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Radical Theory for Group Semiautomata

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Abstract

A Kurosh-Amitsur radical theory is developed for group semiautomata. Radical theory stems from ring theory, it is apt for deriving structure theorems and for a comparative study of properties. Unlikely to conventional radical theories, the radical of a group semiautomaton need not be a subsemiautomaton, so the whole scene will take place in a suitably constructed category. The fundamental facts of the theory are described in § 2. A special feature of the theory, the existence of complementary radicals, is discussed in § 3. Restricting the theory to additive automata, which still comprise linear sequential machines, in § 4 stronger results will be achieved, and also a (sub)direct decomposition theorem for certain semisimple group semiautomata will be proved. Examples are given at appropriate places. The paper may serve also as a framework for future structural investigations of group semiautomata.

Key Words : Kurosh-Amitsur radical, group semiautomaton.

0 Introduction

The purpose of this paper is to develop a Kurosh-Amitsur radical theory for group semiautomata which may serve as a framework for future radical theoretical investigations and for describing the structure of semisimple group semiautomata.

In the variety of group semiautomata there is a one to one correspondence between homomorphisms and kernels, so it is meaningful to designate a kernel of a group semiautomaton as its radical. Doing so, however, there is an obstacle : a kernel is not always a subsemiautomaton, but only a normal subgroup subject to some additional requirement. This shortcoming can be overcome, if we work in an appropriately constructed category comprising group semiautomata and groups as objects. In this way kernels can be considered as subobjects.

The category suitable for a radical theory of group semiautomata will be constructed in § 1 analogously as done for semifields in [12]. Following the framework of [8], the fundamental notions of radical theory along with their characterizations, are given in § 2 in a self-contained way. A special feature of the radical theory of group semiautomata is the existence of complementary radical and semisimple classes which are discussed in § 3. Restricting the investigations to additive group

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semiautomata introduced in [4], we can get more explicit results. We shall see in § 4 that semisimple classes of additive group semiautomata are always hereditary, and we shall prove a subdirect decomposition theorem for additive group semiautomata which are semisimple with respect to a certain radical. Examples are supplied at appropriate places.

1 Preliminaries

A *group semiautomaton* (for short, a GS-*automaton*) is a quadruple $(A, +, X, \delta)$ consisting of an additive (not necessarily commutative) group $(A, +)$ as a set of states, of an input set $X \neq \emptyset$ and a state transition function $\delta : A \times X \to A$. The input set X, as usual, can be extended to the free monoid X^* over X, and then it is required that the transition function δ satisfies

$$\delta(a, xy) = \delta(\delta(a, x), y)$$

for all $x, y \in X^*$.

The notion of GS-automaton is a generalization of that of linear sequential machine [3] or linear sequential automaton [1], and has been investigated, for instance, in [5], [6] (cf. also [9]).

In terms of universal algebra a GS-automaton is nothing but a universal algebra (A, Ω) with underlying set A and a set of operations $\Omega = \{+\} \cup \delta$ where $+$ is a binary operation making $(A, +)$ a group and δ consists of unary operations $f_x : a \to \delta(a, x)$, for all $a \in A$ and $x \in X$. Hence we know that GS-automata over a fixed input set X form a variety, and it is clear what a subsemiautomaton, a homomorphic image, an isomorphism, a direct or subdirect sum, a subdirectly irreducible GS-automaton, etc. means. Also the meaning of the homomorphism theorem and of the isomorphism theorems is obvious.

Throughout this paper the set X of inputs will be fixed, or equivalently, the set δ of unary operations will be a given one, and so a GS-automaton on the set A of states will be denoted by $(A, +, \delta)$, or sometimes briefly by A, if there is no fear of ambiguity. Moreover, for the clumsy notation $\delta(a, x)$ we shall write simply ax.

A congruence relation κ of a GS-automaton $(A, +, \delta)$ is a congruence on the group $(A, +)$, and therefore κ determines uniquely the coset K containing 0, which is a normal subgroup of $(A, +)$. Since κ is a congruence of the GS-automaton $(A, +, \delta)$, κ is compatible with the unary operations $f_x \in \delta, x \in X$, that is, $f_x(a+k)$ is congruent to $f_x(a)$ modulo κ, that is,

$$(\star) \qquad (a + k)x - ax \in K$$

for every $x \in X, a \in A$ and $k \in K$. Conversely, if K is a normal subgroup of $(A, +)$ and satisfies condition $(\star)$, then the equivalence relation κ defined by K on the set A is a congruence on $(A, +, \delta)$. Thus by the homomorphism theorem every homomorphism

$$\varphi : (A, +, \delta) \to (B, +, \delta)$$

has a *kernel* K which is precisely a normal subgroup of $(A, +)$ subject to the requirement $(\star)$.

Let us observe a fact of importance for our investigations. *A kernel of a GS-automaton need not be a subsemiautomaton,* and a subsemiautomaton $(B, +, \delta)$ of a GS-automaton $(A, +, \delta)$ with normal subgroup $(B, +)$ in $(A, +)$, is not necessarily a kernel.

PROPOSITION 1.1. *A kernel* K *is a subsemiautomaton if and only if* $0X \subseteq K$. *If* K *contains a subsemiautomaton, then* K *itself is a subsemiautomaton.*

PROOF: Since

$$kx - 0x = (0 + k)x - 0x \in K$$

holds for arbitrary elements $k \in K$ and $x \in X$, the assertion follows. The second statement is now clear. □

EXAMPLE 1.2. A subsemiautomaton $(B, +, \delta)$ of a GS-automaton $(A, +, \delta)$ need not be a kernel even if $(B, +)$ is normal in $(A, +)$. Let us consider, namely, the Klein 4-group $(A, +) = \{0, a, b, c\}$ as the set of states and $X = \{x\}$ as the set of inputs. Define δ by the following graph

$$c \xrightarrow{x} b \xrightarrow{x} a \xrightarrow{x} 0 \xrightarrow{x} 0.$$

It can be easily seen that $\{0, a\}$, forms a subsemiautomaton (which is trivially a normal subgroup in A with $0X = 0 \in \{0, a\}$), but it is not a kernel, for

$$(b + a)x - bx = cx - bx = b - a = c \notin \{0, a\}.$$

The fact that there is a one-to-one correspondence between kernels and homomorphisms of GS-automata, but kernels are, in general, not subsemiautomata, adds a special flavor to the radical theory of GS-automata. A similar situation occurs also in the case of semifields [7], for which a radical theory has been developed in a category (universal class) comprising semifields and groups as objects [12]. In setting the scene we shall employ ideas of [12] and follow the framework of the Kurosh-Amitsur radical theory as developed in [8]. Thus we shall work in a universal class $\mathfrak{H}$ of GS-automata and groups, and it is our purpose in this note to develop a Kurosh-Amitsur radical theory in $\mathfrak{H}$ yielding specific results for GS-automata. Due to the high level of generality in [8], the adaptation of the results of [8] to our case is not quite straightforward, therefore for the sake of understandability and clarity we shall present the Kurosh-Amitsur radical theory of GS-automata in a self-contained way, though following the pattern of [8] and using ideas of [12].

Our investigations will take place within a suitable category $\mathfrak{C}$, the objects thereof are GS-automata and groups. Let $\mathfrak{A}$ denote the class of all GS-automata over a fixed input set X and $\mathfrak{G}$ the class of all groups, and we set $\mathrm{Ob}\mathfrak{C} = \mathfrak{A} \cup \mathfrak{G}$. For all $A, B \in \mathfrak{A} \cup \mathfrak{G}$ we consider the following three types of morphisms $\varphi: A \to B$:

1) All GS-automaton homomorphisms $\varphi: (A, +, \delta) \to (B, +, \delta)$ for $A, B \in \mathfrak{A}$.
2) All group homomorphisms $\varphi: (A, +) \to (B, +)$ for $A, B \in \mathfrak{G}$.
3) All group homomorphisms $\varphi: (A, +) \to (B, +, \delta)$ for $A \in \mathfrak{G}$ and $B \in \mathfrak{A}$ where one does not care about the transition function δ (or equivalently, about the unary operations $f_x \in \delta, x \in X$) defined on B.

The morphisms of types 1), 2) and 3) will consititute the morphisms of $\mathfrak{C}$. It is clear that $\mathfrak{C}$ has become a category. Designating the subclass

$$\mathcal{E} = \{\text{all surjective morphisms of types 1) and 2) in } \mathfrak{C}\}$$

and

$$\mathcal{M} = \{\text{all injective morphisms in } \mathfrak{C}\},$$

both $\mathcal{E}$ and $\mathcal{M}$, along with the objects of $\mathfrak{C}$, form obviously subcategories in $\mathfrak{C}$. Moreover, $\mathcal{E}$ and $\mathcal{M}$ consist of epimorphisms and monomorphisms, respectively, and $\mathcal{E} \cap \mathcal{M}$ is the class of all isomorphisms in $\mathfrak{C}$. Every morphism $\varphi\colon A \to B$ in $\mathfrak{C}$ factors as

$$A \xrightarrow{\varphi} B = A \xrightarrow{\varepsilon} C \xrightarrow{\mu} B$$

where $\varepsilon \in \mathcal{E}$ and $\mu \in \mathcal{M}$. Thus $\mathfrak{C}$ is endowed with a bicategory structure.

For developing a radical theory, it is sufficient and sometimes also useful to restrict the investigations to a certain subcategory of $\mathfrak{C}$. A non-empty subcategory $\mathfrak{H}$ of $\mathfrak{C}$ is called a *universal class*, if $\mathfrak{H}$ satisfies the following conditions :

(i) $\mathfrak{H}$ is closed with respect to all surjective morphisms $\varphi\colon A \to B$ of types 1) and 2).

(ii) $\mathfrak{H}$ is closed under taking kernels : for any morphisms $\varphi\colon A \to B$ in $\mathfrak{H}$ also $K = \ker\varphi$ is in $\mathfrak{H}$, (or equivalently, if K is a kernel in $A \in \mathfrak{H}$, then also $K \in \mathfrak{H}$).

(iii) $(A, +, \delta) \in \mathfrak{H}$ implies $(A, +) \in \mathfrak{H}$.

Concerning the universal class $\mathfrak{H}$ we shall work with, we make some observations.

