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On Semi-Conditional Grammars with Productions Having either Forbidding or Permitting Conditions

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Abstract

This paper simplifies semi-conditional grammars so their productions have no more than one associated word–either a permitting condition or a forbidding condition. It is demonstrated that this simplification does not decrease the power of semi-conditional grammars.

1 Introduction

A semi-conditional grammar is a context-free grammar with productions having two associated words–a permitting condition and a forbidding condition. Such a production can rewrite a word, w, provided its permitting/forbidding condition is/is not a subword of w. Semi-conditional grammars without erasing productions characterize the family of context-sensitive languages; when erasing productions are allowed, these grammars define all family of recursively enumerable languages.

This paper studies a simplified concept of these grammars, whose productions have no more than one associated word–either a permitting condition or a forbidding condition. It is shown that this simplification does not decrease the generative power of semi-conditional grammars.

2 Definitions and Examples

We assume that the reader is familiar with formal language theory (see [3]).

Let V be an alphabet V^* denotes the free monoid generated by V under the operation of concatenation, where λ denotes the unit of V^*. Let $V^+ = V^* - \{\lambda\}$. Given a word, $w \in V^*$, $|w|$ represents the length of w, and $alph(w)$ denotes the set of symbols occurring in w. We set $sub(w) = \{y : y \text{ is a subword of } w\}$. Given a symbol, $a \in V$, $\#_a w$ denotes the number of occurrences of a in w.

A semi-conditional grammar (an *sc*-grammar for short) is a quadruple, $G = (V, P, S, T)$, where V, T, and S are the total alphabet, the terminal alphabet ($T \subset V$), and the axiom, respectively, and P is a finite set of productions of the form $(A \rightarrow \alpha, \beta, \mu)$ with $A \in V - T, \alpha \in V^*, \beta \in V^+ \cup \{0\}$, and $\mu \in V^+ \cup \{0\}$, where 0

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is a special symbol, $0 \notin V$ (intuitively, 0 means that the production's condition is missing). If $(A \to \alpha, \beta, \mu) \in P$ implies $\alpha \neq \Lambda$, G is said to be propagating. G has degree $(i, 0)$, where i is a natural number, if for every $(A \to \alpha, \beta, \mu) \in P, \beta \in V^+$ implies $|\beta| \leq i$, and $\mu = 0$. G has degree $(0, j)$, where j is a natural number, if for every $(A \to \alpha, \beta, \mu) \in P, \beta = 0$, and $\mu \in V^+$ implies $|\mu| \leq j$. G has degree (i, j), where i and j are two natural numbers, if for every $(A \to \alpha, \beta, \mu) \in P, \beta \in V^+$ implies $|\beta| \leq i$, and $\mu \in V^+$ implies $|\mu| \leq j$. Let $u, v \in V^*$, and $(A \to \alpha, \beta, \mu) \in P$. Then, u directly derives v according to $(A \to \alpha, \beta\mu)$, denoted by

$$u \Rightarrow v \; [(A \to \alpha, \beta, \mu)]$$

provided for some $u_1, u_2 \in V^*$, the following conditions (1) through (4) hold

(1) $u = u_1 A u_2$
(2) $v = u_1 \alpha u_2$
(3) $\beta \neq 0$ implies $\beta \in \mathrm{sub}(u)$
(4) $\mu \neq 0$ implies $\mu \notin \mathrm{sub}(u)$

When no confusion exists, we simply write $u \Rightarrow v$. As usual, we extend $\Rightarrow$ to $\Rightarrow^i$ (where $i \geq 0$), $\Rightarrow^+$, and $\Rightarrow^*$. The language of G, denoted by $L(G)$, is defined by $L(G) = \{w \in T^*; S \Rightarrow^* w\}$.

Now, we introduce the central notion of this paper–a simple semi-conditional grammar. Informally, a simple semi-conditional grammar is an *sc*-grammar in which any production has no more than one condition–either a permitting condition or a forbidding condition. Formally, let $G = (V, P, S, T)$ be an *sc*-grammar. G is a *simple semi-conditional grammar* (an ssc-grammar for short) if $(A \to x, \alpha, \beta) \in P$ implies $\{0\} \subseteq \{\alpha, \beta\}$.

To give an insight into *ssc* grammars, let us present two examples.

Example 1 Let

$$G = (\{S, A, X, C, Y, a, b\}, P, S, \{a, b\})$$

be an *ssc*-grammar, where

$$\begin{aligned} P \; = \; \{&(S \to AC, 0, 0),\\ &(A \to aXb, Y, 0),\\ &(C \to Y, A, 0),\\ &(Y \to Cc, 0, A),\\ &(A \to ab, Y, 0),\\ &(Y \to c, 0, A),\\ &(X \to A, C, 0)\} \end{aligned}$$

Notice that G is propagating, and it has degree (1, 1). Consider *aabbcc*. G derives this word as follows:

$$S \Rightarrow AC \Rightarrow AY \Rightarrow aXbY \Rightarrow aXbCc \Rightarrow aAbCc \Rightarrow aAbYc \Rightarrow aabbYc \Rightarrow aabbcc.$$

Obviously,

$$L(G) = \{a^n b^n c^n; n \geq 1\}.$$

Note that $\{a^n b^n c^n; n \geq 1\}$ is not a context-free language.

Example 2 Let
$$G = (\{S, A, B, X, Y, a\}, P, S, \{a\})$$
be an *ssc*-grammar, where P is defined as follows:

$$\begin{aligned} P \; = \; & \{(S \to a, 0, 0), \\ & (S \to X, 0, 0), \\ & (X \to YB, 0, A), \\ & (X \to aB, 0, A), \\ & (Y \to XA, 0, B), \\ & (Y \to aA, 0, B), \\ & (A \to BB, XA, 0)\} \\ & (B \to AA, YB, 0)\} \\ & (B \to a, a, 0)\}. \end{aligned}$$

G is a propagating *ssc*-grammar of degree (2,1). For $aaaaaaaa$, G makes the following derivation:

$$\begin{aligned} S \; \Rightarrow \; & X \Rightarrow YB \Rightarrow YAA \Rightarrow XAAA \Rightarrow XABBA \Rightarrow XABBBB \Rightarrow XBBBBBB \Rightarrow \\ & aBBBBBB \Rightarrow aBaBBBBB \Rightarrow aBaBBBBa \Rightarrow aaaBBBBa \Rightarrow \\ & aaaBBBaa \Rightarrow aaaaBaaa \Rightarrow aaaaaaaa. \end{aligned}$$

Clearly, G generates $\{a^{2^n}; n \geq 0\}$, that is,
$$L(G) = \{a^{2^n}; n \geq 0\}.$$

Note that $\{a^{2^n}; n \geq 0\}$ is not context-free.

The family of languages generated by *ssc*-grammars of degree (i, j) is denoted by $\mathbf{SSC}(i, j)$. Set
$$\mathbf{SSC} = \bigcup_{i=0}^{\infty} \bigcup_{j=0}^{\infty} \mathbf{SSC}(i, j).$$

To indicate that only propagating grammars are considered, we use the prefix **prop-**; for intance, **prop-SSC** (2, 1) denotes the family of languages generated by propagating *ssc*-grammars of degree (2, 1).

The families of context-free, context-sensitive, and recursively enumerable languages are denoted by **CF**, **CS**, and **RE**, respectively.

Let us finally recall that a context sensitive grammar in Penttonen normal form is a quadruple, $G = (V, P, S, T)$, where V, S, and T have the same meaning as for an *sc*-grammar, and any production in P is either of the form $AB \to AC$ or of the form $A \to \alpha$, where $A, B, C \in V - T, \alpha \in (T \cup (V - T)^2)$ (see [2]). In the standard manner, we define $\Rightarrow, \Rightarrow^i, \Rightarrow^+, \Rightarrow^*$, and $L(G)$. If we want to express that $x \Rightarrow y$ in G according to $p \in P$, we write $x \Rightarrow y\ [p]$.

3 Results

From the definition, the results achieved in [1], and the examples given in the previous section, we see that

$$\mathbf{CF} \subset \mathbf{prop{-}SSC} \subseteq \mathbf{prop{-}SC} = \mathbf{prop{-}SC}(2,1) = \mathbf{prop{-}SC}(1,2) = \mathbf{CS}$$

and

$$\textbf{prop–SSC} \subseteq \textbf{SSC} \subseteq \textbf{SC} = \textbf{SC}(2,1) = \textbf{SC}(1,2) = \textbf{RE}$$

This section states that

$$\begin{gathered}\textbf{CF}\\ \subset \\ \textbf{prop}-\textbf{SSC} = \textbf{prop}-\textbf{SSC}(2,1) = \textbf{prop}-\textbf{SSC}(1,2) = \\ \textbf{prop}-\textbf{SC} = \textbf{prop}-\textbf{SC}(2,1) = \textbf{prop}-\textbf{SC}(1,2) = \textbf{CS}\\ \subset \\ \textbf{SSC} = \textbf{SSC}(2,1) = \textbf{SSC}(1,2) = \textbf{SC} = \textbf{SC}(2,1) = \textbf{SC}(1,2) = \textbf{RE}\end{gathered}$$

In other words, we demonstrate that *ssc*-grammars are as powerful as *sc*-grammars. To establish this result, we first prove that propagating *ssc*-grammars of degree (2,1) generate precisely the family of context-sensitive languages.

