i-2} + \cdots + a_{i-j} = \overline{A}_{i-j}.$$

So (1.7) becomes

$$\frac{\overline{A}_1 + \overline{A}_2 + \cdots + \overline{A}_{n-1}}{\overline{A}_1 + \overline{A}_2 + \cdots + \overline{A}_{n-2}} \le \frac{\overline{A}_1 + \overline{A}_2 + \cdots + \overline{A}_{n-2}}{\overline{A}_1 + \overline{A}_2 + \cdots + \overline{A}_{n-3}} \le \cdots \le \frac{\overline{A}_1 + \overline{A}_2}{\overline{A}_1},$$

which can be shown to be true by just cross-multiplying terms on each side of each inequality and noting that $\overline{A}_i$ is decreasing in i by its definition. Thus we have proved the following Lemma.

Lemma 1.9 *Let* $\mathbf{A}$ *be a position independent policy that moves requested item* i *positions with probability* a_i, $\sum_{i=0}^{n-1} a_i = 1$. *Then* $\mathbf{A} \in S$.

When $a_0 > 0$, we can look at the embedded Markov chain when the items actually change positions. The probability that item 1 is at position i in this embedded Markov chain will be equal to the proportion of time item 1 is at position i in the original chain. The policy governing the embedded chain is characterized by $a_i^* = \frac{a_i}{1-a_0}, i = 1, \ldots, n$, and $a_0^* = 0$. We can, without loss of generality, restrict ourselves from now on to the position independent policies that always move the requested item at least one position closer to the front, unless it is already at the first position.

When two position independent policies $\mathbf{A}$ and $\mathbf{B}$ are compared, (1.7) of Proposition 1.8 becomes, for $k = 1, \ldots, n-1$,

$$\frac{\overline{A}_1 + \overline{A}_2 + \cdots + \overline{A}_k}{\overline{A}_1 + \overline{A}_2 + \ldots + \overline{A}_{n-1}} \ge \frac{\overline{B}_1 + \overline{B}_2 + \cdots + \overline{B}_k}{\overline{B}_1 + \overline{B}_2 + \ldots + \overline{B}_{n-1}}. \tag{1.8}$$

An interpretation of this condition (1.8) is as follows. Let X^A be the renewal time of some renewal process with $\Pr\{X^A = i\} = a_i$, $i = 1, \ldots, n-1$, and $a_0 = 0$. Then the equilibrium renewal time of X^A, called X_e^A, will be distributed by

$$\Pr\{X_e^A \le k\} = \frac{\overline{A}_1 + \overline{A}_2 + \cdots + \overline{A}_k}{\overline{A}_1 + \overline{A}_2 + \cdots + \overline{A}_{n-1}}.$$

Therefore, (1.8) means $X_e^A \le_{st} X_e^B$. Theorem 1.5 combined with Corollary 1.6 can be restated for position independent policies as follows.

Theorem 1.10 *Given two position independent policies* $\mathbf{A}$ *and* $\mathbf{B}$ *such that* $X_e^A \le_{st} X_e^B$, *then* $\{\pi_i^A\} \le_{st} \{\pi_i^B\}$ *and* $C(\mathbf{p}; \mathbf{A}) \le_{st} C(\mathbf{p}; \mathbf{B})$ *for* $\mathbf{p} = (cp, p, \ldots, p), c > 1$.

Proof. Direct application of Theorem 1.5, Corollary 1.6 and Lemma 1.9. □

Note that the condition that A_{ij} is decreasing in i in Theorem 1.5. becomes $\overline{A}_i$ is decreasing in i which is true by its definition. An immediate result of Theorem 1.10 is that moving i positions closer is better than moving $i+1$ positions closer. Formally,

Corollary 1.11 *The steady state cost under move-i-position policy is stochastically smaller than that under move-$(i+1)$-position policy.*

Proof. Direct application of Theorem 1.10. □

Tenembaum and Nemes [9] examine two spectra of policies. For each spectrum, they show that the policies are ordered by their expected steady state cost, having tranposition at one end of the spectrum with minimum expected steady state cost and move-to-front at the other with maximum expected steady state cost. It can be shown that this also results directly from Theorem 1.5 and Corollary 1.6, and not only are the policies ordered by their expected steady state cost but their steady state costs are also stochastically smaller or larger than each other.

