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On isomorphic representation of nondeterministic tree automata*

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Abstract

In this paper we deal with isomorphically complete systems of finite nondeterministic tree automata with respect to the general product and the cube-product. In both cases characterizations of isomorphically complete systems are presented which imply that the general product and the cube-product are equivalent regarding isomorphic completeness.

In the theory of finite automata it is a central problem to characterize such systems from which any automaton can be represented isomorphically or homomorphically under a given composition. Such systems are called isomorphically, respectively, homomorphically complete with respect to the composition considered. From the practical point of view, finite complete systems have great importance. The first composition admitting finite isomorphically complete systems was introduced by V. M. Glushkov in [7], who gave a characterization of the isomorphically complete systems. Later F. Gécseg [2] introduced a product hierarchy, the α_i-products, $i = 0, 1, \ldots$, and Z. Ésik [1] proved that, from the point of view of homomorphic completeness, Glushkov's composition is equivalent to the α_i-product for $i \geq 2$. Regarding isomorphic completeness, it turned out that there is no finite isomorphically complete system with respect to any of the α_i-products. A systematic account of the results on α_i-products including the ones mentioned above can be found in the monograph [3].

The first generalization of Glushkov's result to tree automata was given by M. Steinby in [10]. The generalization of the notion of finite automata to trees has a rigorius mathematical discussion in [6]. Another generalization of Gluskhov's result to nondeterministic automata is given in [4]. In this paper we extend this result to nondeterministic tree automata. Namely, we define the Glushkov-type product of nondeterministic tree automata and characterize the isomorphically complete systems with respect to this composition. Our characterization implies the existence of finite isomorphically complete systems of nondeterministic tree automata with respect to this product.

The cube-product, which is a simpler composition than Glushkov's one, was introduced in [8] where a characterization of the isomorphically complete systems

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with respect to this product was presented as well. From this characterization it follows that the Glushkov-type product and the cube-product are equivalent regarding isomorphic completeness.

The generalization of the cube-product to tree automata and the characterization of the isomorphically complete systems with respect to it is given in [9]. A similar generalization and characterization for nondeterministic automata is presented in [5]. In both cases the characterization of the isomorphically complete systems implies that the Glushkov-type composition and the cube-pruduct are equivalent regarding isomorphic completeness. Here we generalize the cube-product to nondeterministic tree automata and give a characterization of the isomorphically complete systems with respect to it. Our characterization shows that the cube-product and the Glushkov-type product are equivalent regarding isomorphic completeness for the class of nondeterministic tree automata, too.

To start the discussion, we introduce some notions and notations. By a set of *relational symbols* we mean a nonempty union $\Sigma = \Sigma_1 \bigcup \Sigma_2 \bigcup \ldots$, where Σ_m, $m = 1, 2, \ldots$, are pairwise disjoint sets of symbols. For any $m \geq 1$, the set Σ_m is called the set of *m-ary relational symbols.* It is said that the *rank* or *arity* of a symbol $\sigma \in \Sigma$ is m if $\sigma \in \Sigma_m$. Now let a set Σ of relational symbols and a set R of positive integers be given. R is called the *rank-type* of Σ if for any integer $m \geq 0$, $\Sigma_m \neq \emptyset$ if and only if $m \in R$. In the sequel we shall work under a fixed rank-type R.

Now let Σ be a set of relational symbols with rank-type R. By a *nondeterministic Σ-algebra* $\mathcal{A}$ we mean a pair consisting of a nonempty set A and a mapping that assigns to every relational symbol $\sigma \in \Sigma$ an m-ary relation $\sigma^{\mathcal{A}} \subseteq A^m$, where the arity of σ is m. The set A is called the *set of elements of* $\mathcal{A}$ and $\sigma^{\mathcal{A}}$ is the *realization of σ in* $\mathcal{A}$. The mapping $\sigma \to \sigma^{\mathcal{A}}$ will not be mentioned explicitly, we only write $\mathcal{A} = (A, \Sigma)$. For any $m \in R$, $\sigma \in \Sigma_m$, $(a_1, \ldots, a_{m-1}) \in A^{m-1}$, we denote by $(a_1, \ldots, a_{m-1})\sigma^{\mathcal{A}}$ the set $\{a : a \in A \ \& \ \sigma^{\mathcal{A}}(a_1, \ldots, a_{m-1}, a)\}$. If $(a_1, \ldots, a_{m-1})\sigma^{\mathcal{A}}$ is a one-element set $\{a\}$, then we write $(a_1, \ldots, a_{m-1})\sigma^{\mathcal{A}} = a$.