Theorem 1 $\textbf{CS} = \textbf{prop} - \textbf{SSC}(2,1)$.

Proof. Clearly, $\textbf{prop} - \textbf{SSC}(2,1) \subseteq \textbf{CS}$, so it suffices to prove the converse inclusion.

Let $G = (V, P, S, T)$ be a context-sensitive grammar in Penttonen normal form. We construct an *ssc*-grammar, $G' = (V \cup W, P', S, T)$, that generates $L(G)$. Let

$$W = \{\tilde{B}; AB \rightarrow AC \in P, A, B, C \in V - T\}$$

We define P' in the following way:

1. if $A \rightarrow \alpha \in P, A \in V - T, \alpha \in T \cup (V-T)^2$,
   then add $(A \rightarrow \alpha, 0, 0)$ into P',
2. if $AB \rightarrow AC \in P, A, B, C \in V - T$,
   then add
   $$(B \rightarrow \tilde{B}, 0, \tilde{B}), (\tilde{B} \rightarrow C, A\tilde{B}, 0), \text{ and } (\tilde{B} \rightarrow B, 0, 0)$$
   to P'($\tilde{B}$ is the $\sim$ version of B in $AB \rightarrow AC$).

Notice that G is a propagating *ssc*-grammar of degree (2,1). Moreover, from (2), we have for any $\tilde{B} \in W$

$$S \Rightarrow^*_{G'} \alpha \text{ implies } \#_{\tilde{B}}\alpha \leq 1$$

because the only production that can generate $\tilde{B}$ is of the form $(B \rightarrow \tilde{B}, \emptyset, \tilde{B})$.

Let g be the finite substitution from V^* into $(W \cup V)^*$ defined as follows: for all $D \in V$,

1. if $\tilde{D} \in W$($\tilde{D}$ is the $\sim$ version of D), then $g(D) = \{D, \tilde{D}\}$;
2. if $\tilde{D} \notin W$, then $g(D) = \{D\}$.

Next, we will show that for any $w \in V^+$,

$$S \Rightarrow_G^m w \text{ if and only if } S \Rightarrow_{G'}^n v \text{ with } v \in g(w)$$

for some $m, n \geq 0$.

Only if: This is proved by induction on m.

Basis: Let $m = 0$. The only w is S as $S \Rightarrow_G^0 S$. Clearly, $S \Rightarrow_{G'}^n S$ for $n = 0$, and $S \in g(S)$.

Induction Hypothesis: Assume that the claim holds for all derivations of length m or less, for some $m \geq 0$.

Induction Step: Consider a derivation $S \Rightarrow_G^{m+1} \alpha, \alpha \in V^+$. Because $m + 1 \geq 1$, there is some $\beta \in V^*$ and $p \in P$ such that $S \Rightarrow_G^m \beta \Rightarrow_G \alpha$ $[p]$. By the induction hypothesis, $S \Rightarrow_{G'}^n \beta'$ for some $\beta' \in g(\beta)$ and $n \geq 0$. Next, we distinguish two cases, case (i) considers p with one nonterminal on its left-hand side, and case (ii) considers p with two nonterminals on its left-hand side.

(i) Let $p = D \rightarrow \beta_2 \in P, D \in V - T, \beta_2 \in T \cup (V - T)^2, \beta = \beta_1 D \beta_3, \beta_1, \beta_3 \in V^*$, $\alpha = \beta_1 \beta_2 \beta_3, \beta' = \beta_1' X \beta_3', \beta_1' \in g(\beta_1), \beta_3' \in g(\beta_3)$, and $X \in g(D)$. By (1), $(D \rightarrow \beta_2, 0, 0) \in P$. If $X = D$, then $S \Rightarrow_{G'}^n \beta_1' D \beta_3' \Rightarrow_{G'} \beta_1' \beta_2 \beta_3'$ $[(D \rightarrow \beta_2, 0, 0)]$. Because $\beta_1' \in g(\beta_1), \beta_3' \in g(\beta_3)$, and $\beta_2 \in g(\beta_2)$, we obtain $\beta_1' \beta_2 \beta_3' \in g(\beta_1 \beta_2 \beta_3) = g(\alpha)$. If $X = \tilde{D}$, we have $(X \rightarrow D, 0, 0) \in P'$, so $S \Rightarrow_{G'}^n \beta_1' X \beta_3' \Rightarrow_{G'} \beta_1' D \beta_3'$ $[(D \rightarrow \beta_2, 0, 0)]$, and $\beta_1' \beta_2 \beta_3' \in g(\alpha)$.

(ii) Let $p = AB \rightarrow AC \in P, A, B, C \in V - T, \beta = \beta_1 AB \beta_2, \beta_1, \beta_2 \in V^*$, $\alpha = \beta_1 AC \beta_2, \beta' = \beta_1' XY \beta_2', \beta_1' \in g(\beta_1), \beta_2' \in g(\beta - 2), X \in g(A)$, and $Y \in g(B)$. Recall that for any $\tilde{B}$, $\#_{\tilde{B}} \beta' \leq 1$ and $(\tilde{B} \rightarrow B, 0, 0) \in P'$. Then, $\beta' \Rightarrow_{G'}^i \bar{\beta}_1 AB \bar{\beta}_2$ for some $i \in \{0, 1\}$ so $\bar{\beta}_j \in g(\beta_j), j = 1, 2$, and $(g(A) \cup g(B)) \cap alph(\beta_1 AB \beta_2) = \{A, B\}$. At this point, we have:

$$\begin{aligned} S &\Rightarrow_{G'}^* \bar{\beta}_1 AB \bar{\beta}_2 \\ &\Rightarrow_{G'} \bar{\beta}_1 A\tilde{B} \bar{\beta}_2 \quad [(B \rightarrow \tilde{B}, 0, \tilde{B})] \\ &\Rightarrow_{G'} \bar{\beta}_1 AC \bar{\beta}_2 \quad [(\tilde{B} \rightarrow C, A\tilde{B}, 0)] \end{aligned}$$

where $\bar{\beta}_1 \in g(\beta_1), \bar{\beta}_2 \in g(\beta_2), C \in g(C)$, i.e., $\bar{\beta}_1 AC \bar{\beta}_2 \in g(\alpha)$.

If: This is established by induction on n; in other words, we demonstrate that

$$\text{if } S \Rightarrow_{G'}^n v \text{ with } v \in g(w) \text{ for some } w \in V^+, \text{ then } S \Rightarrow_G^* w.$$

Basis: For $n = 0, v$ surely equals S as $S \Rightarrow_{G'}^0 S$. Because $S \in g(S)$, we have $w = S$. Clearly, $S \Rightarrow_G^0 S$.

Induction Hypothesis: Assume the claim holds for all derivations of length n or less, for some $n \geq 0$.

Induction Step: Consider a derivation, $S \Rightarrow_G^{n+1} \alpha', \alpha' \in g(\alpha), \alpha \in V^+$. As $n + 1 \geq 1$, there exists some $\beta \in V^+$ such that $S \Rightarrow_{G'}^n \beta' \Rightarrow_{G'} \alpha'$ $[p], \beta' \in g(\beta)$. By induction hypothesis, $S \Rightarrow_G^* \beta$. Let $\beta' = \beta_1' B' \beta_2', \beta = \beta_1 B \beta_2, \beta_j' \in g(\beta_j), j = 1, 2, \beta_j \in V^*, B' \in g(B), B \in V - T, \alpha' = \beta_1' \mu' \beta_2'$, and $p = (B' \rightarrow \mu', \mu_1, \mu_2) \in P'$. The following three cases — (i), (ii), and (iii) — cover all possible forms of the derivation step $\beta' \Rightarrow_{G'} \alpha'$ $[p]$.

(i) $\mu' \in g(B)$. Then, $S \Rightarrow_G^* \beta_1 B\beta_2, \beta_1'\mu'\beta_2' \in g(\beta_1 B\beta_2)$, i.e., $\alpha' \in g(\beta_1 B\beta_2)$.

