# On the Boolean structure of fuzzy logical systems: a counter example

J. Dombi *† Gy. Lencsés ‡

**Abstract**

The article of Murthy, Pal and Majumder [1] gives a new interpretation of the connectives in fuzzy sets claiming that these connectives preserve the whole Boolean structure of ordinary set theoretic operations. In our paper a counter example is given where the property of associativity is not valid for the new connectives.

## Introduction

Many authors attempt to construct fuzzy logical systems preserving as many Boolean properties as it is possible. It is well known that to preserve the whole Boolean structure of set operations when extending them pointwisely to [0,1] valued membership functions of fuzzy sets is not possible (see e.g. [2]). For instance excluded middle law and idempotence are incompatible for fuzzy sets if we demand that the result of the operation in any point must be dependent only on the value of the membership functions in this point.

C.A. Murthy et al. [1] try to solve this problem by defining operators the result of which may be dependent not only on the value of membership functions but also on their relative natures. They claim that the operators ⓝ and ⓤ defined in their article fulfil all of the Boolean properties. If it were true, then these new operators should be preferred to any other earlier construction.

We will show, however, a counter example where the operators ⓝ and ⓤ do not fulfil some Boolean properties. It will be shown that the operators are ill-defined, and we will point out why it is impossible to prove some of the Boolean properties of the operators ⓝ and ⓤ by the Theorems 1-7 of the cited paper.

*Research Group on the Theory of Automata, Hungarian Academy of Sciences Aradi Vértanúk tere 1., H-6720 Szeged, Hungary

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‡Department of Sociology, University of Szeged, Petőfi S. sgt. 30-34, H-6720 Szeged, Hungary

# 1 Preliminary definitions

First of all we have to recall the definitions of C.A. Murthy et al. [1].

## 1. 1 Properties of the operators

They claim that the operators $\textcircled{\cap}$ and $\textcircled{\cup}$ fulfil the following properties. Here $A, B, C$ are fuzzy sets in a universe $X$, $\mu_A, \mu_B$, etc. are membership functions of $A, B$, etc., $A^C$ is the complement of $A$.

$p_1 : \mu_{A \textcircled{\cap} A^C}(x) = 0 \quad$ for all $x \in X$

$p_2 : \mu_{A \textcircled{\cup} A^C}(x) = 1 \quad$ for all $x \in X$

$p_3$ : commutativity
$\mu_{A \textcircled{\cup} B}(x) = \mu_{B \textcircled{\cup} A}(x)$
$\mu_{A \textcircled{\cap} B}(x) = \mu_{B \textcircled{\cap} A}(x)$

$p_4$ : associativity
$\mu_{A \textcircled{\cup} (B \textcircled{\cup} C)}(x) = \mu_{(A \textcircled{\cup} B) \textcircled{\cup} C}(x)$
$\mu_{A \textcircled{\cap} (B \textcircled{\cap} C)}(x) = \mu_{(A \textcircled{\cap} B) \textcircled{\cap} C}(x)$

$p_5$ : idempotency
$\mu_{A \textcircled{\cup} A}(x) = \mu_A(x)$
$\mu_{A \textcircled{\cap} A}(x) = \mu_A(x)$

$p_6$ : distributive laws
$\mu_{A \textcircled{\cup} (B \textcircled{\cap} C)}(x) = \mu_{(A \textcircled{\cup} B) \textcircled{\cap} (A \textcircled{\cup} C)}(x)$
$\mu_{A \textcircled{\cap} (B \textcircled{\cup} C)}(x) = \mu_{(A \textcircled{\cap} B) \textcircled{\cup} (A \textcircled{\cap} C)}(x)$

$p_7$ : identity
$\mu_{A \textcircled{\cup} \emptyset}(x) = \mu_A(x)$
$\mu_{A \textcircled{\cap} X}(x) = \mu_A(x)$

$p_8$ : a) absorption laws
b) DeMorgan's laws
c) involution laws

$p_9 : 0 \leq \mu_{A \textcircled{\cap} B} \leq \min(\mu_A, \mu_B)$
$1 \geq \mu_{A \textcircled{\cup} B} \geq \max(\mu_A, \mu_B)$

In the following $\mu_A, \mu_B, \mu_C$ are denoted by f,g,h respectively, $\mu_{A \textcircled{\cap} B}$ is denoted by $f \textcircled{\cap} g$, etc.