1. The identical mapping ι of the set of states A induces a bijection $\iota\colon (A, +) \to (A, +, \delta)$ which is not an isomorphism, for its inverse does not exist in $\mathfrak{C}$ (in fact, it is not defined).

2. $\mathfrak{H}$ contains an initial object $(0, +)$ and a terminal object $(0, +, \delta)$ whenever $\mathfrak{H} \cap \mathfrak{A} \neq \emptyset$. We call $(0, +)$ and $(0, +, \delta)$ the *trivial objects* of $\mathfrak{H}$, and we shall write $\mathfrak{T}$ for the class of trivial objects. Since $(0, +)$ and $(0, +, \delta)$ are not isomorphic, in view of [11] we can predict a peculiar feature of the radical theory of GS-automata, and that is the existence of non-trivial complementary radical and semisimple classes (cf. § 3).

3. If $(A, +, \delta) \in \mathfrak{H}$ and $\varphi\colon A \to B$ is a morphism, then $K = \ker\varphi$ is either a subsemiautomaton $(K, +, \delta)$ (this is the case whenever K is a subsemiautomaton) or a normal subgroup $(K, +)$ (this is the case when K is not a subsemiautomaton). In the first case $(K, +)$ is a subobject of $(A, +, \delta)$ which is contained in the subobject $(K, +, \delta)$, but they are not equivalent subobjects.

4. The image of a kernel need not be a kernel. For instance, let $(K, +)$ be a kernel of a group $(A, +)$ and

$$\iota\colon (A, +) \to (A, +, \delta)$$

the identical embedding. Since $(K, +)$ is merely a normal subgroup of $(A, +)$,

$$\iota(K, +) = \begin{cases} (K, +) & \text{if K is not a subsemiautomaton,} \\ (K, +, \delta) & \text{if K is a subsemiautomaton,} \end{cases}$$

but $\iota(K, +)$ need not be a kernel of $\iota(A, +) = (A, +, \delta)$, regardless as whether it is a subsemiautomaton or not (cf. EXAMPLE 1.2).

5. We have to be careful in applying the second isomorphism theorem in $\mathfrak{H}$. Let $(L, +)$ be a subgroup of $(A, +)$ in a GS-automaton $(A, +, \delta)$. If K is a kernel of $(A, +, \delta)$, then $L/(L \cap K)$ is only a group, although $L + K$ may be a subsemiautomaton, for instance, if L is a kernel of $(A, +, \delta)$ and K is also a subsemiautomaton. In this case we have

$$(L/(L \cap K), +) \cong ((L + K)/K, +) \xrightarrow{\iota} ((L + K)/K, +, \delta)$$

and the left hand side is not isomorphic to the right hand side.

6. In the category $\mathfrak{C}$ (and therefore also in $\mathfrak{H}$) direct sums, in general, do not exist; more precisely, the (complete) direct sum $\sum^{\oplus} A_\alpha$ of objects $A_\alpha, \alpha \in \Lambda$, exists in $\mathfrak{C}$ if and only if either all A_α are GS-automata, or all of them are groups.

Kernels of an object A of $\mathfrak{H}$ form clearly a complete lattice isomorphic to the lattice of congruences of A. Unions and intersections in the lattice of kernels will be denoted by $\vee$ and $\wedge$, respectively. As usual, $\vee$ over the empty set and $\wedge$ over the empty set in the lattice of kernels of an object A, will mean the trivial kernel of A and A itself, respectively.

PROPOSITION 1.3. *If* K *and* L *are kernels of a* GS-*automaton* $(A, +, \delta)$, *then either* $K \vee L = (K + L, +)$ *or* $K \vee L = (K + L, +, \delta)$. *In particular, if* K *is a subsemiautomaton, then* $K \vee L = (K + L, +, \delta)$.

PROOF: $K + L$ is obviously a normal subgroup in A. Let $a \in A, k + l \in K + L$ and $x \in X$ be arbitrary elements. Then

$$(a + k + l)x - ax = (a + k + l)x - (a + k)x + (a + k)x - ax \in K + L$$

holds proving the first assertion. Hence in view of PROPOSITION 1.1 the second statement follows. □

2 Radical operator, radical class, semisimple class

In this section we fix a universal class $\mathfrak{H}$. Whenever we consider a subclass **C** of objects of $\mathfrak{H}$, we suppose that **C** is an abstract class (that is, **C** is closed under isomorphisms) and that $\mathfrak{T} \subseteq$ **C**. Moreover, we introduce the following notation :

A—▷B means a nonzero surjective morphism of type 1) or 2),
$K \triangleleft A$ means that K is a nonzero kernel of A.

In the sequel we are going to give the fundamental definitions and characterizations of radical theory in a self-contained way for GS-automata. Further results can be proven in a similar way as in [12] or can be derived from [8].

An operator ϱ which assigns to each object $A \in \mathfrak{H}$ a kernel ϱA of A is called a *radical operator*, if ϱ satisfies the following set of conditions for all $A, B \in \mathfrak{H}$:

(ϱa) if $\varphi \colon A \to B$ is a surjective morphism, then $\varphi(\varrho A) \subseteq \varrho B$ holds,
(ϱb) $|\varrho(A/\varrho A)| = 1$,
(ϱc) if $\varrho B = B \triangleleft A$, then $B \subseteq \varrho A$,
(ϱd) $\varrho\varrho A = \varrho A$.

PROPOSITION 2.1. *Let* ϱ *be a radical operator. The class*

$$\mathbf{R}_\varrho = \{A \in \mathfrak{H} \mid \varrho A = A\}$$

fulfils the following conditions for all $A, B \in \mathfrak{H}$:

(R*a*) *if* $A \in \mathbf{R}_\varrho$, *then for every* A—▷B *there exists a* $K \triangleleft B$ *with* $K \in \mathbf{R}_\varrho$,
(R*b*) *if* $A \in \mathfrak{H}$ *and for every* A—▷B *there exists a* $K \triangleleft B$ *with* $K \in \mathbf{R}_\varrho$, *then*

$A \in \mathbf{R}_\varrho$,

(Rk) *if* $(A, +, \delta) \in \mathfrak{H}$ *and there exists a* $K \triangleleft (A, +)$ *such that* $K \in \mathbf{R}_\varrho$, *then there exists an* $L \triangleleft (A, +, \delta)$ *with* $L \in \mathbf{R}_\varrho$.

PROOF: Let $A \in \mathbf{R}_\varrho$ and $\varphi: A \twoheadrightarrow B$ be arbitrarily chosen. By (ϱa) we have

$$B = \varphi(A) = \varphi(\varrho A) \subseteq \varrho\varphi(A) = \varrho B \subseteq B,$$

and hence $B \in \mathbf{R}_\varrho$. Thus (Ra) is trivially satisfied.

Let $A \in \mathfrak{H} \setminus \mathfrak{T}$ be an object such that for each $A \twoheadrightarrow B$ there exists a $K \triangleleft B$ with $K \in \mathbf{R}_\varrho$. If $A \notin \mathbf{R}_\varrho$, then $\varrho A \neq A$, and so for $B = A/\varrho A$ we have $|B| > 1$. By the hypothesis there exists a $K \triangleleft B$ such that $\varrho K = K$, and hence (ϱc) yields $K \subseteq \varrho B$. Thus we have got

$$1 < |K| \leq |\varrho B| = |\varrho(A/\varrho A)|$$

contradicting (ϱb). Consequently $A \in \mathbf{R}_\varrho$, proving (Rb).

Finally, let us suppose that $(A, +, \delta) \in \mathfrak{H}$ is a GS-automaton such that $K \triangleleft (A, +)$ with some $K \in \mathbf{R}_\varrho$. Then (ϱc) yields $K \subseteq \varrho(A, +)$. Further, for the morphism $\iota: (A, +) \to (A, +, \delta)$ in view of (ϱa) we get

$$\iota(K) \subseteq \iota(\varrho(A, +)) \subseteq \varrho(\iota(A, +)) = \varrho(A, +, \delta),$$

and so

$$1 < |K| = |\iota(K)| \leq |\varrho(A, +, \delta)|$$

holds. Since by (ϱd) we have also $\varrho(A, +, \delta) \in \mathbf{R}_\varrho$, the validity of condition (Rk) has been established. □

PROPOSITION 2.2. *If a subclass* $\mathbf{R}$ *of* $\mathfrak{H}$ *satisfies conditions* (Ra), (Rb), (Rk), *then* $\mathbf{R}$ *fulfils also the following ones :*

(Rh) *the class* $\mathbf{R}$ *is homomorphically closed* : if $A \in \mathbf{R}$ and $\varphi: A \twoheadrightarrow B$, then $B \in \mathbf{R}$,

(Rc) *if* $(A, +, \delta) \in \mathfrak{H}$ *and* $(A, +) \in \mathbf{R}$, *then* $(A, +, \delta) \in \mathbf{R}$,

(Re) *the class* $\mathbf{R}$ *is closed under extensions* : if $K \triangleleft A \in \mathfrak{H}$, $K \in \mathbf{R}$ and $A/K \in \mathbf{R}$, then $A \in \mathbf{R}$,

(Ri) *the class* $\mathbf{R}$ *has the inductive property* : if $K_1 \subseteq \cdots \subseteq K_\alpha \subseteq \cdots$ is any ascending chain of kernels of an object $A \in \mathfrak{H}$ such that $K_\alpha \in \mathbf{R}$ for each index α, then $\vee K_\alpha \in \mathbf{R}$,

(Rt) $\mathfrak{T} \subseteq \mathbf{R}$.

PROOF: Let $A \in \mathbf{R}$ and $\varphi: A \twoheadrightarrow B$, and let us consider an arbitrary $\psi: B \twoheadrightarrow C$. Then also $\psi\varphi: A \twoheadrightarrow C$ holds, and so by (Ra) there exists a $K \triangleleft C$ with $K \in \mathbf{R}$. Hence (Rb) is applicable on B yielding $B \in \mathbf{R}$. This proves (Rh).