(ii) $B' = B \in V - T, \mu' \in T \cup (V-T)^2, \mu_1 = 0 = \mu_2$. Then, there exists a production, $B \rightarrow \mu' \in P$, so $S \Rightarrow_G^* \beta_1 B\beta_2 \Rightarrow_G \beta_1\mu'\beta_2[B \rightarrow \mu']$. Since $\mu' \in g(\mu')$, we have $\alpha = \beta_1\mu'\beta_2$ such that $\alpha' \in g(\alpha)$.

(iii) $B' = \tilde{B}, \mu' = C, \mu_1 = A\tilde{B}, \mu_2 = 0, A, B, C \in V - T$. Then, there exists a production of the form $AB \rightarrow AC \in P$. Since $\#_Z\beta' \leq 1, Z = \tilde{B}$, and $A\tilde{B} \in sub(\beta')$, we have $\beta_1' = \delta' A, \beta_1 = \delta A$ (for some $\delta \in V^*$), and $\delta' \in g(\delta)$. Thus, $S \Rightarrow_G \delta AB\beta_2 \Rightarrow_G \delta AC\beta_2[AB \rightarrow AC], \delta AC\beta_2 = \beta_1 C\beta_2$. Because $C \in g(C)$, we get $\alpha = \beta_1 C\beta_2$ such that $\alpha' \in g(\alpha)$.

By the principle of induction, we have thus established that for any $w \in V^+, S \Rightarrow_G^* w$ if and only f $S \Rightarrow_{G'}^* v$ with $v \in g(w)$. Because $g(x) = \{x\}$, for any $x \in T^*$, we have for every $w \in T^+$,

$$S \Rightarrow_G^* w \text{ if and only if } S \Rightarrow_{G'}^* w.$$

Thus, $L(G) = L(G')$, and the theorem holds. Q.E.D.

Corollary 2 CS = prop − SSC(2, 1) = prop − SSC = prop − SC(2, 1) = prop − SC.

We now turn to the investigation of *ssc*-grammars with erasing productions. We prove that these grammars generate precisely the family of recursively enumerable languages.

Theorem 3 RE = SSC *(2,1)*.

Proof. Clearly, we have the containment SSC(2, 1) ⊆ RE; hence, it suffices to show RE ⊆ SSC(2, 1). Every language $L \in$ RE can be generated by a recursively enumerable grammar, whose productions are of the form $AB \rightarrow AC$ or $A \rightarrow \alpha$ where $A, B, C \in V - T, \alpha \in T \cup (V-T)^2 \cup \{\lambda\}$ (see [2]). Thus, the containment RE ⊆ SSC(2, 1) can be proved by analogy with the proof of Theorem 1 (the details are left to the reader). Q.E.D.

Corollary 4 RE = SSC(2, 1)= SSC = SC(2, 1)= SC.

To demonstrate that propagating *ssc*-grammars of degree (1,2) characterize CS, we first establish a normal form for context-sensitive grammars (see Lemmas 5 and 6).

Lemma 5 Every $L \in CS$ can be generated by a context sensitive grammar, $G = (N_{CF} \cup N_{CS} \cup T, P, S, T)$, where N_{CF}, N_{CS}, and T are pairwise disjoint alphabets, and every production in P is either of the form $AB \rightarrow AC$ or $A \rightarrow x$, where $B \in N_{CS}, A, C \in N_{CF}, x \in N_{CS} \cup T \cup (\cup_{i=1}^2 N_{CF}^i)$.

Proof. Let $L \in$ CS. Without loss of generality, we can assume that L is generated by a context sensitive grammar $G' = (V, P', S, T)$ in Penttonen normal form, that is, every production in P' is either of the form $AB \rightarrow AC$ or $A \rightarrow BC$ or $A \rightarrow a$ (where $A, B, C \in V' - T$ and $a \in T$).

Let $G = (N_{CF} \cup N_{CS} \cup T, P, S, T)$ be the context sensitive grammar defined as follows:

$$\begin{aligned} N_{CF} &= V - T; \\ N_{CS} &= \{\tilde{B}; \tilde{B} \text{ is the tilde version of } B \text{ in } AB \rightarrow AC \in P'\}; \\ P &= \{A \rightarrow x;\ A \rightarrow x \in P',\ A \in V - T,\ x \in T \cup (V-T)^2\} \\ &\quad \cup \{B \rightarrow \tilde{B}, \tilde{B} \rightarrow AC; AB \rightarrow AC \in P', A, B, C \in V - T\}. \end{aligned}$$

Obviously, $L(G') = L(G)$, and G is of the required form. Hence, the lemma holds. Q.E.D.

Lemma 6 Every $L \in CS$ can be generated by a context sensitive grammar $G = (\{S\} \cup N_{CF} \cup N_{CS} \cup T, P, S, T)$, where $\{S\}, N_{CF}, N_{CS}, T$ are pairwise disjoint alphabets, and every production in P is either of the form $S \to aD$ or $AB \to AC$ or $A \to x$, where $a \in T, D \in N_{CF} \cup \{\lambda\}, B \in N_{CS}, A, C \in N_{CF}, x \in N_{CS} \cup T \cup (\cup_{i=1}^{2} N_{CF}^{i})$.

Proof. Let L be a context sensitive language over an alphabet, T. Without loss of generality, we can express L as $L = L_1 \cup L_2$, where $L_1 \subseteq T$ and $L_2 \subseteq TT^+$. Thus, by analogy with the proofs of Theorems 1 and 2 in [2], L_2 can be represented as $L_2 = \cup_{a \in T} aL_a$, where each L_a is a context sensitive language. Let L_a be generated by a context sensitive grammar, $G_a = (N_{CF_a} \cup N_{CS_a} \cup T, P_a, S_a, T)$, of the form of Lemma 5. Clearly, we can assume that for all $a's$, the nonterminal alphabets $(N_{CF_a} \cup N_{CS_a})$ are pairwise disjoint. Let S be a new start symbol. Consider the context sensitive grammar

$$G = (\{S\} \cup N_{CF} \cup N_{CS} \cup T, P, S, T)$$

defined as:

$$N_{CF} = \cup_{a \in T} N_{CF_a};$$
$$N_{CS} = \cup_{a \in T} N_{CS_a};$$
$$P = \cup_{a \in T} P_a \cup \{S \to aS_a; a \in T\} \cup \{S \to a; a \in L_1\}.$$

Obviously, G satisfies the required form, and we have

$$L(G) = L_1 \cup (\cup_{a \in T}\ aL(G_a)) = L_1 \cup (\cup_{a \in T}\ aL_a) = L_1 \cup L_2 = L.$$

Consequently, the lemma holds. Q.E.D.

We are now ready to characterize **CS** by propagating *ssc*-grammars of degree (1,2).

Theorem 7 $\mathbf{CS} = \mathbf{prop - SSC}(1,2)$.

Proof. Clearly, $\mathbf{prop - SSC}(1,2) \subseteq \mathbf{CS}$; hence, it suffices to prove the converse inclusion.

Let L be a context sensitive language. Without loss of generality, we can assume that L is generated by a context sensitive grammar, $G = (\{S\} \cup N_{CF} \cup N_{CS} \cup T, P, S, T)$, of the form of Lemma 6. Set $V = (\{S\} \cup N_{CF} \cup N_{CS} \cup T)$. Let q be the cardinality of V; $q \geq 1$. Furthermore, let f be an (arbitrary, but fixed) bijection from V onto $\{1, \ldots, q\}$, and let f^{-1} be the inverse of f.

Let $G^{\sim} = (V^{\sim}, P^{\sim}, S, T)$ be a propagating *ssc*-grammar of degree (1,2), in which

$$V^{\sim} = (\cup_{i=1}^{4} W_i) \cup V$$

where

$$W_1 = \{< a, AB \to AC, j >; a \in T, AB \to AC \in P, A, C \in N_{CF}, B \in N_{CS}, 1 \leq j \leq 5\};$$
$$W_2 = \{[a, AB \to AC, j]; a \in T, AB \to AC \in P, A, C, \in N_{CF}, B \in N_{CS}, 1 \leq j \leq q+3\};$$
$$W_3 = \{\hat{B}, B', B''; B \in N_{CS}\};$$
$$W_4 = \{\bar{a}; a \in T\}$$

and $P^{\sim}$ is defined as follows:

1. if $S \to aA \in P, a \in T, A \in (N_{CF} \cup \{\lambda\})$,
   then add $(S \to \bar{a}A, 0, 0)$ to $P^{\sim}$;

2. if $a \in T, A \to x \in P, A \in N_{CF}, x \in (V = \{S\}) \cup (N_{CF})^2$,
   then add $(A \to x, \bar{a}, 0)$ to $P^{\sim}$;