## 1. 2 Definition of type $I$ membership functions

Let the domain $Q = [a, b]$ be a closed interval in $R$, and let $f$ be a membership function with the following properties:

a) $f : Q \to [0,1]$ is continuous
b) $f(Q) = [0,1]$
c) $f\{a,b\} \subseteq \{0,1\}$

$f$ is a type $I$ membership function if it fulfils the next assumption:
Let $a < x_0 < b$ such that $f$ increases (decreases) at $x_0$. Then there exist $x_1$ and $x_2$ such that

$$a \le x_1 < x_0 < x_2 \le b \quad \text{and} \quad f(x_1) = 0 \;\; (f(x_1) = 1), \;\; f(x_2) = 1$$

$(f(x_2) = 0)$ and $f$ is nondecreasing (nonincreasing) at all $x \in (x_1, x_2)$.

**1. 3 Definition of ⓝ and ⓤ.**

a) Murthy et al. first define a set $A_x$ for every $f$ membership function and for every point $x \in [a,b]$ as follows:

$$A_x = \begin{cases} [0, f(x)] & \text{if } f \text{ is nondecreasing at } x \\ [1 - f(x), 1] & \text{if } f \text{ is nonincreasing at } x \\ \text{any finite set} & \text{if } f(x) = 0 \\ [0,1] & \text{if } f(x) = 1 \end{cases}$$

$B_x$ and $C_x$ are similarly defined for the functions $g$ and $h$ in any point $x$.

b) Then ⓝ and ⓤ are defined by

$$(f \text{ ⓝ } g)(x) = \lambda(A_x \cap B_x)$$
$$(f \text{ ⓤ } g)(x) = \lambda(A_x \cup B_x)$$

where $\lambda$ is the Lebesgue measure on $R$.

# 2 A counter example

We give an example, where the property of associativity $(P_4)$ of ⓝ does not hold.
Let us consider the ⓝ operator. If we use it two times, one after another

$$((f \text{ ⓝ } g) \text{ ⓝ } h)(x)$$

then according to the definition 1.3 in each step first of all we have to determine sets:

– in the first step: the sets $A_x$ and $B_x$,
– in the second step: the set $D_x$ connected with $(f$ ⓝ $g)(x)$ by definition 1.3a, and the set $C_x$.

But it is easily possible that $D_x = A_x \cap B_x$ is not valid (e.g. when $f$ is increasing and $g$ is decreasing at $x$). In this case the properties of Lebesgue measure in connection with ordinary sets cannot be automatically used to prove associativity and distributivity as it was done in Part VI of the cited paper.
Let us see a counter example where $f$, $g$ and $h$ are type $I$ membership functions and the associativity of ⓝ does not hold. Let $Q = [0,1]$, $f$, $g$ and $h$ be piecewise linear membership functions as shown in Figure 1.

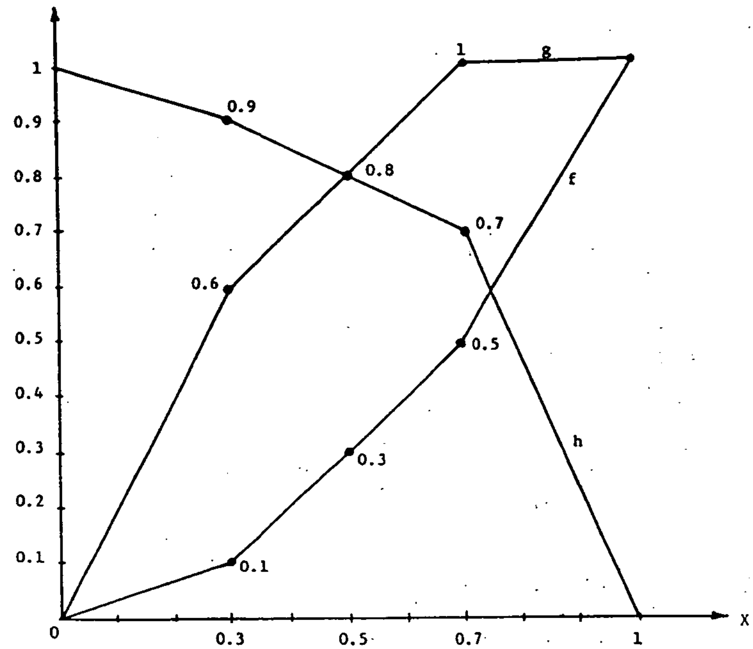


Fig. 1. Functions $f, g, h$

Here $f, g$ and $h$ are type $I$ membership functions. But $g \circledcap h$ does not belong to the same type because $(g \circledcap h)([0,1]) = [0, 0.7]$. See figure 2.

