On the Complexity of Dynamic Tests for Logic Functions

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Abstract

A generalisation of the concept of *dynamic test* is proposed for detecting logic and parametric faults at input / output terminals of logic networks realising $k-$ valued logic functions ($k \geq 2$). Upper and lower bounds on the *complexity* (i.e., length) of minimal dynamic tests are obtained for various classes of logic functions.

1 Introduction

In dynamic testing of combinational logic networks (see [1,2]) the fault-free and faulty circuits are distinguished if they have different dynamic (i.e., time varying) behaviors (output level variations) under the same input stimulation by a transition signal. It should be noted (see [1-3]) that there are statically undetectable logic faults, as well as parametric faults (e.g., inadmissible variations of the magnitude of time delays), which may become detectable only in dynamic testing. In [3] a notion of dynamic test was introduced for input/output terminals (I/O faults) of combinational networks since in many cases faults are more likely to occur at the input/output terminals rather than inside. The dynamic test [3,4] is defined to be a set of input patterns sensitizing the output of the network with respect to simultaneous switching of every feasible subset of input variables. Evidently, the dynamic test for I/O faults does not depend on the internal structure of the network, but depends only on the function realised by the output. In [3-7] some classes of logic and parametric I/O faults are described to be detectable by dynamic tests, and the complexity (i.e., length) of minimal dynamic tests is investigated for various classes of logic functions.

In this paper, a generalisation of the notion of dynamic test called (dynamic) test of regularity, is proposed for $k-$ valued logic functions, $k \geq 2$. As a result, the class of detectable I/O faults is considerably enlarged. A notion of stability dual to that of sensitivity is introduced, and the test of regularity is defined to be a set of input patterns that are sufficient to both sensitize and stabilize the logic function with respect to simultaneous switching of every feasible subset of input variables. Upper and lower bounds on the complexity of minimal tests of regularity are obtained for some classes of k-valued logic functions.

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2 Notations and Definitions

Let $E_k = \{0, 1, ..., k-1\}, k \geq 2$, denote

$$E_k^n = \{\tilde{\alpha}/\tilde{\alpha} = (\alpha_1, ..., \alpha_n), \alpha_i \in E_k, i = \overline{1,n}\};$$

$$P_k(n) = \{f/f : E_k^n \mapsto E_k\};$$

$$G_I(\tilde{\alpha}) = \{\tilde{\beta}/\tilde{\beta} \in E_k^n, (\beta_j \neq \alpha_j) \leftrightarrow (j \in I)\}$$

where $I \subseteq N_n = \{1, 2, ..., n\}, I \neq \emptyset, \tilde{\alpha} \in E_k^n$.

For $k \geq 2, \tilde{\alpha} \in E_k^n, I = \{i_1, ..., i_s\} \subseteq N_n, I \neq \emptyset$, the set $G_I(\tilde{\alpha}) \bigcap E_2^n$ has only one element denoted in the sequel by

$$\tilde{\alpha}^I = (\alpha_1, ..., \alpha_{i_1-1}, \overline{\alpha}_{i_1}, \alpha_{i_1+1}, ..., \alpha_{i_s-1}, \overline{\alpha}_{i_s}, \alpha_{i_s+1}, ..., \alpha_n)$$

where $\overline{\sigma} = 1 - \sigma, \sigma \in E_2$.

Definition 2.1 . The function $f(x_1, ..., x_n) \in P_k(n)$ is *sensitive (stable)* at vector $\tilde{\alpha} \in E_k^n$ with respect to the subset of variables $\{x_{i_1}, ..., x_{i_s}\} \subseteq \{x_1, ..., x_n\}$ if there exists a vector $\tilde{\beta} \in G_{\{i_1,...,i_s\}}(\tilde{\alpha})$ such that $f(\tilde{\alpha}) \neq f(\tilde{\beta})$ (respectively , $f(\tilde{\alpha}) = f(\tilde{\beta})$). The function f is sensitive (stable) with respect to $\{x_{i_1}, ..., x_{i_s}\}$ if there exists a vector $\tilde{\alpha} \in E_k^n$ at which f is sensitive (stable) with respect to $\{x_{i_1}, ..., x_{i_s}\}$.

Definition 2.2. The function $f \in P_k(n)$ is said to be *regular* if it is both sensitive and stable with respect to every nonempty subset of variables $\{x_{i_1}, ..., x_{i_s}\} \subseteq \{x_1, ..., x_n\}$.

Denote by $R_k(n)$ the set of all regular functions $f \in P_k(n)$.