3. if $a \in T, AB \to AC \in P, A, C, \in N_{CF}, B \in N_{CS}$,
   then add to $P^{\sim}$ the following set of productions
   (an informal explanation of these productions can be found below):

$$
\begin{aligned}
&\{(\bar{a} \to < a, AB \to AC, 1 >, 0, 0),\\
&(B \to B', < a, AB \to AC, 1 >, 0),\\
&(B \to \hat{B}, < a, AB \to AC, 1 >, 0),\\
&(< a, AB \to AC, 1 > \to < a, A \to AC, 2 >, 0, B),\\
&(\hat{B} \to B'', 0, B''),\\
&(< a, AB \to AC, 2 > \to < a, AB \to AC, 3 >, 0, \hat{B}),\\
&(B'' \to [a, AB \to AC, 1], < a, AB \to AC, 3 >, 0)\}\\
\cup\ &\{([a, AB \to AC, j] \to [a, AB \to AC, j+1], 0,\\
&f^{-1}(j)[a, AB \to AC, j]); 1 \le j \le q, f(A) \neq j\}\\
\cup\ &\{([a, AB \to AC, f(A)] \to [a, AB \to AC, f(A)+1], 0, 0),\\
&([a, AB \to AC, q+1] \to [a, AB \to AC, q+2], 0,\\
&B'[a, AB \to AC, q+1]),\\
&([a, AB \to AC, q+2] \to [a, AB \to AC, q+3], 0,\\
&< a, AB \to AC, 3 > [a, AB \to AC, q+2]),\\
&(< a, AB \to AC, 3 > \to < a, AB \to AC, 4 >,\\
&[a, AB \to AC, q+3], 0),\\
&(B' \to B, < a, AB \to AC, 4 >, 0),\\
&(< a, AB \to AC, 4 > \to < a, AB \to AC, 5 >, 0, B'),\\
&([a, AB \to AC, q+3] \to C, < a, AB \to AC, 5 >, 0),\\
&(< a, AB \to AC, 5 > \to \bar{a}, 0, [a, AB \to AC, q+3])\}
\end{aligned}
$$

   ($B', \hat{B}$, and B'' correspond to B in $AB \to AC$);

(4) if $a \in T$, then add $(\bar{a} \to a, 0, 0)$ to $P^{\sim}$.

Let us informally explain the basic idea behind point (3)–the heart of all construction. The production introduced in this point simulate the application of productions of the form $AB \to AC$ in G as follows: an occurrence of B is chosen, and its left neighbor is checked *not* to belong to $V^{\sim} - \{A\}$; at this point, the left neighbor necessarily equals A, so B is rewritten with C.

Formally, we define a finite letter-to-letters substitution g from V^* into $(V^{\sim})^*$ as follows:
if $D \in V$, then add D to $g(D)$;
if $< a, AB \to AC, j > \in W_1(a \in T, AB \to AC \in P, B \in N_{CS}, A, C \in N_{CF}$, $j \in \{1, \ldots 5\})$, then add $< a, AB \to AC, j >$ to $g(a)$;
if $[a, AB \to AC, j] \in W_2(a \in T, AB \to AC \in P, B \in N_{CS}, A, C \in N_{CF}$, $j \in \{1, \ldots, q+3\})$, then add $[a, AB \to AC, j]$ to $g(B)$;
if $\{\hat{B}, B', B''\} \subseteq W_3(B \in N_{CS})$, then include $\{\hat{B}, B', B''\}$ to $g(B)$;
if $\bar{a} \in W_4(a \in T)$, then add $\bar{a}$ to $g(a)$.

Let g^{-1} be the inverse of g.
To show that $L(G) = L(G^{\sim})$, we first prove three claims.

Claim 1: $S \Rightarrow^{+} x$ in $G, x \in V^*$, implies $x \in T(V - \{S\})^*$.

Proof of Claim 1.
Observe that the start symbol, S, does not appear on the right side of any production and that $S \rightarrow x \in P$ implies $x \in T \cup T(V - \{S\})$. Hence, the claim holds.

Claim 2: If $S \Rightarrow^{+} x$ in $G^{\sim}, x \in (V^{\sim})^*$, then x has one of the following seven forms:

(i) $x = ay$, where $a \in T, y \in (V - \{S\})^*$;

(ii) $x = \bar{a}y$, where $\bar{a} \in W_4, y \in (V - \{S\})^*$;

(iii) $x = < a, AB \rightarrow AC, 1 > y$, where $< a, AB - AC, 1 > \in W_1$, $y \in ((V - \{S\}) \cup \{B', \hat{B}, B''\})^*, \#_{B''} y \leq 1$;

(iv) $x = < a, AB \rightarrow AC, 2 > y$, where $<, AB \rightarrow AC, 2 > \in W_1$, $y \in ((V - \{S, B\}) \cup \{B', \hat{B}, B'\})^*, \#_{B'} \leq 1$;

(v) $x = < a, AB \rightarrow AC, 3 > y$, where $< a, AB \rightarrow AC, 3 > \in W_1$, $y \in ((V - \{S, B\}) \cup \{B'\})^* \ (\{[a, AB \rightarrow AC, j]; 1 \leq j \leq q + 3\} \cup \{\lambda, B''\})((V - \{S, B\}) \cup \{B'\})^*$;

(vi) $x = < a, AB \rightarrow AC, 4 > z$, where $< a, AB \rightarrow AC, 4 > \in W_1$, $y \in ((V - \{S\}) \cup \{B'\})^*[a, AB \rightarrow AC, q + 3]((V - \{S\}) \cup \{B'\})^*$;

(vii) $x = < a, AB \rightarrow AC, 5 > y$ where $< a, AB \rightarrow AC, 5 > \in W_1$, $y \in (V - \{S\})^*\{[a, AB \rightarrow AC, q_3], \lambda\}(V - \{S\})^*$.

Proof of Claim 2.
The claim is proved by induction on the length of derivations.

Basis: Consider $S \Rightarrow x$. By inspection of the productions, we have $S \Rightarrow \bar{a}A$ $[(S \rightarrow \bar{a}A, 0, 0)]$ for some $\bar{a} \in W_4, A \in (\{\lambda\} \cup N_{CF})$. Therefore, $x = \bar{a}$ or $x = \bar{a}A$ (where $\bar{a} \in W_4$ and $A \in (\{\lambda\} \cup N_{CF})$); in either case, x is a word of the required form.

Induction hypothesis: Assume the claim holds for all derivations of length at most n, for some $n \geq 1$.

Induction step: Consider a derivation of the form $S \Rightarrow^{n+1} x$. Since $n \geq 1$, we have $n + 1 \geq 2$. Thus, there is some z of the required form $(z \in (V^{\sim})^*)$ such that $S \Rightarrow^n z \Rightarrow x\ [p]$ for some $p \in P^{\sim}$.

Let us first prove by contradiction that the first symbol of z does not belong to T. Assume that the first symbol of z belongs to T. As z is of the required form, we have $z = ay$ for some $a \in (V - \{S\})^*$. By inspection of $P^{\sim}$, there is no $p \in P^{\sim}$ such that $ay \Rightarrow x[p]$, where $x \in (V^{\sim})^*$. We have thus obtained a contradiction, so the first symbol of z is not in T.

Because the first symbol of z does not belong to T, z cannot have form (i); as a result, z has one of forms (ii) through (vii). The following cases I through VI demonstrate that if z has one of these six forms, then x (in $S \Rightarrow^n z \Rightarrow x[p]$) has one of the required forms, too.

I. Assume that z is of form (ii), i.e., $z = \bar{a}y, \bar{a} \in W_4$, and $y \in (V - \{S\})^*$. By inspection of the productions in $P^\sim$, we see that p has one of the following forms (a), (b), and (c):

(a) $p = (A \to u, \bar{a}, 0)$ where $A \in N_{CF}$ and $u \in (V - \{S\}) \cup (N_{CF})^2$;
(b) $p = (\bar{a} \to <a, AB \to AC, 1> 0, 0)$ where $<a, AB \to AC, 1> \in W_1$;
(c) $p = (\bar{a} \to a, 0, 0)$ where $a \in T$.
(Note that productions of forms (a), (b), and (c) are introuced in construction steps (2), (3), and (4), respectively.) If p has form (a), then x has form (ii). If p has form (b), then x has form (iii). Finally, if p has form (c), then x has form (i). In any of these three cases, we obtain x that has one of the required forms.

II. Assume that z has form (iii), i.e., $z = <a, AB \to AC, 1> y$ for some $<a, AB \to AC, 1> \in W_1, y \in ((V - \{S\}) \cup \{B'', \hat{B}, B''\})^*$, and $\#_{B''}y \leq 1$. By the inspection of $P^\sim$, we see that z can be rewritten by productions of these four forms:

(a) $(B \to B', <a, AB \to AC, 1>, 0)$;
(b) $(B \to \hat{B}, <a, AB \to AC, 1>, 0)$;
(c) $(\hat{B} \to B'', 0, B)$ (if $B'' \notin alph(y), i.e., \#_{B''}y = 0$);
(d) $(<a, AB \to AC, 1> \to <a, AB \to AC, 2>, 0, B)$ (if $B'' \notin alph(y), i.e., \#_B y = 0$).