Let $(A, +, \delta)$ be a GS-automaton in $\mathfrak{H}$ such that $(A, +) \in \mathbf{R}$, and K be an arbitrary kernel of $(A, +, \delta)$ with $K \neq (A, +, \delta)$. Then we have

$$(A/K, +) \xrightarrow{\iota} (A/K, +, \delta)$$

and also $(A/K, +) \in \mathbf{R} \setminus \mathfrak{T}$ in view of (Rh). Hence (Rk) infers the existence of a kernel L of $(A/K, +, \delta)$ such that $L \in \mathbf{R} \setminus \mathfrak{T}$. Since the choice of K was arbitrary, by (Rb) we conclude $(A, +, \delta) \in \mathbf{R}$, proving the validity of (Rc).

For proving (Re), let L be an arbitrary nonzero kernel of A. We wish to apply (Rb) on A. If $K \subseteq L$, then the isomorphism

$$\frac{A/K}{L/K} \cong A/L$$

and the already demonstrated condition (Rh) yield $A/L \in \mathbf{R}$. If $K \not\subseteq L$, then $|K/(L \cap K)| > 1$ and again by (Rh) also $K/(L \cap K) \in \mathbf{R}$ is valid. Further, by PROPOSITION 1.3 we have

$$K/(L \cap K) \cong (L + K)/L \xrightarrow{\iota} (L \vee K)/L \triangleleft A/L$$

and so by (Rc), if needed, also $(L \vee K)/L \in \mathbf{R}$ holds. Thus A/L possesses always a nonzero kernel in **R**, and therefore (Rb) infers $A \in \mathbf{R}$. This proves (Re).

For demonstrating (Ri), put $L = \vee K_\alpha$. If $L \notin \mathbf{R}$, then in view of (Rb) there exists an $M \triangleleft L$ such that $|L/M| > 1$ and L/M has no nonzero kernel in **R**. Further, by (Rh) we have $K_\alpha/(M \cap K_\alpha) \in \mathbf{R}$ for each α. From PROPOSITION 1.3 we have

$$K_\alpha/(M \cap K_\alpha) \cong (K_\alpha + M)/M \xrightarrow{\iota} (K_\alpha \vee M)/M \triangleleft L/M$$

and so (Rc) infers $(K_\alpha \vee M)/M \in \mathbf{R}$ for every α. Hence by the choice of M it follows $K_\alpha \subseteq M$ for every α, and so also $L = \vee K_\alpha \subseteq M$, contradicting $|L/M| > 1$.

(Rt) is a trivial consequence of (Rb). □

PROPOSITION 2.3. *Let a subclass* **R** *of* $\mathfrak{H}$ *satisfy conditions* (Rh), (Re), (Ri), (Rt), (Rk). *If the operator* ϱ *is defined as*

$$\varrho A = \vee(K \triangleleft A \mid K \in \mathbf{R}), \quad \forall A \in \mathfrak{H},$$

then

i) $\varrho A \in \mathbf{R}$, $\forall A \in \mathfrak{H}$ *and* $\mathbf{R} = \{A \in \mathfrak{H} \mid \varrho A = A\}$,

ii) ϱ *is a radical operator.*

PROOF: First we prove that **R** fulfils (Rc). Suppose the contrary: there exists an automaton $(A, +, \delta) \in \mathfrak{H} \setminus \mathbf{R}$ such that $(A, +) \in \mathbf{R} \setminus \mathfrak{T}$. By (R$i$) and Zorn's Lemma there exists a kernel I of $(A, +, \delta)$ such that $I \in \mathbf{R}$ and I is maximal with respect to this property. Let us consider the automaton $A/I = (A/I, +, \delta)$. Since **R** has (Rh), we have $(A/I, +) \in \mathbf{R}$. Take any kernel L/I of $(A/I, +\delta)$ such that $L/I \in \mathbf{R}$. Then by $I \in \mathbf{R}$ and (Re) we get $L \in \mathbf{R}$. Hence the maximality of I gives us $L = I$. Thus there is no kernel L/K of $(A/I, +, \delta)$ such that $|L/K| > 1$ and $L/K \in \mathbf{R}$. Applying (Rk) we conclude that there is no kernel K/I of $(A/I, +)$ such that $|K/I| > 1$ and $K/I \in \mathbf{R}$. This and $(A/I, +) \in \mathbf{R}$ imply $A = I \in \mathbf{R}$, contradicting $A \in \mathfrak{H} \setminus \mathbf{R}$. Thus (R$c$) has been established.

Now we prove $\varrho A \in \mathbf{R}$. By (Ri) Zorn's Lemma is applicable yielding the existence of a kernel K of A being maximal with respect to $K \in \mathbf{R}$. Let L be any other kernel of A with $L \in \mathbf{R}$. By (Rh) we have $L/(L \cap K) \in \mathbf{R}$ and so in view of

$$L/(L \cap K) \cong (L + K)/K \xrightarrow{\iota} (L \vee K)/K$$

condition (Rc), if needed, yields $(L \vee K)/K \in \mathbf{R}$. Hence by condition (R$e$) we get $L \vee K \in \mathbf{R}$ which implies $L \subseteq K$ by the choice of K. Thus K is the unique kernel of A such that K is maximal with respect to $K \in \mathbf{R}$. This means exactly $\varrho A = K \in \mathbf{R}$.

Now the assertion that $\mathbf{R} = \{A \in \mathfrak{H} \mid \varrho A = A\}$ is obviously true.

For proving that ϱ is a radical operator, we notice that (ϱc) and (ϱd) are clearly satisfied, both by $\varrho A \in \mathbf{R}$.

Next we exhibit (ϱb). Let $L/\varrho A \triangleleft A/\varrho A$ and $L/\varrho A \in \mathbf{R}$. As we have already seen, $\varrho A \in \mathbf{R}$, hence condition (Re) implies $L \in \mathbf{R}$. Thus by the definition of ϱA we conclude $L \subseteq \varrho A$ which implies $|L/\varrho A| = 1$ as well as $|\varrho(A/\varrho A)| = 1$.

For demonstrating (ϱa) it suffices to exhibit its validity for morphisms $\psi: A \longrightarrow\!\triangleright B$ and $\iota: (A,+) \to (A,+,\delta)$ because every surjective morphism φ is a composition of such morphisms or $|A| = 1$, and this latter case is covered by condition (Rt). For any morphism $\psi: A \longrightarrow\!\triangleright B$ we have $\psi(K) \triangleleft B$ or $|\psi(K)| = 1$ whenever $K \triangleleft A$, in particular for $K = \varrho A$. Furthermore, also $\varrho A \in \mathbf{R}$ holds as we have seen, and so condition (Rh) infers $\psi(\varrho A) \in \mathbf{R}$. Thus by definition $\psi(\varrho A) \subseteq \varrho B$ holds. In the case $\iota: (A,+) \to (A,+,\delta)$, let us suppose that (ϱa) is not true, that is, $\iota(\varrho(A,+)) \not\subseteq \varrho(A,+,\delta)$. Then we have

$$|\varrho(A,+)/(\varrho(A,+) \cap \varrho(A,+,\delta))| > 1$$

and

$$\begin{aligned} \varrho(A,+)/(\varrho(A,+) \cap \varrho(A,+,\delta)) &\cong (\varrho(A,+) + \varrho(A,+,\delta))/\varrho(A,+,\delta) \\ &\triangleleft (A/\varrho(A,+,\delta),+). \end{aligned}$$

Moreover, condition (Rh) implies

$$\varrho(A,+)/(\varrho(A,+) \cap \varrho(A,+,\delta)) \in \mathbf{R}.$$

Hence condition (Rk) applies to $K = (\varrho(A,+) + \varrho(A,+,\delta))/\varrho(A,+,\delta)$ yielding the existence of an $L \triangleleft A/\varrho(A,+,\delta)$ with $L \in \mathbf{R}$. This, by $|L| > 1$, contradicts the already demostrated condition (ϱc). Thus $\iota(\varrho(A,+)) \subseteq \varrho(A,+,\delta)$ holds. □

A subclass $\mathbf{R}$ of $\mathfrak{H}$ is called a *radical class* if it satisfies condition (Ra), (Rb), (Rk). PROPOSITION 2.1, 2.2 and 2.3 can be summarized as follows

THEOREM 2.4. *Let ϱ be an operator assigning to each object $A \in \mathfrak{H}$ a kernel ϱA of A, and let $\mathbf{R}$ be a subclass of objects in $\mathfrak{H}$. Then the following three conditions are equivalent :*

1) *ϱ is a radical operator and $\mathbf{R}_\varrho = \mathbf{R}$,*
2) *$\mathbf{R}$ is a radical class and $\varrho A = \vee(K \triangleleft A \mid K \in \mathbf{R})$, $\forall A \in \mathfrak{H}$,*
3) *$\mathbf{R}$ satisfies conditions (Rh), (Re), (Ri), (Rk), (Rt) and $\varrho A = \vee(K \triangleleft A \mid K \in \mathbf{R})$, $\forall A \in \mathfrak{H}$.* □

Let ϱ be a radical operator. The class

$$\mathbf{S}_\varrho = \{A \in \mathfrak{H} \mid |\varrho A| = 1\}$$

is called the *semisimple* class of ϱ (or equivalently, of the radical class $\mathbf{R}_\varrho$). Obviously $\mathbf{R}_\varrho \cap \mathbf{S}_\varrho = \mathfrak{T}$ holds. It is useful introduce the *semisimple operator* $\mathcal{S}$ acting on subclasses $\mathbf{C}$ of objects of $\mathfrak{H}$ and defined by

$$\mathcal{S}\mathbf{C} = \{A \in \mathfrak{H} \mid K \triangleleft A \Rightarrow K \notin \mathbf{C}\}.$$

If ϱ is any radical operator and $\mathbf{R}_\varrho$ the corresponding radical class, then by THEOREM 2.4 we have

$$\mathbf{S}_\varrho = \mathcal{S}\mathbf{R}_\varrho$$

which justifies the terminology.

PROPOSITION 2.5. *If ϱ is a radical operator in $\mathfrak{H}$, then the semisimple class $\mathbf{S}_\varrho$ satisfies the following conditions :*

(S*a*) *if* $A \in \mathbf{S}_\varrho$, *then for every* $K \triangleleft A$ *there exists a* $K \multimap B$ *with* $B \in \mathbf{S}_\varrho$,

(S*b*) *if* $A \in \mathfrak{H}$ *and for every* $K \triangleleft A$ *there exists a* $K \multimap B$ *with* $B \in \mathbf{S}_\varrho$, *then* $A \in \mathbf{S}_\varrho$.