The first is a spectrum of policies $\mathrm{POS}(k), k = 1, \ldots, n$ where the requested item found at position j is moved to position k if $j > k$, and it is moved one position closer to the front if $j \le k$. We can write the matrices $\mathbf{A}$ and $\mathbf{B}$ representing policies $\mathrm{POS}(k+1)$ and $\mathrm{POS}(k)$ respectively as follows.

$$\mathbf{A} = \begin{bmatrix} 1 & & & & & & & & & \\ 1 & 1 & & & & & & & & \\ 0 & 1 & \ddots & & & & & & & \\ \vdots & 0 & \ddots & 1 & & & & & & \\ & \vdots & & 1 & 1 & & & & & \\ & & & 0 & 1 & 1 & & & & \\ & & & \vdots & \vdots & 1 & \ddots & & & \\ & & & & & \vdots & \ddots & 1 & & \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & 1 & 1 & \\ 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 & 1 & 1 \end{bmatrix}$$

Col. $(1) \cdots \quad (k+1) \quad \cdots \quad (n)$

$$
\mathbf{B} = \left[\begin{array}{cccccccccc}
1 & & & & & & & & & \\
1 & 1 & & & & & & & & \\
0 & 1 & \ddots & & & & & & & \\
\vdots & 0 & \ddots & 1 & & & & & & \\
 & \vdots & & 1 & 1 & & & & & \\
 & & & 0 & 1 & 1 & & & & \\
 & & & \vdots & \vdots & 1 & \ddots & & & \\
 & & & & & \vdots & \ddots & 1 & & \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & 1 & 1 & \\
0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 & 1 & 1
\end{array}\right]
$$

$$
\text{Col. } (1) \quad \cdots \quad (k) \quad \cdots \quad (n)
$$

The upper triangles of both matrices **A** and **B** consist of zeros. It can be easily checked that both policies **A** and **B** are in S as they satisfy (1.7) of Proposition 1.8. Moreover, all the conditions of Theorem 1.5 are also satisfied. We can then make a stronger statement that the steady state cost under POS$(k+1)$ is stochastically smaller than the steady state cost under POS(k).

The second is a spectrum of plicies SWITCH$(k), k = 1, \ldots, n$, where the requested item found at j is moved one position closer if $j > k$, and is moved to the first position if $j \leq k$. Al the conditions of Theorem 1.5 and (1.7) of Proposition 1.8 are satisfied by the following matrices **A** and **B** representing SWITCH(k) and SWITHC$(k+1)$ respectively.

$$
\mathbf{A} = \left[\begin{array}{cccccccccc}
1 & & & & & & & & & \\
1 & 1 & & & & & & & & \\
\vdots & 1 & \ddots & & & & & & & \\
 & \vdots & \ddots & 1 & & & & & & \\
1 & 1 & \ldots & 1 & 1 & & & & & \\
0 & 0 & \ldots & 0 & 1 & 1 & & & & \\
\vdots & \vdots & & \vdots & 0 & 1 & \ddots & & & \\
 & & & & \vdots & 0 & \ddots & 1 & & \\
 & & & & & \vdots & & 1 & 1 & \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & 1
\end{array}\right]
\begin{array}{c}
\text{Row} \\
(1) \\
\vdots \\
(k) \\
\vdots \\
\vdots \\
(n)
\end{array}
$$

$$
\mathbf{B} = \left[\begin{array}{cccccccccc}
1 & & & & & & & & & \\
1 & 1 & & & & & & & & \\
\vdots & 1 & \ddots & & & & & & & \\
 & \vdots & \ddots & 1 & & & & & & \\
1 & 1 & \ldots & 1 & 1 & & & & & \\
0 & 0 & \ldots & 0 & 1 & 1 & & & & \\
\vdots & \vdots & & \vdots & 0 & 1 & \ddots & & & \\
 & & & & \vdots & 0 & \ddots & 1 & & \\
 & & & & & \vdots & & 1 & 1 & \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & 1
\end{array}\right]
\qquad
\begin{array}{c}
\textbf{Row} \\
(1) \\
\vdots \\
(k+1) \\
\vdots \\
\vdots \\
(n)
\end{array}
$$

The upper triangles of **A** and **B** also consist of zeros. Similarly, the steady state cost under SWITCH(k) is stochastically smaller than that under SWITCH($k+1$).