We shall say that *almost all* functions from a class $F(n) \subseteq P_k(n)$ have a property R if the fraction of functions from $F(n)$ with property R tends to 1 as $n \to \infty$.

It is easy to prove the following assertion.

Lemma 2.1. Almost all functions $f \in P_k(n)$ are regular.

Definition 2.3. The set of vectors $T^*(s, f) \subseteq E_k^n$ (respectively, $T^{**}(s, f) \subseteq E_k^n$) is called an $s-$ *test of sensitivity (stability)* for $f \in P_k(n)$, if for each subset $\{i_1, ..., i_r\} \subseteq N_n$, $1 \leq r \leq s$, the sensitivity (stability) of f with respect to $\{x_{i_1}, ..., x_{i_r}\}$ implies the existence of a vector $\tilde{\alpha} \in T^*(s, f)$ (respectively, $\tilde{\alpha} \in T^{**}(s, f)$) at which f is sensitive (stable) with respect to $\{x_{i_1}, ..., x_{i_r}\}$.

Definition 2.4. The set of vectors $T(s, f) \subseteq E_k^n$ is called a *(dynamic) $s-$ test of regularity* for the function $f \in P_k(n)$, if it is both an $s-$ test of sensitivity and an $s-$ test of stability for f.

For $s = 1$ (respectively $s = n$) the $s-$ tests will be called *single (complete)* tests. The test $T_0(s, f)$ is called a *minimal* $s-$ test for f, if $|T_0(s, f)| = t(s, f) = \min\{|T(s, f)|/T(s, f) \in Z(s, f)\}$, where $Z(s, f)$ is the set of all $s-$ tests of regularity for f, and $|A|$ denotes the cardinality of the set A.

The main objective of this paper is to find bounds on the complexity measure $t(s, f)$ of minimal $s-$ tests of regularity for logic functions $f \in P_k(n), k \geq 2, 1 \leq s \leq n$.

3 The Complexity of Single Tests

The set of functions $P_k(n)$ may be considered as a probability space with every element $f \in P_k(n)$ having the same probability $\exp_k(-k^n)$. Denote by H_n the Hamming code in the n-cube E_2^n. It is known (see [9]) that $|H_n| = exp_2(n - \lceil \log_2(n+1) \rceil)$. The pair of vectors $\tilde{\alpha}, \tilde{\beta} \in H_n, \tilde{\alpha} \neq \tilde{\beta}$, will be called a regular pair for the Boolean function $f \in P_2(n)$ iff $\bigwedge_{i=1}^n (f(\tilde{\alpha}) \oplus f(\tilde{\alpha}^{(i)}) \oplus f(\tilde{\beta}) \oplus f(\tilde{\beta}^{(i)})) = 1$ where $\oplus$ is the modulo 2 sum. Obviously, if $\{\tilde{\alpha}, \tilde{\beta}\}$ is a regular pair for $f \in P_2(n)$ then $\{\tilde{\alpha}, \tilde{\beta}\}$ is a single test of regularity for f.

For every $\tilde{\alpha}, \tilde{\beta} \in H_n, \tilde{\alpha} \neq \tilde{\beta}, f \in P_2(n)$ define the following random variables

$$\xi_{\tilde{\alpha},\tilde{\beta}}(f) = \begin{cases} 1 & \text{if } \{\tilde{\alpha}, \tilde{\beta}\} \text{ is a regular pair for } f \\ 0 & \text{otherwise} \end{cases}$$

Obviously, $\xi_2(f) = \sum_{\tilde{\alpha},\tilde{\beta} \in H_n, \tilde{\alpha} \neq \tilde{\beta}} \xi_{\tilde{\alpha},\tilde{\beta}}(f)$ determines the number of regular pairs for $f \in P_2(n)$.

From Definition 2.4 with $k \geq 3, s = 1$, it follows that if for every $i, 1 \leq i \leq n$, the function $f \in P_k(n)$ is both sensitive and stable at vector $\tilde{\alpha} \in E_k^n$ with respect to variable x_i, then $\{\tilde{\alpha}\}$ is a minimal single test of regularity for f.

For every $\tilde{\alpha} \in E_k^n$ and $f \in P_k(n)$ define the following random variable

$$\xi_{\tilde{\alpha}}(f) = \begin{cases} 1 & \text{if } \{\tilde{\alpha}\} \text{ is a single test for } f \\ 0 & \text{otherwise} \end{cases}$$

Obviously, the random variable $\xi_k(f) = \sum_{\tilde{\alpha} \in E_k^n} \xi_{\tilde{\alpha}}(f)$ determines the number of single tests of regularity for $f \in P_k(n)$, $k \geq 3$. Now let us compute the expectations $M\xi_k(f)$ and dispersions $D\xi_k(f)$ for the random variables $\xi_k(f), k \geq 2$.