Since for all $x \in [0,1]$ $f(x) \leq g(x)$, and $f$ and $g$ are nondecreasing at all $x \in (0,1)$, so $f \circledcap g = f$ on the whole interval $[0,1]$, according to the definition of $\circledcap$ . On the interval $(0.3, 0.7)$ the functions $f, g, f \circledcap g (= f), g \circledcap h$ are nondecreasing and $h$ is nonincreasing.

So $(f \circledcap g)(0.5) = f(0.5) = 0.3$.

Let the set connected to $f \circledcap g$ at the point 0.5 be $D_{0.5}$ (see definition 1.3a). Then $D_{0.5} = [0, 0.3]$, because $f \circledcap g$ is nondecreasing at 0.5. $C_{0.5} = [0.2, 1]$, because $h$ is nonincreasing at 0.5.

So $((f \circledcap g) \circledcap h)(0.5) = \lambda(D_{0.5} \cap C_{0.5}) = 0.1$

Similarly since $g \circledcap h$ is increasing at 0.5, and $(g \circledcap h)(0.5) = 0.6$, so $E_{0.5} = [0, 0.6]$, where $E_{0.5}$ is the set connected to $g \circledcap h(0.5)$ by definition 1.3a. In addition $f(0.5) = 0.3$, $A_{0.5} = [0, 0.3]$ and so

$$(f \circledcap (g \circledcap h))(0.5) = \lambda(A_{0.5} \cap E_{0.5}) = 0.3$$

That is $((f \circledcap g) \circledcap h)(0.5) \neq (f \circledcap (g \circledcap h)(0.5)$

This result contradicts to the property of associativity of $\circledcap$ .

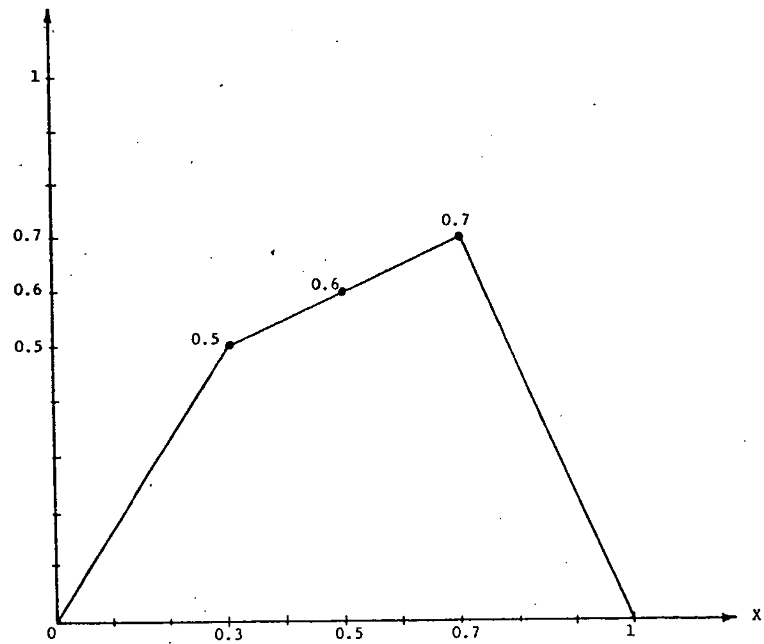


Fig. 2. Function $g$ Ⓝ $h$

# 3 Concluding remarks

1. The definition of Ⓝ and Ⓤ is suitable only for type $I$ membership functions. How can the set $E_x$ be determined for the function $g$ Ⓝ $h$ in Fig. 2. at the point $x = 0.7$? Here $g$ Ⓝ $h$ attains its maximum value, $g$ Ⓝ $h(0.7) = 0.7$, but this value is not equal to 1.

2. Why is the proof of associativity wrong in [1]? The cited paper uses the following argumentation to prove the Boolean properties of Ⓝ and Ⓤ : "the operations are ordinary set operations and the Lebesgue measure satisfies similar properties in connection with ordinary sets". This reasoning would be correct only if in composite operations the sets $A_x$, $B_x$, etc. connected to the membership functions were inherited. That is if $f(x) = (g$ Ⓝ $h)(x)$ and $A_x$, $B_x$, $C_x$ are obtained from definition 1.3a, then $A_x = B_x \cap C_x$ for all $x \in [a, b]$. Murthy et al. prove this only for the case when $f, g$ and $h$ are type $I$ membership functions (see Theorems 1-7 in [1]). If, however, the result function $f$ does not belong to the same type then the above equality for the sets $A_x$, $B_x$, $C_x$ is not true usually. (See e.g. the function $g$ Ⓝ $h$ in our counter example.)

# References

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