2 The Processor Problem

Let $C(\mathbf{p};\mathbf{A})$ now be the steady state cost and $Y_1(\mathbf{p};\mathbf{A})$ the steady state position of processor 1 of the processor problem with success probability vector $\mathbf{p}$ under policy **A**. When only processor 1 has a different success probability, the expected steady state cost, conditioning on the position of processor 1, can be written as

$$
\begin{aligned}
E[C(\mathbf{p};\mathbf{A})] &= \sum_{i=1}^{n} E\Big[C(\mathbf{p};\mathbf{A})|Y_1(\mathbf{p};\mathbf{A}) = i\Big]\pi_i \\
&= \Big[1 + q_1(1 + q + \cdots + q^{n-2})\Big]\pi_1 \\
&+ \sum_{i=2}^{n-1}\Big[(1 + q + \cdots + q^{i-1}) + q_1(q^{i-1} + q^{i} + \cdots + q^{n-2})\Big]\pi_i \\
&+ (1 + q + \cdots + q^{n-1})\pi_n \\
&= \frac{1 - q_1 q^{n-1}}{p} - \frac{p_1 - p}{p}\sum_{i=1}^{n} q^{i-1}\pi_i, \qquad (2.1)
\end{aligned}
$$

where $\mathbf{p} = (p_1, p, \ldots, p)$, $q_1 = 1 - p_1$, $q = 1 - p$ and π_i is the steady state probability that processor 1 is at position i. From (2.1), since q^i is decreasing in i, if $p_1 > p$ and the position of procesor 1 under policy **A** is stochastically smaller than under policy **B**, the expected steady state cost under policy **A** will be smaller than the expected steady state cost under policy **B**. For the rest of the paper, we assume that $p_1 > p$.

2.1 Randomized Policy

Define the randomized policy **A** and its related randomized policy $\mathbf{A}^d$ in exactly the same way as in the list problem. Also let π_i^A be the steady state probability that processor 1 is at position i under policy **A**. Define $K_i^A = \pi_i^A/\pi_n^A$. Lemma 2.1 to Lemma 2.4 below show the relationships between $\{\pi_i^A\}$ and $\{\pi_i^{dA}\}$ under the assumption that $(p_1^d, p_2^d, \ldots, p_{n-1}^d) = (p_1, p, \ldots, p)$.

Lemma 2.1 *Under policy* **A**, *for* $i = 2, \ldots, n$,

$$\pi_i^A/\pi_n^A = K_i^A = \pi_{i-1}^{dA}/\pi_{n-1}^{dA}.$$

Proof. Similar to Lemma 1.1, the Lemma is proved by using the column $r+1$ of the transition matrix, which is given by

$$\begin{bmatrix} \cdots & 0 & \cdots \\ \cdots & \vdots & \cdots \\ \cdots & 0 & \cdots \\ \cdots & q^{r-1}q_1 p \sum_{i=r+1}^{n} q^{i-r-1} \sum_{j=1}^{r} a_{ij} & \cdots \\ \cdots & 1 - q^r + q^r p_1 a_{r+1,r+1} + q^r q_1 p \left(\sum_{i=r+2}^{n} q^{i-r-2} \sum_{j=r+2}^{i} a_{ij} \right) + q^{n-1} q_1 & \cdots \\ \cdots & q^{r+1} p_1 a_{r+2,r+1} & \cdots \\ \cdots & \vdots & \cdots \\ \cdots & q^{r+1} p_1 a_{n,r+1} & \cdots \end{bmatrix}$$

□

Lemma 2.2 *Under policy* **A**, *for* $i = 2, 3, \ldots, n$,

$$\pi_i^A = (1 - \pi_1^A)\pi_{i-1}^{dA}.$$

Proof. Same as Lemma 1.2. □

Lemma 2.3 *Under policy* **A**,

$$\pi_1^A = \frac{p_1 q}{q_1 p} \left(\frac{a_{21}\pi_2^A + q a_{31}\pi_3^A + \cdots + q^{n-2} a_{n1}\pi_n^A}{a_{21} + q a_{31} + \cdots + q^{n-2} a_{n1}} \right).$$

Proof. Similar to Lemma 1.3, the Lemma is proved by using the first column of the transition matrix, which is given by

$$\begin{bmatrix} p_1 + q_1 p \sum_{i=2}^{n} q^{i-2} \sum_{j=2}^{i} a_{ij} + q^{n-1} q_1 & \cdots \\ q p_1 a_{21} & \cdots \\ q^2 p_1 a_{31} & \cdots \\ \vdots & \cdots \\ q^{n-1} p_1 a_{n1} & \cdots \end{bmatrix}$$