Lemma 3.1

$$M\xi_k(f) = \begin{cases} |H_n|(|H_n| - 1)2^{-n-1}, & \text{if } k = 2 \\ k^n(1 - ((k-1)/k)^{k-1} - k^{-k+1})^n & \text{if } k \geq 3 \end{cases}$$

Lemma 3.2 .

$$D\xi_k(f) = \begin{cases} (1 - 2^{-n})M\xi_2(f) & \text{if } k = 2 \\ M\xi_k(f) + (c_1(k)n^2 + c_2(k)n - 1)k^{-n}(M\xi_k(f))^2 & \text{if } k \geq 3 \end{cases}$$

where $c_1(k)$ and $c_2(k)$ depend only on k.

Lemma 3.3. For almost all functions $f \in P_k(n), k \geq 2, n \to \infty$,

$$\xi_k(f) \sim M\xi_k(f).$$

Proof is based on the second - moment method (see, e.g., [8]). From Lemmas 3.1 and 3.2 it follows that $M\xi_k(f) \to \infty$, and $D\xi_k(f) = o((M\xi_k(f))^2)$. Let $\phi(n) \to \infty, \phi(n) = o(\sqrt{M\xi_k(f)})$, then according to Chebyshev's inequality (see [8]) the fraction of functions $f \in P_k(n)$ satisfying the inequality

$|\xi_k(f) - M\xi_k(f)| < (\phi(n))^{-1}M\xi_k(f)$ tends to 1 as $n \to \infty$. Consequently, by definition, $\xi_k(f) \sim M\xi_k(f)$ for almost all functions $f \in P_k(n)$, $n \to \infty$, $k \geq 2$.

Theorem 3.1 . For almost all functions $f \in P_k(n)$

$$t(1,f) = \begin{cases} 2 & \text{if } k = 2 \\ 1 & \text{if } k \geq 3 \end{cases}$$

Proof for $k \geq 3$ follows immediately from Lemma 3.3. For $k = 2$ from Lemma 3.3 it follows that $t(1,f) \leq 2$ for almost all functions $f \in P_2(n)$. From Lemma 2.1 and Definition 2.4 with $k = 2, s = 1$, it follows that $t(1,f) \geq 2$ which completes the proof.

4 Upper Bounds on the Complexity of $s-$ Tests

The set of vectors $Q \subseteq E_k^n$ will be called an $(n, 2s+1, r)-$ code if $|Q| = r$ and $\rho(\tilde{\alpha}, \tilde{\beta}) \geq 2s+1$ for all $\tilde{\alpha}, \tilde{\beta} \in Q, \tilde{\alpha} \neq \tilde{\beta}$, where $\rho(\tilde{\alpha}, \tilde{\beta})$, called the distance between $\tilde{\alpha}$ and $\tilde{\beta}$, is the number of coordinates $i, 1 \leq i \leq n$, such that $\alpha_i \neq \beta_i$.

Lemma 4.1 . Let $Q \subseteq E_k^n$ be an $(n, 2s+1, r)-$ code,$s \geq 2, k \geq 2$ and $\psi(n) \to \infty$ as $n \to \infty$. If

$$r = \begin{cases} \lfloor \log_2 \sum_{i=1}^{s} \binom{n}{i} + \psi(n) \rfloor & \text{for } k = 2 \\ \lfloor (k-1)^{-1} \log_{k/(k-1)} n + \psi(n) \rfloor & \text{for } k \geq 3 \end{cases}$$

then for almost all functions $f \in P_k(n)$ Q is an $s-$ test of regularity.