(S*c*) *if* $(A, +, \delta) \in \mathbf{S}_\varrho$, *then* $(A, +) \in \mathbf{S}_\varrho$.

PROOF: For exhibiting (S*a*), let us consider an object $A \in \mathbf{S}_\varrho = \mathcal{S}\mathbf{R}_\varrho$ and an arbitrary $K \triangleleft A$. Now we have $\varrho K \in \mathbf{R}_\varrho$, and so $|K/\varrho K| > 1$. Also $K/\varrho K \in \mathbf{S}_\varrho$ holds in view of (ϱb). Hence $K \multimap B \in \mathbf{S}_\varrho$ is satisfied with $B = K/\varrho K$.

Next, let us suppose that for every $K \triangleleft A$ there exists a $K \multimap B$ with $B \in \mathbf{S}_\varrho$, but $A \notin \mathbf{S}_\varrho$. Then $|\varrho A| > 1$. In particular, for $K = \varrho A$ there exists a $\varrho A \multimap C \in \mathbf{S}_\varrho$, and by ($\varrho a$) (or (R*h*)) we conclude also $\varrho C = C$ (or $C \in \mathbf{R}_\varrho$). Thus $C \in \mathbf{S}_\varrho \cap \mathbf{R}_\varrho = \mathfrak{T}$, contradicting $\varrho A \multimap C$. This proves the validity of (S*b*).

Finally, assume that $(A, +, \delta) \in \mathfrak{H}$ and $(A, +) \notin \mathbf{S}_\varrho$, that is, $\varrho(A, +) \triangleleft (A, +)$ and $\varrho(A, +) \in \mathbf{R}_\varrho$. By THEOREM 2.4 condition (R*k*) is applicable yielding the existence of an $L \triangleleft (A, +, \delta)$ with $L \in \mathbf{R}_\varrho$. Hence by (ϱb) it follows $L \subseteq \varrho(A, +, \delta)$ implying $(A, +, \delta) \notin \mathbf{S}_\varrho$. This proves (S*c*). □

For any subclass $\mathbf{C} \subseteq \mathfrak{H}$ we define an operator $\mathcal{U}$ as

$$\mathcal{U}\mathbf{C} = \{A \in \mathfrak{H} \mid A \multimap B \Rightarrow B \notin \mathbf{C}\}.$$

The operator $\mathcal{U}$, which is defined dually to the semisimple operator $\mathcal{S}$, is called the *upper radical operator.*

PROPOSITION 2.6. *If a subclass* $\mathbf{S} \subseteq \mathfrak{H}$ *satisfies conditions* (S*a*), (S*b*), (S*c*), *then* $\mathbf{R} = \mathcal{U}\mathbf{S}$ *is a radical class, and* $\mathbf{S} = \mathcal{S}\mathbf{R} = \mathbf{S}_\varrho$ *where ϱ denotes the radical operator correponding to the radical class* $\mathbf{R}$.

PROOF: Since the relation $\multimap$ is transitive, the class $\mathbf{R} = \mathcal{U}\mathbf{S}$ is homomorphically closed, that is, $\mathcal{U}\mathbf{S}$ satisfies (R*h*) and hence also the weaker condition (R*a*).

For demonstrating (R*b*), let us consider and object $A \in \mathfrak{H} \setminus \mathfrak{T}$ such that for every $A \multimap B$ there exists a $K \triangleleft B$ with $K \in \mathcal{U}\mathbf{S}$. If $A \notin \mathcal{U}\mathbf{S}$, then there exists an $A \multimap B$ with $B \in \mathbf{S}$ and by (S*a*) to every $K \triangleleft B$ there exists a $K \multimap C \in \mathbf{S}$, that is, $K \notin \mathcal{U}\mathbf{S}$. This contradicts the assumption on A, and so (R*b*) is satisfied. Let us notice that an object $A \in \mathfrak{T}$ trivially satisfies (R*b*).

Let $(A, +, \delta) \in \mathfrak{H}$ be an object such that $K \triangleleft (A, +)$ and $K \in \mathbf{R} = \mathcal{U}\mathbf{S}$ for some kernel of $(A, +)$. To prove (R*k*) we have to show that $(A, +, \delta) \notin \mathbf{S}$, because then by (S*b*) there exists an $L \triangleleft (A, +, \delta)$ such that $L \in \mathcal{U}\mathbf{S} = \mathbf{R}$, and this means exactly the validity of (R*k*). Suppose that $(A, +, \delta) \in \mathbf{S}$. Then by (S*c*) also $(A, +) \in \mathbf{S}$ is valid, and so by (S*a*) we have $K \multimap B \in \mathbf{S}$ for the kernel K of $(A, +)$ with an appropriate $B \in \mathfrak{H}$. This means $K \notin \mathcal{U}\mathbf{S}$, contradicting $K \in \mathcal{U}\mathbf{S}$. Thus (R*k*) has been established.

Since $\mathbf{R} = \mathcal{U}\mathbf{S}$ satisfies (R*a*), (R*b*) and (R*k*), by THEOREM 2.4 we conclude that $\mathbf{R}$ is a radical class.

As one readily checks, (S*a*) is equivalent to $\mathbf{S} \subseteq \mathcal{S}\mathcal{U}\mathbf{S}$ and (S*b*) is equivalent to $\mathcal{S}\mathcal{U}\mathbf{S} \subseteq \mathbf{S}$. Hence $\mathbf{S} = \mathcal{S}\mathbf{R}$ as well as $\mathbf{S} = \mathbf{S}_\varrho$ hold by the remark proceding PROPOSITION 2.5. □

PROPOSITIONS 2.5 and 2.6 infer immediately

COROLLARY 2.7. *A subclass* $\mathbf{S} \subseteq \mathfrak{H}$ *is the semisimple class of a radical class (or equivalently, of a radical operator) if and only if* $\mathbf{S}$ *satisfies conditions* (S*a*), (S*b*) *and* (S*c*).

For a subclass S of objects, let us define the operator η as

$$\eta A = \wedge(K \triangleleft A \mid A/K \in S)$$

which assigns to each $A \in \mathfrak{H}$ a kernel of A.

PROPOSITION 2.8. *If* S *is the semisimple class corresponding to a radical operator* ϱ, *then*

(S*s*) S *is closed under subdirect sums :* $A = \sum_{\text{subdirect}} A_\alpha$ and $A_\alpha \in S$ for all α imply $A \in S$, or equivalently : $A/\eta A \in S$,

(Sϱ) $\eta A = \varrho A$ *for all* $A \in \mathfrak{H}$,

(Sη) $\eta\eta A$ *is a kernel of* A *for all* $A \in \mathfrak{H}$,

(S*e*) S *is closed under extensions.*

PROOF: Firstly we prove (S*s*). Let us consider an object $A \in \mathfrak{H}$ such that A is a subdirect sum of objects A_α, $\alpha \in \Lambda$, each in S. Then there exists a set $\{K_\alpha \mid \alpha \in \Lambda\}$ of kernels of A such that $A/K_\alpha \cong A_\alpha \in S$ and $|\wedge K_\alpha| = 1$. Let $L \triangleleft A$ be arbitrary. Now by $|L| > 1$ there exists an index α such that $L \not\subseteq K_\alpha$, and hence

$$(L \vee K_\alpha)/K_\alpha \triangleleft A/K_\alpha \in S.$$

Thus, condition (S*a*) infers the existence of an

$$(L \vee K_\alpha)/K_\alpha \longrightarrow\!\!\triangleright B \in S.$$

Hence either

$$L \longrightarrow\!\!\triangleright L/(L \wedge K_\alpha) \cong (L \vee K_\alpha)/K_\alpha \longrightarrow\!\!\triangleright B \in S$$

or by (S*c*)

$$(L,+) \longrightarrow\!\!\triangleright (L/(L \wedge K_\alpha),+) \cong ((L+K_\alpha)/K_\alpha,+) \longrightarrow\!\!\triangleright (B,+) \in S$$

yields $L \longrightarrow\!\!\triangleright C \in S$ where $C = (B,+,\delta)$ or $(B,+)$. Hence by (S*b*) we conclude that $A \in S$, proving (S*s*).

For demonstrating (S*e*), let us consider a $K \triangleleft A$ such that $K \in S$ and $A/K \in S$. Further, let $L \triangleleft A$ be arbitrary. If $L \subseteq K$, then by $L \triangleleft K$ and $K \in S$ condition (S*a*) implies the existence of an $L \longrightarrow\!\!\triangleright B \in S$. If $L \not\subseteq K$, then we have

$$(L \vee K)/K \triangleleft A/K \in S,$$

and so by (S*a*), $(L \vee K)/K \longrightarrow\!\!\triangleright B \in S$ with an appropriate $B \in \mathfrak{H}$. The isomorphism

$$L/(L \wedge K) \cong (L+K)/K \xrightarrow{\iota} (L \vee K)/K$$

and condition (S*c*), if necessary, infer $L \longrightarrow\!\!\triangleright C \in S$ where either $C = (B,+,\delta)$ or $C = (B,+)$. Thus by (S*b*) we obtain $A \in S$ which proves (S*c*).

Next, we are going to prove (Sϱ). By condition (ϱb) we have $|\varrho(A/\varrho A)| = 1$, and therefore $A/\varrho A \in S_\varrho = S$. Hence $\eta A \subseteq \varrho A$ holds by the definition of η. Suppose that $\eta A \neq \varrho A$. Then $\varrho A/\eta A \triangleleft A$ is valid as $|\varrho A/\eta A| > 1$. Moreover, by (ϱa) and (ϱd) we obtain

$$\varrho A/\eta A = \varrho\varrho A/\eta A \subseteq \varrho(\varrho A/\eta A),$$

yielding $\varrho A/\eta A \in \mathbf{R}_\varrho = \mathcal{U}S$. Since $A/\eta A \in S$ by (S*s*), condition (S*a*) applied to $\varrho A/\eta A \triangleleft A/\eta A$ yields the existence of a $\varrho A/\eta A \longrightarrow\!\!\triangleright C \in S$, contradicting $\varrho A/\eta A \in \mathcal{U}S$. Thus $\eta A = \varrho A$ has been proved.