□

Lemma 2.4 *Under policy* **A**, *for* $j = 1, 2, \ldots, n-1$,

$$K_j^A = \frac{p_1 q}{q_1 p}\left(\frac{A_{j+1,j}K_{j+1}^A + qA_{j+2,j}K_{j+2}^A + \cdots + q^{n-j-1}A_{nj}K_n^A}{A_{j+1,j} + qA_{j+2,j} + \cdots + q^{n-j-1}A_{nj}}\right).$$

Proof. Same as Lemma 1.4. □

2.2 Comparison of the Steady State Costs and Probabilities of Two Problems under Two Different Policies

We can now state a result similar to Theorem 1.5 that compares the steady state probability $\{\pi_i\}$ under two different policies. As in the list problem, let S be the set of policies under which the resulting probability distribution $\{\pi_i\}$ is decreasing in i when $p_1 > p$ and increasing in i otherwise.

Theorem 2.5 *Let* **A** *and* **B** *be two policies such that, for* $j = 1, 2, \ldots, n-1$, $k = j+1, \ldots, n$,

$$\frac{A_{j+1,j} + qA_{j+2,j} + \cdots + q^{k-j-1}A_{kj}}{A_{j+1,j} + qA_{j+2,j} + \cdots + q^{n-j-1}A_{nj}} \geq \frac{B_{j+1,j} + qB_{j+2,j} + \cdots + q^{k-j-1}B_{kj}}{B_{j+1,j} + qB_{j+2,j} + \cdots + q^{n-j-1}B_{nj}}, \tag{2.2}$$

and at least one of these two conditions holds:

(a) $\mathbf{A} \in S$ *and* B_{ij} *is decreasing in* i *for all* $j = 1, \ldots, n$,
(b) $\mathbf{B} \in S$ *and* A_{ij} *is decreasing in* i *for all* $j = 1, \ldots, n$.

Then $\{\pi_i^A\} \leq_{st} \{\pi_i^B\}$ *for any* $\mathbf{p} = (p_1, p, \ldots, p), p_1 > p$.

Proof. Same as Theorem 1.5 because if A_{ij} is decreasing in i for all j then so is $q^{i-j-1}A_{ij}$. □

It should be noted that (1.6) and (2.2) are not equivalent when A_{ij} and B_{ij} are decreasing in i for all j, even though (2.2) gives (1.6) when $q = 1$. A simple counterexample can be constructed as follows. Suppose (1.6) is true. Let $j = 1$ and $A_{21} + A_{31} + \cdots + A_{n1} = B_{21} + B_{31} + \cdots + B_{n1}$, with $A_{21} = B_{21}$. So by (1.6), $(A_{21}, A_{31}, \ldots, A_{n1})$ majorizes $(B_{21}, B_{31}, \ldots, B_{n1})$. With the fact that q^i is decreasing in i, we have

$$A_{21} + qA_{31} + \cdots + q^{n-2}A_{n1} \geq B_{21} + qB_{31} + \cdots + q^{n-2}B_{n1},$$

which means

$$\frac{A_{21}}{A_{21} + qA_{31} + \cdots + q^{n-2}A_{n1}} \leq \frac{B_{21}}{B_{21} + qB_{31} + \cdots + q^{n-2}B_{n1}}.$$

This contradicts (2.2) for $j = 1$ and $k = 2$.

A consequence of Theorem 2.5 is that the steady state cost under policy **A** is stochastically smaller than the steady state cost under policy **B**.

Corollary 2.6 *Under the conditions of Theorem 2.5,* $C(\mathbf{p}; \mathbf{A}) \leq_{st} C(\mathbf{p}; \mathbf{B})$.

Proof. Same as Corollary 1.6. □

By exactly the same reason as in Corollary 1.7, transposition has a steady state cost stochastically smaller than any randomized policy.

Corollary 2.7 *For any policy* $\mathbf{A}, C(\mathbf{p}; T) \leq_{st} C(\mathbf{p}; \mathbf{A})$.

Proof. Same as Corollary 1.7. □

A counterexample similar to that in Section 1.2 can be made to show that not every randomized policy is in S. A sufficient condition for a policy **A** to be in S turns out to be the same as in the list problem. That is, when $p_1 > p$, $\{\pi_i^A\}$ is decreasing in i when (2.3) below, which is (1.7) of Proposition 1.8, holds.