Clearly, in cases (a) and (b), we obtain x of form (iii). If $z \Rightarrow x$ [p] in $G^\sim$, where p is of form (c), then $\#_{B''}x = 1$, so we get x of form (iii). Finally, if we use the production of form (d), then we obtain x of form (iv) because $\#_B z = 0$.

III. Assume that z is of form (iv), i.e., $z = <a, AB \to AC, 2> y$, where $<a, AB \to AC, 2> \in W_1, y \in ((V - \{S, B\}) \cup \{B', \bar{B}, B''\})^*$, and $\#_{B''}y \leq 1$. By inspection of $P^\sim$, we see that the follwoing two productions can be used to rewrite z:

(a) $(\bar{B} \to B'', 0, B'')$ $(if B'' \notin alph(y))$;
(b) $(<a, AB \to AC, 2> \to <a, AB \to AC, 3>, 0, \bar{B})$ $(if \bar{B} \notin alph(y))$.

In case (a), we get x of form (iv). In case (b), we have $\#_B y = 0$, so $\#_B x = 0$. Moreover, notice that $\#_{B''}x \leq 1$ in this case. Indeed, the symbol B'' can be generated only if there exists no occurrence of B'' in a given rewritten word, so no more that one occurrence of B'' appears in any sentential form. As a result, we have $\#_{B''} <a, AB \to AC, 3> y \leq 1$, i.e., $\#_{B''}x \leq 1$. In other words, we get x of form (v).

IV. Assume that z is of form (v), i.e., $z = <a, AB \to AC, 3> y$ for some $<a, AB \to AC, 3> \in W_1, y \in ((V - \{S, B\}) \cup \{B'\})^*(\{[a, AB \to AC, j]; 1 \leq j \leq q + 3\} \cup \{B'', \lambda\}) ((V - \{S, B\}) \cup \{B'\})^*$. Assume that $y = y_1 Y y_2$ with $y_1, y_2 \in ((V - \{S, B\}) \cup \{B'\})^*$. If $Y = \lambda$, then we can use no production from $P^\sim$ to rewrite z. Because $z \Rightarrow x$, we have $Y \neq \lambda$. The following cases (A) through (F) cover all possible forms of Y.

(A) Assume $Y = B''$. By inspection of $P^\sim$, we see that the only production that can rewrite z has the form $(B'' \to [a, AB \to AC, 1], <a, AB \to AC, 3>, 0)$. In this case, we get x of form (v).

(B) Assume $Y = [a, AB \to AC, j]w, j \in \{1, \ldots, q\}$, and $f(A) \neq j$. Then z can be rewritten only according to the production $([a, AB \to AC, j] \to [a, AB \to AC, j + 1], 0, f^{-1}(j)[a, AB \to AC, j])$ (which can be used unless the rightmost symbol of $<a, AB \to AC, 3> y_1$ is $f^{-1}(j)$). Clearly, in this case we again get x of form (v).

(C) Assume $Y = [a, AB \rightarrow AC, j], j \in \{1, \ldots, q\}, f(A) = j$. This case forms an analogy to case (B), except that the production of the form $([a, AB \rightarrow AC, f(A)] \rightarrow [a, AB \rightarrow AC, f(A) + 1], 0, 0)$ is now used.

(D) Assume $Y = [a, AB \rightarrow AC, q + 1]$. This case forms an analogy to case (B); the only change is the application of the production $([a, AB \rightarrow AC, q + 1] \rightarrow [a, AB \rightarrow AC, q + 2], 0, B'[a, AB \rightarrow AC, q + 1])$.

(E) Assume $Y = [a, AB \rightarrow AC, q + 2]$. This case forms an analogy to case (B) except that the production $([a, AB \rightarrow AC, q + 2] \rightarrow [a, AB \rightarrow AC, q + 3], 0, < a, AB \rightarrow AC, 3 > [a, AB \rightarrow AC, q + 2])$ is used.

(F) Assume $X = [a, AB \rightarrow AC, q + 3]$. By inspection of $P^{\sim}$, we see that the only production that can rewrite z is $(< a, AB \rightarrow AC, 3 > \rightarrow < a, AB \rightarrow AC, 4 >, [a, AB \rightarrow AC, q + 3], 0)$. If this production is used, we get x of form (vi).

V. Assume that z is of form (vi), i.e., $z = < a, AB \rightarrow AC, 4 > y$, where $< a, AB \rightarrow AC, 4 > \in W_1$ and $y \in ((V - \{S\}) \cup \{B'\})^* \ [a, AB \rightarrow AC, q + 3]((V - \{S\}) \cup \{B'\})^*$. By inspection of $P^{\sim}$, these two productions can rewrite z:

(a) $(B' \rightarrow B, < a, AB \rightarrow AC, 4 >, 0)$;

(b) $(< a, AB \rightarrow AC, 4 > \rightarrow < a, AB \rightarrow AC, 5 >, 0, B')$ (if $B' \notin \text{alph}(y)$).

Clearly, in case (a), we get x of form (vi). In case (b), we get x of form (vii) because $\#_{B'} y = 0$, so $y \in (V - \{S\})^* \{[a, AB \rightarrow AC, q + 3], \lambda\}(V - \{S\})^*$.

VI. Assume that z is of form (vii), i.e., $z = < a, AB \rightarrow AC, 5 > y$, where $< a, AB \rightarrow AC, 5 > \in W_1$ and $y \in (V - \{S\})^* \{[a, AB \rightarrow AC, q + 3], \lambda\}(V - \{S\})^*$. By inspection of $P^{\sim}$, one of the following two productions can be used to rewrite z:

(a) $([a, AB \rightarrow AC, q + 3] \rightarrow C, < a, AB \rightarrow AC, 5 >, 0)$;

(b) $(< a, AB \rightarrow AC, 5 > \rightarrow \bar{a}, 0, [a, AB \rightarrow AC, q + 3])$ (if $[a, AB \rightarrow AC, q + 3] \notin alph(z)$).

In case (a), we get x of form (vii). Case (b) implies $\#_{[a, AB \rightarrow AC, q+3]} y = 0$; thus, x is of form (ii).

This completes the induction step and establishes Claim 2.

Claim 3: It holds that

$$S \Rightarrow^m w \text{ in } G \text{ if and only if } S \Rightarrow^n v \text{ in } G^{\sim}$$

where $v \in g(w)$ and $w \in V^+$, for some $m, n \geq 0$.

Proof of Claim 3.

Only if: The only-if part is established by induction on m; that is, we have to demonstrate that $S \Rightarrow^m w$ in G implies $S \Rightarrow^* v$ in $G^{\sim}$ for some $v \in g(w)$ and $w \in V^+$.

Basis: Let $m = 0$. The only w is S because $S \Rightarrow^0 S$ in G. Clearly, $S \Rightarrow^0 S$ in $G^{\sim}$, and $S \in g(S)$.

Induction Hypothesis: Suppose that our claim holds for all derivations of length m or less, for some $m \geq 0$.

Induction Step: Let us consider a derivation, $S \Rightarrow^{m+1} x$, in $G, x \in V^+$. Because $m + 1 \geq 1$, there are $y \in V^+$ and $p \in P$ such that $S \Rightarrow^m y \Rightarrow x \ [p]$ in G, and by the induction hypothesis, there is also a derivation $S \Rightarrow^n y^\sim$ in $G^\sim$ for some $y^\sim \in g(y)$. The following cases (i) through (iii) cover all possible forms of p.

(i) Let $p = S \rightarrow aA \in P$ for some $a \in T, A \in N_{CF} \cup \{\lambda\}$. Then, by Claim 1, $m = 0$, so $y = S$ and $x = aA$. By (1) in the construction of $G^\sim, (S \rightarrow \bar{a}A, 0, 0) \in P^\sim$. Hence, $S \Rightarrow a^\sim A$ in $G^\sim$ where $a^\sim A \in g(aA)$.

(ii) Let us assume that $p = D \rightarrow y_2 \in P, D \in N_{CF}, y_2 \in (V - \{S\}) \cup (N_{CF})^2, y = y_1 D y_3, y_1, y_3 \in V^*$ and $x = y_1 y_2 y_3$. From the definition of g, it is clear that $g(Z) = \{Z\}$ for all $Z \in N_{CF}$; therefore, we can express $y^\sim = z_1 D z_3$ where $z_1 \in g(y_1)$ and $z_3 \in g(y_3)$. Without loss of generality, we can also assume that $y_1 = ar, a \in T, r \in (V - \{S\})^*$ (see Claim 1), so $z_1 = a''r'', a'' \in g(a)$, and $r'' \in g(r)$. Moreover, by (2) in the construction, we have $(D \rightarrow y_2, \bar{a}, 0) \in P^\sim$. The following cases (a) through (e) cover all possible forms of a''.