Proof. Let $I = \{i_1, ..., i_m\} \subseteq N_n, 1 \leq m \leq s$. Denote by $\Phi_k(I)$ (respectively, $\Psi_k(I)$) the number of functions $f \in P_k(n)$ that are not sensitive (stable) at each vector $\tilde{\alpha} \in Q$ with respect to the subset of variables $\{x_{i_1}, ..., x_{i_m}\}$. Then it is easy to compute

$$\Phi_k(I) = \exp_k(k^n - r(k-1)^m), \Psi_k(I) = k^{k^n} \exp_{(k-1)/k}(r(k-1)^m).$$

Let $p_k(n, Q)$ be the probability of an event that Q is an $s-$ test of regularity for a random function from $P_k(n)$. Now it is easy to verify that if r satisfies the conditions of the lemma, then

$$p_k(n, Q) \geq 1 - \exp_k(-k^n) \sum_{I \subseteq N_n, 1 \leq |I| \leq s} (\Phi_k(I) + \Psi_k(I)) =$$

$$= 1 - \sum_{i=1}^{s} \binom{n}{i} (\exp_k(-r(k-1)^i + \exp_{(k-1)/k}(r(k-1)^i)) = 1 - o(1).$$

Denote by $F_k(n, 2s+1)$ a code in E_k^n of maximal cardinality with a code distance $2s+1$. The following statement is a straight-forward generalization of a well-known result for $k = 2$ (see [9]).

Lemma 4.2 . For all $k \geq 2$

$$|F_k(n, 2s+1)| \geq k^n / \sum_{i=0}^{2s} \binom{n}{i} (k-1)^i.$$

Theorem 4.1 . For almost all functions $f \in P_2(n)$ and $n \to \infty$

$$t(s,f) \lesssim \begin{cases} s \log_2 \frac{n}{s} & \text{if } s = o(n) \\ nH(\lambda) & \text{if } s = \lfloor \lambda n \rfloor, 0 < \lambda < 1/4 \end{cases}$$

where $H(\lambda) = -\lambda \log_2 \lambda - (1-\lambda)\log_2(1-\lambda)$.

Proof. Applying the well - known (*see* [10]) inequality $\sum_{i=0}^{m} \binom{n}{i} \le \exp_2(nH(m/n))$ where $m \le n/2$, and taking into account Lemma 4.2 we obtain that if $s = \lfloor \lambda n \rfloor, 0 < \lambda < 1/4, n \to \infty$ and r satisfies the conditions of Lemma 4.1 for $k = 2$, then

$$|F_2(n, 2s+1)| \ge \exp_2(n(1 - H(2s/n))) > \exp_2(n(1 - H(2\lambda))) > r.$$

Thus, if the conditions mentioned above are satisfied, then there can be constructed an $(n, 2s+1, r)-$ code which according to Lemma 4.1 will be an s-test of regularity for almost all functions $f \in P_2(n)$. Hence, $t(s,f) \le r = \lfloor \log_2 \sum_{i=1}^{s} \binom{n}{i} + \psi(n) \rfloor$, whence the proof follows directly.

Theorem 4.2 . For almost all functions $f \in P_k(n), k \ge 3$,

$$2 \le s \le \lfloor n(\log_2 k - 1)/(2\log_2(k-1)) \rfloor, n \to \infty,$$

$$t(s,f) \lesssim (k-1)^{-1} \log_{k/(k-1)} n.$$

Proof is analogous to that of Theorem 4.1.

5 Lower Bounds on the Complexity of $s-$ Tests

Let $\{i_1, ..., i_r\} \subseteq N_n, c_j \in E_k, j = \overline{1,r}, 1 \le r \le n, k \ge 2$. Denote by $E_k^n(i_1,\ c_1;\ ...;\ i_r,\ c_r)$ the set of all vectors $\tilde{\beta} \in E_k^n$ with $\beta_{i_j} = c_j,\ j = \overline{1,r}$, called an $(n-r)-$ dimensional subcube in E_k^n. The set of indices $\{i_1, ..., i_r\}$ will be called the set of *fixed indices* of the subcube. Any two subcubes in E_k^n will be called *parallel* if they have the same set of fixed indices. Obviously, any two parallel subcubes do not intersect, and $|E_k^n(i_1, c_1; ...; i_r, c_r)| = k^{n-r}$.

The following statement is a straightforward generalisation of a lemma from [10].

Lemma 5.1 . For any set $M \subseteq E_k^n, |M| = m \le n+1, k \ge 2$, there exists a family of $(n-m+1)-$ dimensional parallel subcubes of cardinality k^{m-1}, with each subcube containing at most one vector from M.

Let $\pi_k(m)$ be the probability of an event that a random function from $P_k(n)$ has an $s-$ test of regularity consisting of m vectors.