Proposition 2.8 *A policy* $\mathbf{A} \in S$ *if, for* $j = 1, \ldots, n-1$,

$$\frac{A_{j+1,j} + \cdots + A_{nj}}{A_{j+2,j+1} + \cdots + A_{n,j+1}} \leq \frac{A_{j+1,j} + \cdots + A_{n-1,j}}{A_{j+2,j+1} + \cdots + A_{n-1,j+1}} \leq \cdots \leq \frac{A_{j+1,j} + A_{j+2,j}}{A_{j+2,j+1}}. \tag{2.3}$$

Proof. By the same argument as in Proposition 1.8, $\mathbf{A} \in S$ if, for $k = j+3, \ldots, n$,

$$\frac{A_{j+1,j} + qA_{j+2,j} + \cdots + q^{k-j-1}A_{kj}}{A_{j+2,j+1} + qA_{j+3,j+1} + \cdots + q^{k-j-2}A_{k,j+1}} \leq \frac{A_{j+1,j} + qA_{j+2,j} + \cdots + q^{k-j-2}A_{k-1,j}}{A_{j+2,j+1} + qA_{j+3,j+1} + \cdots + q^{k-j-3}A_{k-1,j+1}}. \tag{2.4}$$

It is then sufficient to show that (2.3) implies (2.4). By cross-multiplying and rearranging terms, (2.4) is equivalent to

$$\frac{qA_{kj}}{A_{j+1,j} + qA_{j+2,j} + \cdots + q^{k-j-2}A_{k-1,j}} \frac{A_{j+1,j} + \cdots + A_{k-1,j}}{A_{j+1,j} + \cdots + A_{k-1,j}}$$

$$\leq \frac{A_{k,j+1}}{A_{j+2,j+1} + qA_{j+3,j+1} + \cdots + q^{k-j-3}A_{k-1,j+1}} \frac{A_{j+2,j+1} + \cdots + A_{k-1,j+1}}{A_{j+2,j+1} + \cdots + A_{k-1,j+1}}. \tag{2.5}$$

Now,

$$\frac{A_{kj}}{A_{j+1,j} + \cdots + A_{k-1,j}} \leq \frac{A_{k,j+1}}{A_{j+2,j+1} + \cdots + A_{k-1,j+1}}$$

$$\Leftrightarrow \frac{A_{j+1,j} + \cdots + A_{kj}}{A_{j+2,j+1} + \cdots + A_{k,j+1}} \leq \frac{A_{j+1,j} + \cdots + A_{k-1,j}}{A_{j+2,j+1} + \cdots + A_{k-1,j+1}}, \tag{2.6}$$

where the inequality on the right hand side of the equivalence is given by (2.3). Also from (2.3), for $m \leq k-1$,

$$\frac{A_{j+1,j}+\cdots+A_{mj}}{A_{j+1,j}+\cdots+A_{k-1,j}} \geq \frac{A_{j+2,j+1}+\cdots+A_{m,j+1}}{A_{j+2,j+1}+\cdots+A_{k-1,j+1}},$$

and because q^i is decreasing in i we have

$$\begin{aligned} & \frac{A_{j+1,j}+qA_{j+2,j}+\cdots+q^{k-j-2}A_{k-1,j}}{A_{j+1,j}+A_{j+2,j}+\cdots+A_{k-1,j}} \\ \geq \; & \frac{qA_{j+2,j+1}+q^2A_{j+3,j+1}+\cdots+q^{k-j-2}A_{k,j+1}}{A_{j+2,j+1}+A_{j+3,j+1}+\cdots+A_{k,j+1}}. \end{aligned} \tag{2.7}$$

Then (2.5) follows from (2.6) and (2.7). □

Thus for the processor problem, by the same argument as in Lemma 1.9, any position independent policy is also in S. Formally,

Lemma 2.9 *Let* $\mathbf{A}$ *be a position independent policy that moves the succesful processor I positions with probability* a_i, $\sum_{i=0}^{n-1} a_i = 1$. *Then* $\mathbf{A} \in S$.

Proof. Same as Lemma 1.9. □

We can then restate Theorem 2.5 combined with Corollary 2.6 for position independent policies as follows.