(a) Let $a'' = \bar{a}$ (see (ii) in Claim 2). Then, we have $S \Rightarrow^n \bar{a}r''Dz_3 \Rightarrow \bar{a}r''y_2z_3$ $[(D \rightarrow y_2, \bar{a}, 0)]$, and $\bar{a}r''y_2z_3 = z_1y_2z_3 \in g(y_1y_2y_3) = g(x)$.

(b) Let $a'' = a$ (see (i) in Claim 2). By (4) in the construction of $G^\sim$, we can express the derivation in $G^\sim : S \Rightarrow^n ar''Dz_3$ as $S \Rightarrow^{n-1} \bar{a}r''Dz_3 \Rightarrow ar''Dz_3$ $[(\bar{a} \rightarrow a, 0, 0)]$; thus, there exists this derivation in $G^\sim : S \Rightarrow^{n-1} \bar{a}r''Dz_3 \Rightarrow \bar{a}r''y_2z_3[(D \rightarrow y_2, \bar{a}, 0)]$ with $\bar{a}r''y_2z_3 \in g(x)$.

(c) Let $a'' = < a, AB \rightarrow AC, 5 >$ for some $AB \rightarrow AC \in P$ (see (vii) in Claim 2), and let $r''Dz_3 \in (V - \{S\})^*$, i.e., $[a, AB \rightarrow AC, q+3] \notin$ alph $(r''Dz_3)$. Then, there exists this derivation in $G^\sim : S \Rightarrow^n < a, AB \rightarrow AC, 5 > r''Dz_3 \Rightarrow \bar{a}r''Dz_3$ $[(< a, AB \rightarrow AC, 5 > \rightarrow \bar{a}, 0, [a, AB \rightarrow AC, q+3])] \Rightarrow \bar{a}r''y_2z_3[(D \rightarrow y_2, \bar{a}, 0)]$, and $\bar{a}r''y_2z_3 \in g(x)$.

(d) Let $a'' = < a, AB \rightarrow AC, 5 >$ (see (vii) in Claim 2). Let $[a, AB \rightarrow AC, q+3] \in$ alph $(r''Dz_3)$. Without loss of generality, we can assume that $y^\sim = < a, AB \rightarrow AC, 5 > r''Ds''[a, AB \rightarrow AC, q+3]t''$, where $s''[a, AB \rightarrow AC, q+3]t'' = z_3, sBt = y_3, s'' \in g(t), s, t \in (V - \{S\})^*$. By inspection of $P^\sim$ (see (3) in the construction of $G^\sim$), we can express the derivation in $G^\sim : S \Rightarrow^n y^\sim$ as:

$$
\begin{array}{lll}
S & \Rightarrow^* & \bar{a}r''Ds''Bt'' \\
 & \Rightarrow & < a, AB \rightarrow AC, 1 > r''Ds''Bt'' \\
 & & [(\bar{a} \rightarrow < a, AB \rightarrow AC1 >, 0, 0)] \\
 & \Rightarrow^{1+|m_1m_2|} & < a, AB \rightarrow AC, 1 >' Ds'\hat{B}t' \\
 & & [m_1(B \rightarrow \hat{B}, < a, AB \rightarrow AC, 1 >, 0)m_2] \\
 & \Rightarrow & < a, AB \rightarrow AC, 2 > r'Ds'\bar{B}t' \\
 & & [(< a, AB \rightarrow AC, 1 > \rightarrow < a, AB \rightarrow AC, 2 >, 0, B)] \\
 & \Rightarrow & < a, AB \rightarrow AC, 2 > r'Ds'B''t' \\
 & & [\hat{B} \rightarrow B'', 0, B'')] \\
 & \Rightarrow & < a, AB \rightarrow AC, 3 > r'Ds'B''t' \\
 & & [(< a, AB \rightarrow AC, 2 > \rightarrow < a, AB \rightarrow AC, 3 >, 0, \hat{B})] \\
 & \Rightarrow & < a, AB \rightarrow AC, 3 > r'Ds'[a, AB \rightarrow AC, 1]t' \\
 & & [(B'' \rightarrow [a, AB \rightarrow AC, 1], < a, AB \rightarrow AC, 3 >, 0)] \\
 & \Rightarrow^{q+2} & < a, AB \rightarrow AC, 3 > r'Ds'[a, AB \rightarrow AC, q+3]t'
\end{array}
$$

$$
\begin{array}{ll}
 & [([a, AB \to AC, 1] \to [a, AB \to AC, 2], 0, f^{-1}(1) \\
 & [a, AB \to AC, 1]) \ldots \\
 & ([a, AB \to AC, f(A) - 1] \to [a, AB \to AC, f(A)], 0, \\
 & f^{-1}(f(A) - 1)[a, AB \to AC, f(A) - 1]) \\
 & ([a, AB \to ACf(A) \to [a, AB \to AC, f(A) + 1], 0, 0) \\
 & ([a, AB \to AC, f(A) + 1] \to [a, AB \to AC, f(A) + 2], 0, \\
 & f^{-1}(f(A) + 1)[a, AB \to AC, f(A) + 1]) \ldots \\
 & ([a, AB \to AC, q] \to [a, AB \to AC, q + 1], 0, \\
 & f^{-1}(q)[a, AB \to AC, q]) \\
 & ([a, AB \to AC, q + 1] \to [a, AB \to AC, q + 2], 0, B' \\
 & [a, AB \to AC, q + 1]) \\
 & ([a, AB \to AC, q + 2] \to [a, AB \to AC, q + 3], 0, \\
 & < a, AB \to AC, 3 > [a, AB \to AC, q + 2])] \\
\Rightarrow & < a, AB \to AC, 4 > r'Ds'[a, AB \to AC, q + 3]t' \\
 & [(< a, AB \to AC, 3 > \to < a, AB \to AC, 4 >, \\
 & [a, AB \to AC, q + 3], 0)] \\
\Rightarrow^{|m_3|} & < a, AB \to AC, 4 > r''Ds''[a, AB \to q + 3]t'' \ [m_3] \\
\Rightarrow & < a, AB \to AC, 5 > r''Ds''[a, AB \to AC, q + 3]t'' \\
 & [(< a, AB \to AC, 4 > \to < a, AB \to AC, 5 >, 0, B')]
\end{array}
$$

where $m_1, m_2 \in \{(B \to B', < a, AB \to AC, 1 >, 0)\}^*, m_3 \in \{(B' \to B, < a, AB \to AC, 4 >, 0)\}^*$, $|m_3| = |m_1 m_2|, r' \in ((\text{alph}(r'') - \{B\}) \cup \{B'\})^*, g^{-1}(r) - r, s' \in ((\text{alph}(s'') - \{B\}) \cup \{B''\})^*, g^{-1}(s') = g^{-1}(s'') = s, t' \in ((\text{alph}(t'') - \{B\}) \cup \{B'\})^*, g^{-1}(t') = g^{-1}(t'') = t$.

Clearly, $\bar{a}r''Ds''Bt'' \in g(arDsBt) = g(arDy_3) = g(y)$. Thus, there exists this derivation in $G^{\sim}$: $S \Rightarrow^* \bar{a}r''Ds''Bt'' \Rightarrow \bar{a}r''y_2s''Bt''$ $[(D \to y_2, \bar{a}, 0)]$ where $z_1 y_2 z_3 = \bar{a}r''y_2s''Bt'' \in g(ary_2sBt) = g(y_1y_2y_3) = g(x)$.

(e) Let $a'' =< a, AB \to AC, i >$ for some $AB \to AC \in P$ and $i \in \{1, \ldots, 4\}$ (see (iii) - (vi) in Claim 2). By analogy with (d), we can construct the derivation $S \Rightarrow^* \bar{a}r''Ds''Bt'' \Rightarrow \bar{a}r''y_2s''Bt''$ $[(D \to y_2, \bar{a}, 0)]$ such that $\bar{a}r''y_2s''Bt'' \in g(y_1y_2y_3) = g(x)$ (the details of this construction are left to the reader).

(iii) Let $p = AB \to AC \in P, A, C \in N_{CF}, B \in N_{CS}, y = y_1ABy_3, y_1, y_3 \in V^*, x = y_1ACy_3, y^{\sim} = z_1AYz_3, Y \in g(B), z_i \in g(y_i)$ where $i \in \{1, 3\}$. Moreover, let $y_1 = ar$ (see Claim 1), $z_1 = a''r'', a'' \in g(a), r'' \in g(r)$. The following cases (a) through (e) cover all possible forms of a''.