Lemma 5.2 . For $k \ge 2, m \le n+1$ and $n \to \infty$

$$\pi_k(m) \le \binom{k^n}{m} \prod_{i=1}^{s} \left(1 - \left(\frac{k-1}{k}\right)^{m(k-1)^i} - k^{-m(k-1)^i}\right)^{\binom{n-m+1}{i}} + o(1).$$

Proof. Denote by $\mathcal{M}(M)$ the set of functions $f \in R_k(n)$ having M as an $s-$ test of regularity. Then, taking into account Lemma 2.1 , it is easy to verify that

$$\pi_k(m) \le \exp_k(-k^n) \sum_{M \subseteq E_k^n, |M|=m} |\mathcal{M}(M)| + o(1).$$

According to Lemma 5.1 there can be found a family of $(n-m+1)-$ dimensional parallel subcubes of cardinality k^{m-1} with each subcube containing at most one vector from M. Let $\{j_1, j_2, ..., j_{m-1}\}$ be the set of fixed indices corresponding to every subcube from the family. Denote $N_n^* = N_n \setminus \{j_1, ..., j_{m-1}\}$. Let $\mathcal{M}^*(M)$ be the set of functions $f \in R_k(n)$ having M as an $s-$ test of regularity with respect to N_n^*, i.e., for every subset $\{l_1, ..., l_v\} \subseteq N_n^*$, $1 \le v \le s$, f is both sensitive and stable with respect to the set of variables $\{x_{l_1}, ..., x_{l_v}\}$. Obviously, $\mathcal{M}(M) \subseteq \mathcal{M}^*(M)$.

It is easy to compute that

$$|\mathcal{M}^*(M)| \le \prod_{i=1}^{s} \left(\exp_k\left(m(k-1)^i\right) - \exp_{k-1}\left(m(k-1)^i\right) - 1\right)^{\binom{n-m+1}{i}} \times$$

$$\times \exp_k\left(k^n - m\sum_{i=1}^{s} \binom{n-m+1}{i}(k-1)^i\right) =$$

$$k^{k^n} \prod_{i=1}^{s} \left(1 - (\frac{k-1}{k})^{m(k-1)^i} - k^{-m(k-1)^i}\right)^{\binom{n-m+1}{i}}.$$

Whence the proof of the lemma follows immediately.

The proof of the following statement is obvious.

Lemma 5.3 . If $\pi_k(m) = o(1)$ for $k \ge 2, m = m(n), n \to \infty$, then for almost all functions $f \in P_k(n)$

$$t(s, f) \ge m + 1.$$

Theorem 5.1 . For $n \to \infty$ and almost all functions $f \in P_2(n)$

$$t(s,f) \gtrsim \begin{cases} (s-1)\log_2 n & \text{if } s = const \ge 2 \\ s\log_2 \frac{n}{s} & \text{if } s = o(n), s \to \infty \\ nH(\lambda)/(1+H(\lambda)) & \text{if } s = \lfloor \lambda n \rfloor, 0 < \lambda < 1/2 \\ n/2 & \text{if } s \ge n/2 \end{cases}$$

Proof. From Lemma 5.2 with $k = 2$ we obtain

$$\pi_k(m) \le 2^{mn} \exp_e(-2^{-m+1} \sum_{i=1}^{s} \binom{n-m+1}{i}) + o(1).$$

Putting

$$m_0 = \begin{cases} \lfloor (s-1)\log_2 n - \log_2 \log_2 n - r(n) \rfloor, & \\ r(n) = o(\log n), r(n) \to \infty & \text{if } s = const \ge 2 \\ \lfloor s\log_2 \frac{n}{s} - 2\log_2 n + s\log_2(1 - \frac{s}{n}\log_2 \frac{n}{s}) \rfloor & \text{if } s = o(n), s \to \infty \\ \lfloor nH(\lambda)/(1+H(\lambda)) - 3\log_2 n \rfloor & \text{if } s = \lfloor \lambda n \rfloor, 0 < \lambda < 1/2 \\ \lfloor n/2 - \nu \log_2 n \rfloor, \nu = const > 5/4 & \text{if } s \ge n/2 \end{cases}$$

it is easy to verify that $\pi_k(m_0) = o(1), n \to \infty$. Hence in view of Lemma 5.3 we obtain the assertion of the theorem.

Corollary 5.1 . If $s = const \geq 2, n \to \infty$, then for almost all functions $f \in P_2(n)$,

$$t(s, f) \asymp \log n.$$

Corollary 5.2 . If $s = o(n), s \to \infty, n \to \infty$, then for almost all functions $f \in P_2(n)$,

$$t(s, f) \sim s \log_2 \frac{n}{s}.$$

Corollary 5.3 . If $s = \lfloor \lambda n \rfloor, 0 < \lambda < 1/4, n \to \infty$, then for almost all functions $f \in P_2(n)$,

$$t(s, f) \asymp n.$$

Corollaries 5.1-5.3 are obtained from Theorems 4.1 and 5.1.