Finally, condition (Sη) is a trivial consequence of $\eta A = \varrho A$ and condition (ϱd). □

PROPOSITION 2.9 . *Let* $\mathbf{S}$ *be a subclass of* $\mathfrak{H}$ *which fulfils conditions* (Sa), (Sc), (Se), (Ss), (St), (Sη). *Then* $\mathbf{S}$ *is a semisimple class.*

PROOF: In view of COROLLARY 2.7 all what we have to prove is the validity of condition (Sb). So, let us consider an object $A \in \mathfrak{H}$ such that for every $K \triangleleft A$ there exists a $K \multimap B$ with $B \in \mathbf{S}$. By way of contradiction, let us suppose that $A \notin \mathbf{S}$. Then by (Ss) we have $A/\eta A \in \mathbf{S}$, and so $|\eta A| > 1$, that is, $\eta A \triangleleft A$. By (Sη) also $\eta A/\eta\eta A \triangleleft A/\eta\eta A$ holds, and by (Ss) we have $\eta A/\eta\eta A \in \mathbf{S}$. Since

$$\frac{A/\eta\eta A}{\eta A/\eta\eta A} \cong A/\eta A \in \mathbf{S},$$

condition (Se) yields $A/\eta\eta A \in \mathbf{S}$ which implies $\eta A \subseteq \eta\eta A$. Thus by the definition of η, ηA has no non-zero isomorphic image in $\mathbf{S}$, contradicting (Sa). □

COROLLARY 2.10. *A subclass* $\mathbf{S}$ *of* $\mathfrak{H}$ *is a semisimple class if and only if* $\mathbf{S}$ *satisfies conditions* (Sa), (Sc), (Se), (Ss), (St) *and* (Sη). *Moreover, the operator* η *occuring in condition* (Sη) *is just the radical operator corresponding to the semisimple class* $\mathbf{S}$.

PROOF: Trivial by PROPOSITIONS 2.8 and 2.9. □

THEOREM 2.11. *The subclasses* $\mathbf{R}$ *and* $\mathbf{S}$ *are corresponding radical and semi-simple classes (that is,* $\mathbf{R} = \mathcal{U}\mathbf{S}$ *and* $\mathbf{S} = \mathcal{S}\mathbf{R}$*) if and only if*

- *a)* $A \in \mathbf{R}$ *and* $A \multimap B$ *imply* $B \notin \mathbf{S}$, *that is,* $\mathbf{R} \subseteq \mathcal{U}\mathbf{S}$,
- *b)* $A \in \mathbf{S}$ *and* $B \triangleleft A$ *imply* $B \notin \mathbf{R}$, *that is,* $\mathbf{S} \subseteq \mathcal{S}\mathbf{R}$,
- *c) for each* $A \in \mathfrak{H}$ *there exists a kernel* K *of* A *such that* $K \in \mathbf{R}$ *and* $A/K \in \mathbf{S}$.
- *d)* $\mathbf{S}$ *fulfils* (Sc) *or* $\mathbf{R}$ *satisfies* (Rk).

PROOF: We already know that these properties hold true for a radical class $\mathbf{R}$ with semisimple class $\mathbf{S} = \mathcal{S}\mathbf{R}$.

Conversely, we apply c) to each $A \in \mathcal{S}\mathbf{R}$. Since $A \in \mathcal{S}\mathbf{R}$ implies $B \notin \mathbf{R}$ for all $B \triangleleft A$, necessarily $|K| = 1$ and hence $A \in \mathbf{S}$, that is $\mathcal{S}\mathbf{R} \subseteq \mathbf{S}$. This together with b) yields $\mathbf{S} = \mathcal{S}\mathbf{R}$. Applying c) to each $A \in \mathcal{U}\mathbf{S}$, from $A/B \notin \mathbf{S}$ for all kernel $B \neq A$, we get $|A/K| = 1$, and so $A = K \in \mathbf{R}$, that is, $\mathcal{U}\mathbf{S} \subseteq \mathbf{R}$. This and a) gives us $\mathbf{R} = \mathcal{U}\mathbf{S}$. Thus $\mathbf{R} = \mathcal{U}\mathcal{S}\mathbf{R}$ and $\mathbf{S} = \mathcal{S}\mathcal{U}\mathbf{S}$ hold. As one easily sees, $\mathbf{R} = \mathcal{U}\mathcal{S}\mathbf{R}$ is equivalent to (Ra) and (Rb) and $\mathbf{S} = \mathcal{S}\mathcal{U}\mathbf{S}$ is equivalent to (Sa) and (Sb). This along with d) proves that $\mathbf{R}$ and $\mathbf{S}$ are corresponding radical and semisimple classes in view of COROLLARY 2.7 or by the definition of $\mathbf{R}$. □

Before giving explicit examples, let us notice that there are plenty of concrete radical classes, for instance, to every partition of simple GS-automata there is a radical class containing exactly one class of the partition (and the other class will be included in the corresponding semisimple class).

EXAMPLE 2.12. We say that a GS-automaton $(A, +, \delta)$ has the *relative 0-reset property,* if to every element $a \in A$ there exists an $x \in X^*$ depending on a, such that $ax = 0$. *The class*

$$\mathbf{R} = \{A \in \mathfrak{A} \mid A \textit{ has the relative 0-reset property}\} \cup \{(0, +)\}$$

is a radical class. Conditions (Rh), (Ri), (Rk), (Rt) are trivially fulfilled. In view of THEOREM 2.4 we still have to show the validity of (Re). Let K be a kernel of

$A \in \mathfrak{H}$ such that $K \in \mathbf{R}$ and $A/K \in \mathbf{R}$. If $|K| = 1$, then we are done. So let $K \triangleleft A$. Since $K \in \mathbf{R}$ and $|K| > 1$, K is a subsemiautomaton, and therefore A has to be GS-automaton. Let $a \in A$ be an arbitrary element. Since $A/K \in \mathbf{R}$, there exists an $x \in X^*$ such that $(a + K)x \subseteq K$, that is,

$$(a + k)x \in K, \quad \forall k \in K.$$

K is a kernel of A, so also

$$(a + k)x - ax \in K$$

holds. These together yield

$$ax \in K \in \mathbf{R}.$$

Hence there exists a $y \in X^*$ such that $(ax)y = 0$, that is, $a(xy) = 0$ with $xy \in X^*$, proving that A has the relative 0-reset property. Thus $\mathbf{R}$ satisfies also condition $(\mathrm{R}e)$, and consequently $\mathbf{R}$ is a radical class.

EXAMPLE 2.13. In a GS-automaton A, 0 is a *reset* if there exists an $x \in X^*$ such that $Ax = 0$. Restricting the universal class to

$$H = \{\text{all finite GS} - \text{automata}\} \cup \{\text{all finite groups}\},$$

the class

$$\mathbf{R} = \{A \in \mathfrak{A} \cap \mathfrak{H} \mid 0 \textit{ is a reset in } A\} \cup \{(0, +)\}$$

is a radical class. Again, conditions $(\mathrm{R}h), (\mathrm{R}i), (\mathrm{R}k), (\mathrm{R}t)$ are trivially satisfied. Notice that $(\mathrm{R}i)$ would not be satisfied for infinite GS-automata. The same proof as in EXAMPLE 2.12 infers the validity of condition $(\mathrm{R}e)$, because there the element $x \in X^*$ may be chosen such that $Ax \subseteq K$ and $y \in X^*$ such that $Ky = 0$, whence $A(xy) = 0$.

EXAMPLE 2.14. A GS-automaton $(A, +, \delta)$ is said to be 0-*connected*, if for every $a \in A$ there exists an $x \in X$ such that $0x = a$. Then

$$\mathbf{R} = \{A \in \mathfrak{A} \mid A \text{ is 0-connected}\} \cup \{(0, +)\}$$

is a radical class. Conditions $(\mathrm{R}h), (\mathrm{R}t), (\mathrm{R}k), (\mathrm{R}i)$ are trivially satisfied, only $(\mathrm{R}e)$ needs verification. So, let $K \triangleleft A$ such that $K \in \mathbf{R}$ and $A/K \in \mathbf{R}$. Now K has to be a subsemiautomaton, and therefore $KX \subseteq K$. Since $A/K \in \mathbf{R}$, for each $a \in A$ there exists an $x \in X$ such that $Kx \subseteq a + K$. Hence $KX \subseteq K$ implies $a \in K$, and by $K \in \mathbf{R}$ there exists a $y \in X$ with $0y = a$. Clearly, we have also

$$\mathbf{R} = \{A \in \mathfrak{A} \mid A = 0X\} \cup \{(0, +)\}.$$

3 Complementary radical and semisimple classes

We start this section with

EXAMPLE 3.1. The class

$$\mathbf{R} = \{A \in \mathfrak{A} \mid (0, +, \delta) \text{ is a subsemiautomaton in } A\} \cup \{(0, +)\}$$

is a radical class and

$$\mathbf{S} = \mathcal{S}\mathbf{R} = \{A \in \mathfrak{A} \mid 0 \text{ is not a subsemiautomaton in } A\} \cup \mathfrak{G}$$

is the corresponding semisimple class in the universal class $\mathfrak{C}$, as one readily checks. Moreover, $\mathbf{R} \cup \mathbf{S} = \mathfrak{C}$, though $\mathbf{R} \neq \mathfrak{C}$ and $\mathbf{S} \neq \mathfrak{C}$.

Motivated by this EXAMPLE we introduce the following definition.

Let ϱ be a radical operator in $\mathfrak{H}$ with corresponding radical class $\mathbf{R}$ and semisimple class $\mathbf{S}$. We say that ϱ is *complementary*, or that ($\mathbf{R}$ and $\mathbf{S}$) are *complementary*, if

$$\varrho \mathrm{A} = \mathrm{A} \quad or \quad |\varrho \mathrm{A}| = 1, \quad for\ all\ \mathrm{A} \in \mathfrak{H},$$

or equivalently,

$$\mathbf{R} \cup \mathbf{S} = \mathfrak{H}.$$

The existence of non-trivial complementary radical operators (here non-trivial means $\mathbf{R} \neq \mathfrak{T} \neq \mathbf{S}$) is a consequence of the fact that in the category $\mathfrak{H}$ the initial object $(0,+)$ is not equivalent to the terminal object $(0,+,\delta)$ (cf. [11]).