(a) Let $a'' = \bar{a}$. Then, by Claim 2, $Y = B$. By (3) in the construction of $G^{\sim}$, there exists the following derivation in $G^{\sim}$:

$$
\begin{array}{lll}
S & \Rightarrow^n & \bar{a}r''ABz_3 \\
 & \Rightarrow & < a, AB \to AC, 1 > r''ABu_3 \\
 & & [(\bar{a} \to < a, AB \to AC, 1 > 0, 0)] \\
 & \Rightarrow^{1+|m_1|} & < a, AB \to AC, 1 > r'A\hat{B}z_3 \\
 & & [m_1(B \to \hat{B}, < a, AB \to AC, 1 >, 0)] \\
 & \Rightarrow & < a, AB \to AC, 2 > r'A\hat{B}u_3
\end{array}
$$

$$
\begin{array}{ll}
 & [(<a, AB \to AC, 1> \to <a, AB \to AC, 2>, 0, B)] \\
\Rightarrow & <a, AB \to AC, 2> r'AB''u_3 \\
 & [(\hat{B} \to B'', 0, B'')] \\
\Rightarrow & <a, AB \to AC, 3> r'AB''u_3 \\
 & [(<a, AB \to AC, 2> \to <a, AB \to AC, 3>, 0, \hat{B})] \\
\Rightarrow & <a, AB \to AC, 3> r'A[a, AB \to AC, 1]u_3 \\
 & [(B'' \to [a, AB \to AC, 1], <a, AB \to AC, 3>, 0)] \\
\Rightarrow^{q+2} & <a, AB \to AC, 3> r'A[a, AB \to AC, q+3]u_3 \\
 & [([a, AB \to AC, 1] \to [a, AB \to AC, 2], 0, \\
 & f^{-1}(1)[a, AB \to AC, 1]) \ldots \\
 & ([a, AB \to AC, f(A)-1] \to [a, AB \to AC, f(A)], 0, \\
 & f^{-1}(f(A)-1)[a, AB \to AC, f(A)-1]) \\
 & ([a, AB \to AC, f(A)] \to [a, AB \to AC, f(A)+1], 0, 0) \\
 & ([a, AB \to AC, f(A)+1]) \to [a, AB \to AC, f(A)+2], 0, \\
 & f^{-1}(f(A)+1)[a, AB \to AC, f(A)+1]) \ldots \\
 & ([a, AB \to AC, q] \to [a, AB \to AC, q+1], 0, \\
 & f^{-1}(q)[a, AB \to AC, q]) \\
 & ([a, AB \to AC, q+1] \to [a, AB \to AC, q+2], 0, B' \\
 & [a, AB \to AC, q+1]) \\
 & ([a, AB \to AC, q+2] \to [a, AB \to AC, q+3], 0, \\
 & <a, AB \to AC, 3> [a, AB \to AC, q+2])] \\
\Rightarrow & <a, AB \to AC, 4> r'A[a, AB \to AC, q+3]u_3 \\
 & [(<a, AB \to AC, 3> \to <a, AB \to AC, 4>, \\
 & [a, AB \to AC, q+3], 0)] \\
\Rightarrow & <a, AB \to AC, 4> r''A[a, AB \to AC, q+3]z_3 \quad [m_2] \\
\Rightarrow & <a, AB \to AC, 5> r''A[a, AB \to AC, q+3]z_3 \\
 & [(<a, AB \to AC, 4> \to <a, AB \to AC, 5>, 0, B')] \\
\Rightarrow & <a, AB \to AC, 5> r''ACz_3 \\
 & [([a, AB \to AC, q+3] \to C, <a, AB \to AC, 5>, 0)]
\end{array}
$$

where $m_1 \in \{(B \to B', <a, AB \to AC, 1>, 0)\}^*, m_2 \in \{(B' \to B, <a, AB \to AC, 4>, 0)\}^*, |m_1| = |m_2|, u_3 \in ((\text{alph}(z_3) - \{B\}) \cup \{B'\})^*, g^{-1}(u_3) = g^{-1}(z_3) = y_3, r' \in ((\text{alph}(r'') - \{B\}) \cup \{B'\})^*, g^{-1}(r') = g^{-1}(r'') = r$.

It is clear that $<a, AB \to AC, 5> \in g(a)$; thus, $<a, AB \to AC5> r''ACz_3 \in g(arACy_3) = g(x)$.

(b) Let $a'' = a$. Then, by Claim 2, $Y = B$. By analogy with (ii.b) and (iii.a) in the proof of this claim (see above), we obtain: $S \Rightarrow^{n-1} \bar{a}r''ABz_3 \Rightarrow^* <a, AB \to AC, 5> r''ACz_3$ so $<a, AB \to AC, 5> r''ACz_3 \in g(x)$.

(c) Let $a'' = <a, AB \to AC, 5>$ for some $AB \to AC \in P$ (see (vii) in Claim 2), and let $r''AYz_3 \in (V - \{S\})^*$. At this point, $Y = B$. By analogy with (ii.c) and (iii.a) in the proof of this claim (see above), we can construct $S \Rightarrow^{n+1}$

$\bar{a}r''ABz_3 \Rightarrow^* < a, AB \rightarrow AC, 5 > r''ACz_3$ so $< a, AB \rightarrow AC, 5 > r''ACz_3 \in g(x)$.

(d) Let $a'' =< a, AB \rightarrow AC, 5 >$ for some $AB \rightarrow AC \in P$ (see (vii) in Claim 2), and let $[a, AB \rightarrow AC, q+3] \in alph(r''AY_3)$. By analogy with (ii.d) and (iii.a) in the proof of this claim (see above), we can construct $S \Rightarrow^* \bar{a}r''ABz_3$ and, then, $S \Rightarrow^* \bar{a}r''ABz_3 \Rightarrow^* < a, AB \rightarrow AC, 5 > r''ACz_3$ so $< a, AB \rightarrow AC, 5 > r''ACz_3 \in g(arACy_3) = g(x)$.

(e) Let $a'' =< a, AB \rightarrow AC, i >$ for some $AB \rightarrow AC \in P, i \in \{i, \ldots, 4\}$, see (III) - (IV) in Claim 2. By analogy with (ii.e) and (iii.d) in the proof of this claim (see above), we can construct $S \Rightarrow^* \bar{a}r''ACz_3$, where $\bar{a}r''ACz_3 \in g(x)$.

If: By induction on n, we next prove that
if $S \Rightarrow^n v$ in $G^\sim$ with $v \in g(w)$ and $w \in V^*$ (for some $n \geq 0$),
then $S \Rightarrow^* w$ in G.

Basis: For $n = 0$, the only v is S as $S \Rightarrow^0 S$ in $G^\sim$. Because $\{S\} = g(S)$, we have $w = S$. Clearly, $S \Rightarrow^0 S$ in G.

Induction hypothesis: Assume the claim holds for all derivations of length n or less, for some $n \geq 0$. Let us show that it is also true for $n+1$.

Induction step: For $n + 1 = 1$ (i.e. $n = 0$), there only exists a direct derivation of the form $S \Rightarrow \bar{a}A[(S \rightarrow \bar{a}A, 0, 0)]$ where $A \in N_{CF} \cup \{\lambda\}, a \in T$, and $\bar{a}A \in g(aA)$.

By (1), we have in P a production of the form $S \rightarrow aA$ and, thus, a direct derivation $S \Rightarrow aA$.

Suppose $n + 1 \geq 2$ (i.e. $n \geq 1$). Consider a derivation in $G^\sim : S \Rightarrow^{n+1} x'$ where $x' \in g(x), x \in V^*$. As $n + 1 \geq 2$, there exist $\bar{a} \in W_4, A \in N_{CF}, y \in V^+$, such that $S \Rightarrow \bar{a}A \Rightarrow^{n-1} y' \Rightarrow x'[p]$ in $G^\sim$, where $p \in P^\sim, y' \in g(y)$, and by induction hypothesis, $S \Rightarrow^* y$ in G.

Let us assume that $y' = z_1Zz_2, y = y_1Dy_2, z_j \in g(y_j), y_j \in (V - \{S\})^*, j = 1, 2,$ $Z \in g(D), D \in V - \{S\}, p = (Z \rightarrow r', r_1, r_2) \in P', r_1 = 0$ or $r_2 = 0, x' = z_1r'z_2, r' \in g(r)$ for some $r \in V^*$ (i.e. $x' \in g(y_1ry_2)$). The following cases (i) through (iii) cover all possible forms of $y' \Rightarrow x'[p]$ in $G^\sim$.

(i) Let $Z \in N_{CF}$. By inspection of $P^\sim$, we see that $Z = D, p = (D \rightarrow r', \bar{a}, 0) \in P^\sim, D \rightarrow r \in P$ and $r = r'$. Thus, $S \Rightarrow^* y_1By_2 \Rightarrow y_1ry_2[B \rightarrow r]$ in G.