Theorem 5.2 . If $s \geq 2, k \geq 3, n \to \infty$, then for almost all functions $f \in P_k(n)$,

$$t(s, f) \gtrsim (k-1)^{-2} \log_{k/(k-1)} n.$$

Proof. From Lemma 5.2 with $k \geq 3$ we obtain

$$\pi_k(m) \leq k^{mn} \exp_e \left(-\binom{n-m+1}{2} \left(\frac{k-1}{k}\right)^{m(k-1)^2} \right) + o(1).$$

It is easy to verify that $\pi_k(m) = o(1)$ for $m = \lfloor (k-1)^{-2} \log_{k/(k-1)} n - \log_k \log_k n \rfloor$, $n \to \infty$. Thus, in view of Lemma 5.3, the theorem is proved.

Corollary 5.4 . If $k \geq 3, 2 \leq s \leq \lfloor n(\log_2 k - 1)/(2 \log_2 (k-1)) \rfloor$, $n \to \infty$, then for almost all functions $f \in P_k(n)$,

$$t(s, f) \asymp \log n.$$

Proof follows directly from Theorems 4.2 and 5.2 .

6 Upper Bounds on the Complexity of Complete Tests

For Boolean functions $f \in P_2(n)$ denote $\omega_I^f(\tilde{x}) = f(\tilde{x}) \oplus f(\tilde{x}^I)$, $I \subseteq N_n, I \neq \emptyset$. Now let us describe an algorithm for constructing a complete test of regularity for an arbitrary function $f \in P_2(n)$.

Algorithm 6.1 . **Step 1.** Choose an arbitrary vector $\tilde{\alpha}_1 \in E_2^n$ and put

$$T_1(f) = \{\tilde{\alpha}_1\};$$

$$T_1^0 = \{I / I \subseteq N_n, I \neq \emptyset, \omega_I^f(\tilde{\alpha}_1) = 0\};$$

$$\mathcal{T}_1^1 = \{I/I \subseteq N_n, I \neq \emptyset, \omega_I^f(\tilde{\alpha}_1) = 1\}.$$

If $\mathcal{T}_1^0 \bigcup \mathcal{T}_1^1 \neq \emptyset$ and there exists a subset $I_2 \in \mathcal{T}_1^\sigma$ for some $\sigma \in \{0,1\}$ such that $\omega_{I_2}^f(\tilde{x}) \not\equiv \sigma$, then we pass to the next step, otherwise the algorithm terminates.

Step $i (i \geq 2)$. Choose a vector $\tilde{\alpha}_i \in E_2^n$ such that $\omega_{I_i}^f(\tilde{\alpha}_i) = \overline{\sigma}$, where $I_i \in \mathcal{T}_{i-1}^\sigma, \omega_{I_i}^f(\tilde{x}) \not\equiv \sigma, \sigma \in \{0,1\}$. If

$$\sum_{\sigma=0}^{1} |\{I/I \in \mathcal{T}_{i-1}^\sigma, \omega_I^f(\tilde{\alpha}_i) = \overline{\sigma}\}| \geq \lfloor \frac{1}{2} \sum_{\sigma=0}^{1} |\mathcal{T}_{i-1}^\sigma| \rfloor + 1$$

then put

$$T_i(f) = T_{i-1}(f) \bigcup \{\tilde{\alpha}_i\};$$

$$\mathcal{T}_i^0 = \{I/I \in \mathcal{T}_{i-1}^0, \omega_I^f(\tilde{\alpha}_i) = 0\};$$

$$\mathcal{T}_i^1 = \{I/I \in \mathcal{T}_{i-1}^1, \omega_I^f(\tilde{\alpha}_i) = 1\},$$

otherwise

$$T_i(f) = \{\tilde{\alpha}^{I_i}/\tilde{\alpha} \in T_{i-1}(f)\} \bigcup \{\tilde{\alpha}_i^{I_i}\};$$

$$\mathcal{T}_i^0 = \{I \Delta I_i / I \in \mathcal{T}_{i-1}^\sigma, I \neq I_i, \omega_I^f(\tilde{\alpha}_i) = \overline{\sigma}\};$$

$$\mathcal{T}_i^1 = \{I \Delta I_i / I \in \mathcal{T}_{i-1}^{1-\sigma}, \omega_I^f(\tilde{\alpha}_i) = \sigma\},$$

where Δ is the set-theoretical operation of symmetric difference.