THEOREM 3.2. *Let ϱ be a radical operator in $\mathfrak{H}$ with radical class $\mathbf{R}$ and semisimple class $\mathbf{S}$. If*

1) $\mathbf{R}$ *contains at least one nonzero* GS*-automaton and all* GS*-automata of* $\mathfrak{H}$ *with one-element subsemiautomaton,*

and

2) $\mathbf{S}$ *contains all groups of* $\mathfrak{H}$,

then ϱ is a non-trivial complementary radical operator.

If $\mathfrak{H}$ is closed under forming finite direct sums (in the sense of 6 *of §1) and ϱ is a non-trivial complementary radical operator in $\mathfrak{H}$, then* $\mathbf{R}$ *fulfils* 1) *and* $\mathbf{S}$ *fulfils* 2).

PROOF: Assume that 1) and 2) are satisfied, and let $\mathrm{A} \in \mathfrak{H}$ be an arbitrary nonzero object. If $|\varrho \mathrm{A}| = 1$, then $\mathrm{A} \in \mathbf{S}$. Suppose that $|\varrho \mathrm{A}| > 1$. Since all groups are in $\mathbf{S}$, we conclude by $\varrho \mathrm{A} = \varrho\varrho \mathrm{A} \in \mathbf{R}$ that $\varrho \mathrm{A}$ is a GS-automaton with subsemiautomaton $(0,+,\delta)$, and hence so is A. Thus $\mathrm{A} \in \mathbf{R}$, proving that ϱ is complementary.

Next, suppose that $\mathfrak{H}$ has finite direct sums and ϱ is a non-trivial complementary radical operator. In virtue of (Sc) the semisimple class $\mathbf{S}$ contains at least one group $(\mathrm{A},+) \notin \mathfrak{T}$. Let $(\mathrm{B},+) \in \mathfrak{H} \setminus \mathfrak{T}$ be arbitrary. By the assumption on $\mathfrak{H}$ we have $(\mathrm{A},+) \oplus (\mathrm{B},+) \in \mathfrak{H}$. Now $(\mathrm{A},+) \oplus (\mathrm{B},+) \in \mathbf{R}$ is not possible because then $(\mathrm{A},+)\oplus(\mathrm{B},+) \twoheadrightarrow (\mathrm{A},+)$ and (Rh) would imply $(\mathrm{A},+) \in \mathbf{R}$. Thus $(\mathrm{A},+)\oplus(\mathrm{B},+) \in \mathbf{S}$, as ϱ is complementary. Since $\mathbf{S} \cap \mathfrak{G}$ is a semisimple class of groups, $\mathbf{S} \cap \mathfrak{G}$ is hereditary, and hence

$$(\mathrm{B},+) \triangleleft (\mathrm{A},+) \oplus (\mathrm{B},+) \in \mathbf{S} \cap \mathfrak{G}$$

yields $(\mathrm{B},+) \in \mathbf{S} \cap \mathfrak{G} \subseteq \mathbf{S}$, proving that $\mathbf{S}$ contains all groups of $\mathfrak{H}$.

Since $\mathbf{S}$ contains all groups and ϱ is non-trivial, $\mathbf{R}$ has to contain at least one nonzero GS-automaton. Assume that $\mathbf{R}$ does not contain all GS-automata of $\mathfrak{H}$ with one-element subsemiautomaton. Then there exists an $(\mathrm{A},+,\delta_\mathrm{A}) \in \mathfrak{H}$ such that $(0,+,\delta_\mathrm{A})$ is a subsemiautomaton of $(\mathrm{A},+,\delta_\mathrm{A})$ and $(\mathrm{A},+,\delta_\mathrm{A}) \notin \mathbf{R}$, that is, $(\mathrm{A},+,\delta_\mathrm{A}) \in \mathbf{S}$ by ϱ complementary. Let $(\mathrm{B},+,\delta_\mathrm{B})$ be an arbitrary GS-automaton in $\mathfrak{H}$. By the assumption on $\mathfrak{H}$ the direct sum $(\mathrm{A},+,\delta_\mathrm{A})\oplus(\mathrm{B},+,\delta_\mathrm{B})$ is in $\mathfrak{H}$. By (Rh) and $(\mathrm{A},+,\delta_\mathrm{A})\oplus(B,+,\delta_\mathrm{B}) \twoheadrightarrow (\mathrm{A},+,\delta_\mathrm{A}) \in \mathbf{S}$ the relation $(\mathrm{A},+,\delta_\mathrm{A})\oplus(\mathrm{B},+,\delta_\mathrm{B}) \in \mathbf{R}$ is not possible whence by ϱ complementary it follows $(\mathrm{A},+,\delta_\mathrm{A}) \oplus (\mathrm{B},+,\delta_\mathrm{B}) \in \mathbf{S}$. Thus by $\delta_\mathrm{A}(0,x) = 0$, $(\mathrm{B},+,\delta_\mathrm{B}) \triangleleft (\mathrm{A},+,\delta_\mathrm{A}) \oplus (\mathrm{B},+,\delta_\mathrm{B})$ and hence by (Sa), there exists a $(\mathrm{B},+,\delta_\mathrm{B}) \twoheadrightarrow (\mathrm{C},+,\delta_\mathrm{C}) \in \mathbf{S}$. Thus by (R$h$) we get $(\mathrm{B},+,\delta_\mathrm{B}) \notin \mathbf{R}$, and since ϱ is complementary, we conclude $(\mathrm{B},+,\delta_\mathrm{B}) \in \mathbf{S}$. Hence $\mathbf{S} = \mathfrak{H}$ and $\mathbf{R} = \mathfrak{T}$

follows, contradicting the assumption that ϱ is non-trivial. Thus $\mathbf{R}$ contains all GS-automata of $\mathfrak{H}$ with one-element subsemiautomaton. □

COROLLARY 3.3. *The class*

$$\mathbf{R}_0 = \{A \in \mathfrak{H} \cap \mathfrak{A} \mid (0,+,\delta) \text{ is a subsemiautomaton of } A\} \cup \{(0,+)\}$$

is a complementary radical class in $\mathfrak{H}$. *If* $\mathfrak{H}$ *has finite direct sums and* $\mathbf{R} \neq \mathfrak{T}$ *is a complementary radical class, then* $\mathbf{R}_0 \subseteq \mathbf{R}$.

PROOF: The first statement follows from EXAMPLE 3.1 and the second one from THEOREM 3.2. □

THEOREM 3.4. $\mathbf{R}$ *is a complementary radical class in* $\mathfrak{H}$ *if and only if* $\mathbf{R}$ *satisfies* $(\mathrm{R}h), (\mathrm{R}c), (\mathrm{R}t)$ *and*

(C) $B \in \mathbf{R}$ *and* $B \triangleleft A \in \mathfrak{H}$ *imply* $A \in \mathbf{R}$.

$\mathbf{S}$ *is a complementary semisimple class in* $\mathfrak{H}$ *if and only if* $\mathbf{S}$ *satisfies* $(\mathrm{S}c)$,

$(\mathrm{S}t)$ $\mathfrak{T} \subseteq \mathbf{S}$,

$(\mathrm{S}h)$ $A \in \mathbf{S}$ *and* $B \triangleleft A$ *imply* $B \in \mathbf{S}$,

(D) $B \in \mathbf{S}$, $A \in \mathfrak{H}$ *and* $A \longrightarrow\!\triangleright B$ *imply* $A \in \mathbf{S}$.

PROOF: Let $\mathbf{R}$ be a complementary radical class. If $A \in \mathfrak{H} \setminus \mathbf{R}$, then $A \in \mathcal{S}\mathbf{R}$ and hence $B \notin \mathbf{R}$ for any $B \triangleleft A$, which implies (C).

Conversely, let us assume that $\mathbf{R}$ satisfies $(\mathrm{R}h), (\mathrm{R}c), (\mathrm{R}t)$ and (C). Condition (C) readily implies $(\mathrm{R}e)$ and $(\mathrm{R}i)$. To show $(\mathrm{R}k)$, let us consider a GS-automaton $(A,+,\delta)$ such that $K \triangleleft (A,+)$ and $K \in \mathbf{R}$ for some $K \in \mathfrak{H}$. Now condition (C) implies $(A,+) \in \mathbf{R}$ and condition $(\mathrm{R}c)$ infers $(A,+,\delta) \in \mathbf{R}$, proving $(\mathrm{R}k)$. Thus by THEOREM 2.4 $\mathbf{R}$ is a radical class. Suppose that $A \notin \mathbf{R}$ for some $A \in \mathfrak{H}$. Then (C) yields $A \in \mathbf{S}$, and hence $\mathbf{R}$ is complementary.

Assume that $\mathbf{S}$ is a complementary semisimple class. $(\mathrm{S}t)$ is always satisfied by $(\mathrm{S}a)$ and $(\mathrm{S}b)$ or $(\mathrm{S}\eta)$. If $B \triangleleft A$ and $B \notin \mathbf{S}$, then $B \in \mathbf{R}$ and hence $A \notin \mathbf{S}$. This proves $(\mathrm{S}h)$. If $A \in \mathfrak{H} \setminus \mathbf{S}$, then $A \in \mathbf{R}$ and hence $A \longrightarrow\!\triangleright B$ implies $B \in \mathbf{R}$ by $(\mathrm{R}h)$. This means that (D) is satisfied.