(ii) Let $r = D$. Then, by induction hypothesis, we have the derivation $S \Rightarrow^* y_1Dy_2$ and $y_1Dy_2 = y_1ry_2$ in G.

(iii) Let $p = ([a, AB \rightarrow AC, q+3] \rightarrow C, < a, AB \rightarrow AC, 5 >, 0), Z = [a, AB \rightarrow AC, q+3]$. Thus, $r' = C$ and $D = B \in N_{CS}$. By case (VI) in Claim 2 and the form of p, we have $z_1 =< a, AB \rightarrow AC, 5 > t$ and $y_1 = au$, where $t \in g(u), < a, AB \rightarrow AC, 5 > \in g(a), u \in (V - \{S\})^*$, and $a \in T$. From (3) in the construction of $G^\sim$, it follows that there exists a production of the from $AB \rightarrow AC \in P$. Moreover, (3) and Claim 2 imply that the derivation in $G^\sim$:

$$S \Rightarrow \bar{a}A \Rightarrow^{n-1} y' \Rightarrow x'[p]$$

can be expressed in the form

$$\begin{array}{lll} S & \Rightarrow & \bar{a}A \\ & \Rightarrow^* & \bar{a}tBz_2 \\ & \Rightarrow & < a, AB \rightarrow AC, 1 > vtBz_2 \\ & & \quad [(\bar{a} \rightarrow < a, AB \rightarrow AC, 1 >, 0, 0)] \end{array}$$

$$\Rightarrow^{|\theta'|} \quad <a, AB \to AC, 1> v\hat{B}w_2$$
$$[\theta']$$
$$\Rightarrow \quad <a, AB \to AC, 1> vB''w_2$$
$$[(\bar{B} \to B'', 0, B'')]$$
$$\Rightarrow \quad <a, AB \to AC, 2> vB''w_2$$
$$[(a, AB \to AC, 1> \to <a, AB \to AC, 2>, 0, B)]$$
$$\Rightarrow \quad <a, AB \to AC, 3> vB''w_2$$
$$[(<a, AB \to AC, 2> \to <a, AB \to AC, 3>, 0, \hat{B})]$$
$$\Rightarrow \quad <a, AB \to AC, 3> v[a, AB \to AC, 1]w_2$$
$$[(B'' \to [a, AB \to AC, 1], <a, AB \to AC, 3>, 0)]$$
$$\Rightarrow^{|\theta|+2} \quad <a, AB \to AC, 3> v[a, AB \to AC, q+3]w_2$$
$$[\theta]$$
$$\Rightarrow \quad <a, AB \to AC, 4> v[a, AB \to AC, q+3]w_2$$
$$[(<a, AB \to AC, 3> \to <a, AB \to AC, 4>, [a, AB \to AC, q+3], 0)]$$
$$\Rightarrow^{|\theta'|-1} \quad <a, AB \to AC, 4> t[a, AB \to AC, q+3]z_2$$
$$[\theta'']$$
$$\Rightarrow \quad <a, AB \to AC, 5> t[a, AB \to AC, q+3]z_2$$
$$[(<a, AB \to AC, 4> \to <a, AB \to AC, 5>, 0, B')]$$
$$\Rightarrow \quad <a, AB \to AC, 5> tCz_2$$
$$[([a, AB \to AC, q+3] \to C, <a, AB \to AC, 5>, 0)]$$

where $\theta' \in \{(B \to B', <a, AB \to AC, 1>, 0)\}^*\{(B \to \hat{B}, <a, AB \to AC, 1>, 0)\}$ $\{(B \to B', <a, AB \to AC, 1>, 0)\}^*, g(B) \cap alph(vw_2) \subseteq \{B'\}$, $g^{-1}(v) = g^{-1}(t), g^{-1}(w_2) = g^{-1}(z_2)$,
$\theta = \theta_1([a, AB \to AC, f(A)] \to [a, AB \to AC, f(A)+1], 0, 0)\theta_2([a, AB \to AC, q+1] \to [a, AB \to AC, q+2], 0, B'[a, AB \to AC, q+1])([a, AB \to AC, q+2] \to [a, AB \to AC, q+3], 0, <a, AB \to AC, 3> [a, AB \to AC, q+2])$,
$\theta_1 = ([a, AB \to AC, 1] \to [a, AB \to AC, 2], 0, f^{-1}(1)[a, AB \to AC, 1])$ $([a, AB \to AC, 2] \to [a, AB \to AC, 3], 0, f^{-1}(2)[a, AB \to AC, 2]) \dots$ $([a, AB \to AC, f(A)-1] \to [a, AB \to AC, f(A)], 0, f^{-1}(f(A)-1)[a, AB \to AC, f(A)-1])$,
where $f(A)$ implies $q_1 = \lambda$,
$\theta_2 = ([a, AB \to AC, f(A)+1] \to [a, AB \to AC, f(A)+2], 0, f^{-1}(f(A)+1)\,[a, AB \to AC, f(A)+1]) \dots ([a, AB \to AC, q] \to [a, AB \to AC, q+1], 0, f^{-1}(q)[a, AB \to AC, q])$, where $f(A) = q$ implies $q_2 = \lambda, \theta'' \in \{(B' \to B, <a, AB \to AC, 4>, 0)\}^*$.

The above derivation implies that the rightmost symbol of t must be A. As $t \in g(u)$, the rightmost symbol of u must be A as well. That is, $t = s'A, u = sA$ and $s' \in g(s)$ (for some $s \in (V - \{S\})^*$). By the induction hypothesis, there exists a derivation in $G : S \Rightarrow^* asABy_2$. Because $AB \to AC \in P$, we get $S \Rightarrow^* asABy_2 \Rightarrow asACy_2[AB \to AC]$, where $asACy_2 = y_1ry_2$.

By (i), (ii), (iii) and inspection of $P^\sim$, we see we have considered all possible derivations of the form $S \Rightarrow^{n+1} x'$ (in $G^\sim$), so we have established Claim 3 by the principle of induction.

The equivalence of G and $G^{\sim}$ can be easily derived from Claim 3. By the definition of g, we have $g(a) = \{a\}$ for all $a \in T$. Thus, by Claim 3, we have for all $x \in T^*$:

$$S \Rightarrow^* x \text{ in } G \text{ if and only if } S \Rightarrow^* x \text{ in } G^{\sim}$$

Consequently, $L(G) = L(G^{\sim})$. We conclude that

$$\mathbf{CS} = \mathbf{prop} - \mathbf{SSC}(1,2)$$

and the theorem holds. Q.E.D.

Corollary 8
$\mathbf{CS} = \mathbf{prop} - \mathbf{SSC}(1,2) = \mathbf{prop} - \mathbf{SSC} = \mathbf{prop} - \mathbf{SC}(1,2) = \mathbf{prop} - \mathbf{SC}$.

We now turn to the investigation of *ssc*-grammars of degree (1,2) with erasing productions.

Theorem 9 $\mathbf{RE} = \mathbf{SSC}(1,2)$.

Proof. Clearly, we have the containment $\mathbf{SSC}(1,2) \subseteq \mathbf{RE}$; hence, it suffices to show $\mathbf{RE} \subseteq \mathbf{SSC}(1,2)$. Every language $L \in \mathbf{RE}$ can be generated by a grammar $G = (V,T,P,S)$ in which each production is of the form $AB \to AC$ or $A \to x$, where $A, B, C \in V - T, x \in \{\lambda\} \cup T \cup (V-T)^2$ (see [2]). Thus, the containment $\mathbf{RE} \subseteq \mathbf{SSC}(1,2)$ can be established by analogy with the proof of Theorem 7 (the details are left to the reader) Q.E.D.

Corollary 10 $\mathbf{RE} = \mathbf{SSC}(1,2) = \mathbf{SSC} = \mathbf{SC}(1,2) = \mathbf{SC}$.

Corollaries 2,4, 8, and 11 imply the main result of this paper:

Corollary 11

$$\begin{gathered}\mathbf{CF}\\ \subset\\ \mathbf{prop} - \mathbf{SSC} = \mathbf{prop} - \mathbf{SSC}(2,1) = \mathbf{prop} - \mathbf{SSC}(1,2) =\\ \mathbf{prop} - \mathbf{SC} = \mathbf{prop} - \mathbf{SC}(2,1) = \mathbf{prop} - \mathbf{SC}(1,2) = \mathbf{CS}\\ \subset\\ \mathbf{SSC} = \mathbf{SSC}(2,1) = \mathbf{SSC}(1,2) = \mathbf{SC} = \mathbf{SC}(2,1) = \mathbf{SC}(1,2) = \mathbf{RE}\end{gathered}$$

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