If $\mathcal{T}_i^0 \bigcup \mathcal{T}_i^1 \neq \emptyset$ and there exists a subset $I_{i+1} \in \mathcal{T}_i^\sigma$ for some $\sigma \in \{0,1\}$ such that $\omega_{I_{i+1}}^f(\tilde{x}) \not\equiv \sigma$, then we pass to Step $i+1$, otherwise the algorithm terminates.

Finally, Algorithm 6.1 will determine a set $T_m(f)$ of $m \geq 1$ vectors which, as we are going to prove below, is a complete test of regularity for $f \in P_2(n)$.

We shall say that the subset $I \subseteq N_n, I \neq \emptyset$, is a feasible fault of sensitivity (stability) for $f \in P_2(n)$ if $\omega_I^f(\tilde{x}) \not\equiv 0$ (respectively, $\omega_I^f(\tilde{x}) \not\equiv 1$), and the vector $\tilde{\alpha} \in E_2^n$ detects the fault of sensitivity (stability) I for f if $\omega_I^f(\tilde{\alpha}) = 1$ (respectively, $\omega_I^f(\tilde{\alpha}) = 0$).

Theorem 6.1 . For all functions $f \in P_2(n)$,

$$t(n,f) \leq n+1.$$

Proof. Let $m = m(f)$ be the number of steps performed by Algorithm 6.1 for f. It is easy to see that for each $i, 1 \leq i \leq m$, the vectors from $T_i(f)$ do not detect the faults of sensitivity $I \in \mathcal{T}_i^0$ and the faults of stability $J \in \mathcal{T}_i^1$, and the total number of faults not detected by the vectors from $T_i(f)$ is reduced more than twice after each step. Since the algorithm terminates iff $T_m(f)$ detects all feasible faults of sensitivity and stability for f, then $T_m(f)$ is a complete test of regularity for f. Consequently, $t(n,f) \leq |T_m(f)| = m$. It is easy to prove by induction on $i, 1 \leq i \leq m$, that $|\mathcal{T}_i^0| + |\mathcal{T}_i^1| \leq 2^{n+1-i} - 1$. The conditions causing Algorithm 6.1 to terminate imply $0 \leq |\mathcal{T}_m^0| + |\mathcal{T}_m^1| \leq 2^{n+1-m} - 1$ whence the bound $m \leq n+1$ is derived directly.

Corollary 6.1 . For almost all functions $f \in P_2(n)$,

$$n/2 \lesssim t(n, f) \leq n + 1.$$

Proof follows directly from Theorems 5.1 and 6.1.

Let $t^*(s, f)$ (respectively, $t^{**}(s, f)$) be the complexity of a minimal $s-$ test of sensitivity (stability) for $f \in P_k(n)$.

Lemma 6.1 [7]. For almost all functions $f \in P_k(n), k \geq 3$,

$$t^*(n, f) \leq n.$$

We will say that the function $f \in P_k(n), k \geq 3$, is stable in E_2^n if for every $I \subseteq N_n, I \neq \emptyset$, there exists a vector $\tilde{\alpha} \in E_2^n$ detecting the fault of stability I for f in E_2^n, i.e. , $f(\tilde{\alpha}) = f(\tilde{\alpha}^I)$, where $\tilde{\alpha}^I$ is the sole vector from $G_I(\tilde{\alpha}) \bigcap E_2^n$. Denote by $\theta(f, I)$ the number of vectors $\tilde{\alpha} \in E_2^n$ detecting the fault of stability I for f.

Lemma 6.2 . For almost all functions $f \in P_k(n), k \geq 3$, the inequality $\theta(f, I) \geq \frac{1}{k} 2^{n-1}$ holds for all $I \subseteq N_n, I \neq \emptyset$.

Lemma 6.3 . Almost all functions $f \in P_k(n), k \geq 3$, are stable in E_2^n.

Lemma 6.4 . Almost all functions $f \in P_k(n), k \geq 3$, take all k values from E_k at vectors from E_2^n.

Proofs of Lemmas 6.2-6.4 are not difficult, so they are omitted.