Conversely, let us suppose that $\mathbf{S}$ satisfies $(\mathrm{S}c), (\mathrm{S}t), (\mathrm{S}h)$ and (D). Condition $(\mathrm{S}h)$ implies trivially $(\mathrm{S}a)$. We want to see the validity of $(\mathrm{S}b)$. Assume that $A \in \mathfrak{H}$ is such an object that for every $B \triangleleft A$ there exists a $B \longrightarrow\!\triangleright C \in \mathbf{S}$. From $B \longrightarrow\!\triangleright C \in \mathbf{S}$ and (D) we get $B \in \mathbf{S}$ for every $B \triangleleft A$, in particular for $B = A$. If there is no $B \triangleleft A$, then $|A| = 1$, and $(\mathrm{S}t)$ infers $A \in \mathbf{S}$. Thus $(\mathrm{S}b)$ holds and so by COROLLARY 2.7 $\mathbf{S}$ is a semisimple class. We still have to see that $\mathbf{S}$ is complementary. If $A \in \mathfrak{H} \setminus \mathcal{U}\mathbf{S}$, then there exists an $A \longrightarrow\!\triangleright B \in \mathbf{S}$ and so (D) yields $A \in \mathbf{S}$. Thus $\mathbf{S}$ is complementary. □

4 Additive automata

An element $x_0 \in X$ is called a *zero-input*, if $0x_0 = 0$. A GS-automaton $(A,+,\delta)$ is said to be *additive*, if there exists a zero-input $x_0 \in X$ with the following properties :

i) *decomposition property* : $ax = ax_0 + 0x, \quad \forall a \in A, \quad \forall x \in X$,

ii) *zero-input additivity* : $(a+b)x_0 = ax_0 + bx_0, \quad \forall a, b \in A$.

Obviously on every additive group $(A,+)$ of at least two elements one can define at least two non-isomorphic GS-automata. Concerning additive GS-automata the reader is refered to [4].

In the sequel we suppose that *all the* GS-*automata* in the universal class $\mathfrak{H}$ considered, *are additive ones.*

PROPOSITION 4.1. *Let* A *be an additive* GS-*automaton and* $C \triangleleft (B, +, \delta) \triangleleft (A, +, \delta)$. *If* $(C, +)$ *is a normal subgroup of* $(A, +)$, *then* C *is a kernel of* A.

PROOF: We have to show that

$$(a + k)x - ax \in C$$

holds for all $a \in A$, $k \in C$ and $x \in X$. Since A is additive, we have

$$(a + k)x - ax = (a + k)x_0 + 0x - 0x - ax_0 = ax_0 + kx_0 - ax_0.$$

Taking into account that C is a kernel of $(B, +, \delta)$, it follows

$$bx_0 + kx_0 - bx_0 = (b + k)x_0 + 0x - 0x - bx_0 = (b + k)x_0 - bx_0 \in C$$

for all $b \in B$. Since $(C, +)$ is a normal subgroup of $(A, +)$, we may conjugate by $(ax_0 - bx_0) \in A$ obtaining

$$ax_0 + kx_0 - ax_0 = (ax_0 - bx_0) + (bx_0 + kx_0 - bx_0) - (ax_0 - bx_0) \in C,$$

regardless as whether kx_0 is in C or not. Thus also

$$(a + k)x - ax \in C$$

holds proving the assertion. □

PROPOSITION 4.2. *Let* ϱ *be an operator assigning to each* $A \in \mathfrak{H}$ *a kernel* ϱA *of* A *and satisfying condition* (ϱa). *If* $(B, +, \delta) \triangleleft (A, +, \delta) \in \mathfrak{H}$, *then* $\varrho(B, +, \delta)$ *is a kernel of* A.

PROOF: In virtue of PROPOSITION 4.1 we have to prove that $(\varrho B, +)$ is a normal subgroup of $(A, +)$. Since $(B, +)$ is normal in $(A, +)$, for every element $a \in A$ the mapping

$$\varphi_a(b) = a + b - a, \qquad \forall b \in B,$$

is an isomorphism of $(B, +)$ onto itself. Hence condition (ϱa) yields

$$\varphi_a(\varrho B) \subseteq \varrho \varphi_a(B) = \varrho B,$$

proving that $(\varrho B, +)$ is a normal subgroup in $(A, +)$. □

THEOREM 4.3. *Every semisimple class* S *in* $\mathfrak{H}$ *is hereditary, that is,* S *satisfies* (Sh).

PROOF: Let ϱ be the radical operator corresponding to S. If $(B, +, \delta) \triangleleft (A, +, \delta) \in S$, then by PROPOSITION 4.2 $\varrho(B, +, \delta)$ is a kernel of $(A, +, \delta)$ and by (ϱc) and (ϱd) we have

$$\varrho(B, +, \delta) \subseteq \varrho(A, +, \delta) \in \mathfrak{T}.$$

Thus also $(B, +, \delta) \in S$ holds.

If $(B, +) \triangleleft (A, +, \delta)$, then also $(B, +) \triangleleft (A, +)$ is valid. Moreover, condition (Sc) infers $(A, +) \in S$. As is well-known, semisimple classes of groups are hereditary. Hence we conclude $(B, +) \in S$, and the Theorem is proved. □

From THEOREMS 2.11 and 4.3 we obtain immediately.

COROLLARY 4.4. *Subclasses* **R** *and* **S** *of* $\mathfrak{H}$ *are corresponding radical and semisimple classes if and only if*

a) $\mathbf{R} \cap \mathbf{S} \in \mathfrak{T}$,
b) **R** *is homomorphically closed, that is,* (Rh) *is fulfilled,*
c) **S** *is strongly hereditary, that is,* **S** *satisfies* (Sc) *and* (Sh),
d) *for each* $A \in \mathfrak{H}$ *there exists a kernel* K *of* A *such that* $K \in \mathbf{R}$ *and* $A/K \in \mathbf{S}$. □

In order to get more explicit results and derive a structure theorem (COROLLARY 4.7) for semisimple objects, we shall restrict our investigations to a universal class $\mathfrak{H}$ in which all the groups are commutative. This class still includes linear sequential machines.

PROPOSITION 4.5. *Let us suppose that* $L \triangleleft K \triangleleft (A,+,\delta) \in \mathfrak{H}$. *If* x_0 *is a 0-input for* A *then* $L \triangleleft (A,+,\delta)$ *if and only if* $Lx_0 \subseteq L$. *If* $L \triangleleft (K,+,\delta) \triangleleft (A,+,\delta)$, *then* $L \triangleleft (A,+,\delta)$.

PROOF:

$$\begin{aligned} L \triangleleft (A,+,\delta) \quad &\Leftrightarrow \quad (a+l)x - ax \in L \text{ for all } a \in A, l \in L \text{ and } x \in X, \\ &\Leftrightarrow \quad lx_0 \in L \text{ for all } l \in L, \\ &\Leftrightarrow \quad Lx_0 \subseteq L. \end{aligned}$$

For the second assertion, note that by $L \triangleleft (K,+,\delta)$ it follows $(k+l)x - kx \in L$ for all $k \in K$, $l \in L$ and $x \in X$, and hence $lx_0 \in L$ for all $l \in L$. Thus the first statement yields $L \triangleleft (A,+,\delta)$. □

A kernel K of an object $A \in \mathfrak{H}$ is said to be *essential* in A, if for any other kernel $L \triangleleft A$ it follows $K \wedge L \notin \mathfrak{T}$. This fact will be denoted by $K \triangleleft \circ A$. A subclass **M** of $\mathfrak{H}$ is said to be *closed under essential extensions*, if $K \triangleleft \circ A$ and $K \in \mathbf{M}$ imply $A \in \mathbf{M}$.

THEOREM 4.6. *Let* **M** *be a subclass of* $\mathfrak{H} \cap \mathfrak{A}$ *such that* **M** *is hereditary, closed under essential extensions and satisfies condition*

(F) $L \triangleleft K \triangleleft A \in \mathfrak{H}$ *and* $K/L \in \mathbf{M}$ *imply* $L \triangleleft A$.

If $\overline{\mathbf{M}}$ *denotes the subdirect closure of* **M** *that is*

$$\overline{\mathbf{M}} = \{A \in \mathfrak{H} \mid A \textit{ is a subdirect sum of objects from } \mathbf{M}\}$$

then the class $\mathbf{S} = \overline{\mathbf{M}} \cup (\mathfrak{H} \cap \mathfrak{G})$ *is a semisimple class.*

PROOF: First, we show that every kernel K of a GS-automaton $A \in \overline{\mathbf{M}}$ is a subsemiautomaton. For this end it suffices to prove that $0X = 0$. Since $A \in \overline{\mathbf{M}}$, there are kernels I_α, $\alpha \in \Lambda$, of A such that $A/I_\alpha \in \mathbf{M}$ for each $\alpha \in \Lambda$ and $\wedge(I_\alpha \mid \alpha \in \Lambda) \in \mathfrak{T}$. The class **M** consists of GS-automata, so by the hereditariness of **M** every kernel, in particular the trivial kernel of A/I_α is a subsemiautomaton, and therefore $I_\alpha X \subseteq I_\alpha$ for each $\alpha \in \Lambda$. This implies

$$0X = (\wedge I_\alpha)X \subseteq \wedge I_\alpha = 0.$$

Next, we are going to prove that $\overline{\mathbf{M}}$ is hereditary. Let us consider an arbitrary kernel K of a GS-automaton $A \in \overline{\mathbf{M}}$. If $K \in \mathfrak{T}$, then by the previous statement

$K = (0, +, \delta)$ holds, and so $K \in \mathbf{M} \subseteq \overline{\mathbf{M}}$. So, let us assume that $K \triangleleft A$. Since K is a subsemiautomaton, we have

$$K/(K \wedge I_\alpha) \cong (K \vee I_\alpha)/I_\alpha \triangleleft A/I_\alpha \in \mathbf{M}$$

for all $\alpha \in \Lambda$. Thus the hereditariness of $\mathbf{M}$ yields $K/(K \wedge I_\alpha) \in \mathbf{M}$. Moreover, by $\bigwedge_\alpha (K \wedge I_\alpha) = (\bigwedge_\alpha I_\alpha) \wedge K = 0$ We conclude $K \in \overline{\mathbf{M}}$, proving that $\overline{\mathbf{M}}$ is, in fact, hereditary.

The hereditariness of $\overline{\mathbf{M}}$ readily yields that of the class $\mathbf{S} = \overline{\mathbf{M}} \cup (\mathfrak{H} \cap \mathfrak{G})$, and therefore $\mathbf{S}$ satisfies (Sa) trivially.