Theorem 2.10 *Given two position independent policies* $\mathbf{A}$ *and* $\mathbf{B}$ *such that, for* $k = 1, \ldots, n-1$,

$$\frac{\overline{A}_1+q\overline{A}_2+\cdots+q^{k-1}\overline{A}_k}{\overline{A}_1+q\overline{A}_2+\cdots+q^{n-2}\overline{A}_{n-1}} \geq \frac{\overline{B}_1+q\overline{B}_2+\cdots+q^{k-1}\overline{B}_k}{\overline{B}_1+q\overline{B}_2+\cdots+q^{n-2}\overline{B}_{n-1}}, \tag{2.8}$$

then $\{\pi_i^A\} \leq_{st} \{\pi_i^B\}$ *and* $C(\mathbf{p};\mathbf{A}) \leq_{st} C(\mathbf{p};\mathbf{B})$ *for any* $\mathbf{p} = (p_1, p, \ldots, p), p_1 > p$.

Proof. Direct application of Theorem 2.5 Corollary 2.6 and Lemma 2.9. □

There is no obvious interpretation of (2.8), unlike (1.8), as in the list problem. However, (2.8) yields the same monotonicity result as in the list problem that move-i-position has a steady state cost stochastically smaller than move-$(i+1)$-position. Let $\mathbf{A}$ and $\mathbf{B}$ represent the move-i-position and move-$(i+1)$-position policies respectively. Then,

$$\overline{A}_1 = \overline{A}_2 = \cdots = \overline{A}_i = 1, \overline{A}_{i+1} = \overline{A}_{i+2} = \cdots = \overline{A}_{n-1} = 0$$

$$\overline{B}_1 = \overline{B}_2 = \cdots = \overline{B}_{i+1} = 1, \overline{B}_{i+2} = \overline{B}_{i+3} = \cdots = \overline{B}_{n-1} = 0.$$

Therefore, for $k = 1, \ldots, n-1$,

$$\frac{\overline{A}_1 + q\overline{A}_2 + \cdots + q^{k-1}\overline{A}_k}{\overline{A}_1 + q\overline{A}_2 + \cdots + q^{n-2}\overline{A}_{n-1}} = \frac{1 + q + \cdots + q^{k-1}}{1 + q + \cdots + q^{i-1}}$$
$$\geq \frac{1 + q + \cdots + q^{k-1}}{1 + q + \cdots + q^{i}}$$
$$= \frac{\overline{B}_1 + q\overline{B}_2 + \cdots + q^{k-1}\overline{B}_k}{\overline{B}_1 + q\overline{B}_2 + \cdots + q^{n-2}\overline{B}_{n-1}}. \qquad (2.9)$$

We have proved the following Corollary.

Corollary 2.11 *The steady state cost under the move-i-position policy is stochastically smaller than that under the move-$(i+1)$-position policy.*

Proof. By (2.9) and Theorem 2.10. □

By Theorem 2.5, it also holds, as in the case of the list problem shown in Section 1.2, that the policies in the two spectra of Tenenbaum and Nemes [9] are ordered such that the policies in each spectrum have steady state costs stochastically smaller or larger than each other.

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References

[1] Anderson, E.J., Nash, P., and Weber, R.R. A counterexample to a conjecture on optimal list ordering. *Journal of Applied Probability,* Vol. 19, No. 3 (1982), 730-732.

[2] Gonnett, G.H., Munro, J.I., and Suwanda, H. Exegesis of self-organizing linear search. *SIAM Journal on Computing,* Vol. 10 (1981), No. 3, 613-637.

[3] Hendricks, W.J. The stationary distribution of an interesting Markov chain. *Journal of Applied Probability,* Vol. 9, No. 1 (1972), 231–23.

[4] Hendricks, W.J. An account of self-organizing systems. *SIAM Journal on Computing,* Vol. 5 (1976), No. 4, 715-723.

[5] Hester, J.H., and Hirschberg, D.S. Self-organizing Linear Search. *Computing Surveys,* Vol. 17, No. 3 (1985), 295-311.

[6] Kan, Y.C., and Ross, S.M. Optimal list order under partial memory constraints. *Journal of Applied Probability,* Vol. 17, No. 4 (1980), 1004-1015.

[7] Phelps, R.I., and Thomas, L.C. On optimal performance in self-organizing paging algorithms. *Acta Cybernetica,* Vol. 5, No. 1 (1980), 88-85.

[8] Ross, S.M. Processor reordering rules. *Probability in the Engineering and Information Science,* Vol. 4, No. 2 (1990), 181-186.

[9] Tenenbaum, A., and Nemes, R.M. Two spectra of self-organizing sequential search algorithms. *SIAM Journal on Computing,* Vol. 11, No. 3 (1982), 557-566.

[10] Topkis, D.M. Reordering heuristics for routing in communication networks. *Journal of Applied Probability,* Vol. 23, No. 1 (1986), 130-143.

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