Now let us describe an algorithm for constructing a complete test of stability for almost all functions $f \in P_k(n), k \geq 3$. To this end, each function $f \in P_k(n)$ is associated to a table $J(f)$ with 2^n rows, one for each vector from E_2^n, and $2^n - 1$ columns, one for each feasible fault of stability $I \subseteq N_n, I \neq \emptyset$. At the intersection of the ith row and jth column corresponding to $\tilde{\alpha} \in E_2^n$ and $I \subseteq N_n$, respectively, there stands a '1'('0') iff $f(\tilde{\alpha}) = f(\tilde{\alpha}^I)$ (respectively, $f(\tilde{\alpha}) \neq f(\tilde{\alpha}^I)$). Let $T_0(f) = \emptyset$ and $J_0(f) = J(f)$.

Algorithm 6.2 . **Step i** ($i \geq 1$) . Select a vector $\tilde{\alpha}_i \in E_2^n$ with the corresponding row in $J_{i-1}(f)$ having the maximum number of 1's, and put $T_i(f) = T_{i-1}(f) \bigcup \{\tilde{\alpha}_i\}$. Denote by $J_i(f)$ the table obtained from $J_{i-1}(f)$ by deleting all the columns having 1's in the row corresponding to $\tilde{\alpha}_i$. If $J_i(f) = \emptyset$ or $J_i(f)$ has only 0's , then the algorithm terminates, otherwise we pass to Step $i + 1$.

Note that according to Lemma 6.3, for almost all functions $f \in P_k(n)$ Algorithm 6.2 terminates iff $J_m(f) = \emptyset$ for some $m \geq 1$. Hence, the following assertion holds.

Lemma 6.5 . For almost all functions $f \in P_k(n), k \geq 3$, Algorithm 6.2 constructs a complete test of stability.

Lemma 6.6 . For almost all functions $f \in P_k(n), k \geq 3$,

$$t^{**}(n, f) \leq \lceil n \log_{2k/(2k-1)} 2 \rceil + 1.$$

Proof. Let μ_r be the fraction of faults of stability $I \subseteq N_n$, $I \neq \emptyset$, detected by vectors $\tilde{\alpha}_1, ..., \tilde{\alpha}_r \in E_2^n$ which are selected after the rth step of Algorithm 6.2. Then, obviously, $(1 - \mu_r)(2^n - 1)$ is the number of faults of stability remaining still undetected , i.e., the number of nonempty subsets $I \subseteq N_n$ such that $f(\tilde{\alpha}_j) \neq f(\tilde{\alpha}_j^I)$ for all $j, 1 \leq j \leq r$. From Lemmas 6.2 and 6.3 it follows that for almost all functions $f \in P_k(n)$ the total number of feasible faults of stability, detectable by the remaining vectors from E_2^n is not less than $\frac{1}{k}(1-\mu_r)(2^n-1)2^{n-1}$. Consequently, among the remaining vectors there can be found a vector detecting not less than $(1-\mu_r)(2^n-1)/(2k)$ faults of stability which are not detected by the first r selected vectors. Thus, we obtain

$$\mu_{r+1} \geq \mu_r + \frac{1}{2k}(1-\mu_r) = (1-\frac{1}{2k})\mu_r + \frac{1}{2k} \geq \ldots \geq$$

$$\geq (1-\frac{1}{2k})^r \mu_1 + \frac{1}{2k}\sum_{i=0}^{r-1}(1-\frac{1}{2k})^i \geq$$

(since Lemma 6.4 implies $\mu_1 \geq \frac{1}{k} > \frac{1}{2k}$)

$$\geq \frac{1}{2k}\sum_{i=0}^{r}(1-\frac{1}{2k})^i = 1-(1-\frac{1}{2k})^{r+1}.$$

Thus, $\mu_r \geq 1-(1-\frac{1}{2k})^r$. Putting $r_0 = \lceil \log_{1-\frac{1}{2k}} \frac{1}{2^n-1} \rceil \leq \lceil n \log_{2k/(2k-1)} 2 \rceil$, we find out that after the choice of r_0 vectors there will still remain undetected not more than

$$(1-\mu_{r_0})(2^n-1) \leq (1-\frac{1}{2k})^{r_0}(2^n-1) \leq 1$$

faults of stability. Taking into account Lemma 6.5, we obtain that for almost all functions $f \in P_k(n)$

$$t^{**}(n,f) \leq r_0 + (1-\mu_{r_0})(2^n-1) \leq \lceil n \log_{2k/(2k-1)} 2 \rceil + 1.$$

Theorem 6.2 . For almost all functions $f \in P_k(n), k \geq 3$,

$$t(n,f) \lesssim n(1+\log_{2k/(2k-1)} 2).$$

Proof follows directly from Lemmas 6.1 and 6.6 .

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