To prove the validity of (Sb), let us consider an object $A \in \mathfrak{H}$ such that every $K \triangleleft A$ has a nonzero homomorphic image K/L in $\mathbf{S} = \overline{\mathbf{M}} \cup (\mathfrak{H} \cup \mathfrak{G})$. If $A = (A, +) \in \mathfrak{H} \cap \mathfrak{G}$, then $A \in \mathbf{S}$. Hence we shall consider the case $A = (A, +, \delta)$. Let us suppose that $A \notin \mathbf{S}$. Since A is a GS-automaton, we get $A \notin \overline{\mathbf{M}}$. Hence

$$K = \wedge(K_\beta \triangleleft A \mid A/K_\beta \in \mathbf{M}) \neq 0.$$

Since $\mathbf{M}$ is hereditary and consists of GS-automata, the trivial kernel of A/K_β is a subsemiautomaton, which implies $K_\beta X \subseteq K_\beta$, and so each K_β is a subsemiautomaton. Hence so is K as well. By the hypothesis on A, K has a nonzero homomorphic image K/L in S and also $K/L \in \overline{\mathbf{M}}$ holds, for K is a subsemiautomaton. Hence there exists a kernel J/L of K/L such that

$$K/J \cong \frac{K/L}{J/L} \in \mathbf{M} \setminus \mathfrak{T}.$$

Using condition (F) we conclude that J is a kernel of A. Let us choose a kernel M of A being maximal with respect to the property $M \wedge K = J$. By Zorn's Lemma such a kernel M does exist. Now we have

$$K/J = K/(M \wedge K) \cong (K + M)/M \triangleleft A/M.$$

For any $Q/M \triangleleft A/M$ the choice of M yields $J \subsetneq Q \wedge K$, and therefore

$$(Q \wedge K)/J \triangleleft K/J \in \mathbf{M}.$$

Thus the hereditariness of $\mathbf{M}$ infers that $(Q \wedge K)/J$ is a GS-automaton, and so

$$0 \neq (Q \wedge K)/J \cong ((Q \wedge K) \vee M)/M \subseteq ((K \vee M)/M) \wedge (Q/M),$$

proving that $(K \vee M)/M$ is essential in A/M. Hence by

$$(K \vee M)/M \cong K/J \in \mathbf{M}$$

and by $\mathbf{M}$ being closed under essential extensions, we conclude $A/M \in \mathbf{M}$. This implies by the definition K that $K \subseteq M$, and so $J = M \wedge K = K$ holds, yielding $K/J \in \mathfrak{T}$, contradicting $K/J \notin \mathfrak{T}$. Thus $A \in \mathbf{S}$ has been proved, establishing the validity of condition (Sb).

Since $\mathfrak{G} \cap \mathfrak{H} \subseteq \mathbf{S}$ by definition, the class $\mathbf{S}$ fulfils condition (Sc), too. Thus in view of COROLLARY 2.7 $\mathbf{S}$ is a semisimple class. □

COROLLARY 4.7. *Let* $\mathbf{M}$ *be a subclass of* $\mathfrak{H} \cap \mathfrak{A}$ *such that* $\mathbf{M}$ *is hereditary, closed under essential extensions and satisfies condition* (F). *Then*

$$\mathbf{R} = \{A \in \mathfrak{H} \cap \mathfrak{A} \mid A \longrightarrow\!\!\triangleright B \Rightarrow B \notin \mathbf{M}\} \cup \{(0,+)\}$$

is a radical class. Denoting by ϱ *the corresssponding radical operator,* $\varrho A = 0$ *for a* GS*-automaton* $A \in \mathfrak{H} \cap \mathfrak{A}$ *if and only if* $A = \sum_{\text{subdirect}} (A_\alpha \mid A_\alpha \in \mathbf{M})$. *In particular if* A *satisfies also the descending chain condition on kernels, then* A *is a finite direct sum of* GS*-automata from the class* $\mathbf{M}$.

PROOF: The assertions are immediate consequences of THEOREM 4.6, because $\mathbf{R} = \mathcal{U}\mathbf{S}$. The last assertions can be proved by standard reasoning (cf. [2] Corollary 5). □

PROPOSITION 4.8. *The radical class* $\mathbf{R}$ *of* COROLLARY 4.7 *has the following hereditary property :*

$$\textit{if } (K,+,\delta) \triangleleft (A,+,\delta) \in \mathbf{R}, \textit{ then } (K,+,\delta) \in \mathbf{R}.$$

PROOF: Suppose that $(K,+,\delta) \notin \mathbf{R}$. Then there exists a kernel L of K such that $K/L \in \mathbf{S} \setminus \mathfrak{T}$. Since K is a GS-automaton, necessarily $K/L \in \overline{\mathbf{M}}$ holds. Thus also $K/J \in \mathbf{M}$ holds with an appropriate $L \subseteq J \triangleleft K$. Applying condition (F) on $\mathbf{M}$, it follows $J \triangleleft A$. Let M be a kernel of A being maximal relative to the property $K \wedge M = J$. As we have seen in the proof of THEOREM 4.6,

$$K/J = K/(K \wedge M) \cong (K \vee M)/M \triangleleft \circ A/M.$$

Since $K/J \in \mathbf{M}$ and $\mathbf{M}$ is closed under essential extensions, we get $A/M \in \mathbf{M} \subseteq \mathbf{S}$. Thus $A/M \subseteq \mathbf{S} \cap \mathbf{R} = \mathfrak{T}$, which yields $A = M$, and also $J = K \wedge M = K$, as well as $K/J \in \mathfrak{T}$, a contradiction. Thus $K = (K,+,\delta) \in \mathbf{R}$ has been proved. □

Recall that an object $A \in \mathfrak{H}$ is said to be *subdirectly irreducible*, if $H = \wedge(K \triangleleft A) \notin \mathfrak{T}$. The kernel H of A is referred to as the *heart* of A.

In the sequel we give a concrete class $\mathbf{M}$ of GS-automata which satisfies the conditions required in THEOREM 4.6, COROLLARY 4.7 and PROPOSITION 4.8.

THEOREM 4.9. *The class*

$$\mathbf{M} = \{A = (A,+,\delta) \in \mathfrak{H} \mid A \textit{ is subdirectly irreducible and } 0X = 0\}$$

is hereditdary, closed under essential extensions and satisfies condition (F).

Furthermore, for the radical class

$$\mathbf{R} = \{A \in \mathfrak{H} \cap \mathfrak{A} \mid A \longrightarrow\!\!\triangleright B \Rightarrow B \notin \mathbf{M}\} \cup \{(0,+)\}$$

the following two conditions are equivalent :

(i) $A \in \mathbf{R} \setminus \mathfrak{T}$

(ii) $A \in \mathfrak{H} \cap \mathfrak{A}$ *and if* $K \triangleleft A$ *and* $K \longrightarrow\!\!\triangleright L$, *then* L *is not a simple* GS*-automaton with subsemiautomaton* 0.

In analogy with ring theory we may call this radical $\mathbf{R}$ the *antisimple radical* of commutative additive GS-automata.

PROOF: Since $0X = 0$, every kernel K of any $A \in \mathbf{M}$ is a subsemiautomaton. Hence by PROPOSITION 4.5 every kernel L of K is also a kernel of A. Thus the heart of A is contained in every $L \triangleleft K$, and therefore K is subdirectly irreducible, proving that **M** is hereditary.

For proving that **M** is closed under essential extensions, let us consider a $K \in \mathbf{M}$ and $K \triangleleft \circ A$. We have to show that A is subdirectly irreducible. Let $I \triangleleft A$ be arbitrary. Since $K \triangleleft \circ A$, it follows $K \wedge I \notin \mathfrak{T}$, and so the heart H of K is contained in $K \wedge I$ and also in I. Since I was arbitrary, also $H \subseteq \wedge(I \triangleleft A)$ holds, proving that A is subdirectly irreducible.

In order to show the validity of condition (F), let us suppose that $L \triangleleft K \triangleleft A \in \mathfrak{H}$ and that $K/L \in \mathbf{M}$. Since $K/L \in \mathbf{M}$, we have $LX \subseteq L$. Thus L is a subsemiautomaton of A, and consequently K as well as A are GS-automata. Hence PROPOSITION 4.5 yields $L \triangleleft A$.

By COROLLARY 4.7 **R** is a radical class. Assume that $K \triangleleft A \in \mathbf{R}$ and $K \longrightarrow\!\triangleright L$. If K is merely a group, then so is L too, and condition (ii) is trivially fulfilled. So we may suppose that K is a subsemiautomaton. By PROPOSITION 4.8 it follows that $K \in \mathbf{R}$ which implies $L \notin \mathbf{M}$. Since simple GS-automata are subdirectly irreducible, by the definition of **M** we conclude that either L is not simple or 0 is not a subsemiautomaton of L or both, proving the validity of (ii).

Suppose that $A \notin \mathbf{R}$. Then either A is a group or $A/J \in \mathbf{M}$ with a suitable kernel J of A. In the second case, since the class **M** is hereditary, also the heart L of A/J is in **M** and in view of PROPOSITION 4.5 L has to be a simple GS-automaton. Since $L = K/J$ with an appropriate kernel K of A, we see that (ii) is not satisfied. □

As is well known [10] the subdirectly irreducible abelian groups are precisely the (quasi)-cyclic groups $C(p^n)$, $n = 1, 2, \ldots, \infty$ for all primes p. Obviously, on every subdirectly irreducible abelian group we may define an additive GS-automaton by assigning a homomorphism $x_0 : C(p^n) \to C(p^n)$, which will be a 0-input, and by defining $0X = 0$. There are, however, subdirectly irreducible additive GS-automata the additive group thereof is not subdirectly irreducible. Consider, for instance, the direct sum $C(p) \oplus C(p)$ of two copies of a simple cyclic group, the automorphism x_0 interchanging the components of $C(p) \oplus C(p)$. x_0 can be regarded as a 0-input of $C(p) \oplus C(p)$, further define $0X = 0$. Thus we have got a simple and hence subdirectly irreducible additive GS-automaton $(C(p) \oplus C(p), +, \delta)$, though $(C(p) \oplus C(p), +)$ is not a subdirectly irreducible group. Moreover, there are subdirectly irreducible additive GS-automata, which are not in **M**, for instance $(C(p), +, \delta)$ where the 0-input x_0 may be any homomorphism $x_0 : C(p) \to C(p)$, but $0X \neq 0$ for some $x \in X$. These observations demonstrate that COROLLARY 4.7 applied to the class **M** of THEOREM 4.9 provides a subdirect decomposition for some additive GS-automata only, and that the subdirectly irreducible components are not necessarily subdirectly irreducible groups.

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