It is said that a nondeterministic Σ-algebra $\mathcal{A}$ is *finite* if A is finite, and it is of *finite type* if Σ is finite. By a *nondeterministic tree automaton* we mean a finite nondeterministic algebra of finite type. Finally, it is said that the *rank-type of a nondeterministic tree automaton* $\mathcal{A} = (A, \Sigma)$ is R if the rank-type of Σ is R.

Let $\mathcal{A} = (A, \Sigma)$ and $\mathcal{B} = (B, \Sigma)$ be nondeterministic tree automata with rank-type R. $\mathcal{B}$ is called a *subautomaton* of $\mathcal{A}$ if $B \subseteq A$ and, for all $m \in R$ and $\sigma \in \Sigma_m$, $\sigma^{\mathcal{B}}$ is the restriction of $\sigma^{\mathcal{A}}$ to B^m. A one-to-one mapping μ of A onto B is called an *isomorphism* of $\mathcal{A}$ onto $\mathcal{B}$ if $\sigma^{\mathcal{A}}(a_1, \ldots, a_m)$ if and only if $\sigma^{\mathcal{B}}(\mu(a_1), \ldots, \mu(a_m))$, for all $m \in R$, $(a_1, \ldots, a_m) \in A^m$, $\sigma \in \Sigma_m$. In this case it is said that $\mathcal{A}$ and $\mathcal{B}$ are *isomorphic*. It is easy to see that μ is an isomorphism of $\mathcal{A}$ onto $\mathcal{B}$ if and only if $(a_1, \ldots, a_{m-1})\sigma^{\mathcal{A}}\mu = (\mu(a_1), \ldots, \mu(a_{m-1}))\sigma^{\mathcal{B}}$ holds, for all $m \in R$, $\sigma \in \Sigma_m$, $(a_1, \ldots, a_{m-1}) \in A^{m-1}$.

Now let us denote by $\mathfrak{U}_R$ the class of all nondeterministic tree automata with rank-type R. A composition of nondeterministic tree automata from $\mathfrak{U}_R$ can be represented as a network in which each vertex denotes a nondeterministic tree automaton and the actual relation of a component automaton may depend only on those automata which have a direct connection to it.

In order to define this notion of composition let $\mathfrak{D}$ denote an arbitrary nonempty fixed set of finite directed graphs. We assume that the vertices of any graph in $\mathfrak{D}$ having n vertices are denoted by the numbers $1, \ldots, n$. Let $\mathcal{A} = (A, \Sigma) \in \mathfrak{U}_R$ and $\mathcal{A}_j = (A_j, \Sigma^{(j)}) \in \mathfrak{U}_R$, $j = 1, \ldots, n$. Furthermore, take a family Ψ of mappings

$$\Psi_{mj} : (A_1 \times \ldots \times A_n)^{m-1} \times \Sigma_m \to \Sigma_m^{(j)}, \quad m \in R, \ 1 \le j \le n .$$

It is said that the nondeterministic tree automaton $\mathcal{A}$ is a $\mathfrak{D}$*-product* of the automata $\mathcal{A}_j$, $j = 1, \ldots, n$, with respect to Ψ if the following conditions are satisfied:

(i) $\mathcal{A} = \prod_{j=1}^{n} \mathcal{A}_j$

(ii) there exists a graph $D = (\{1, \ldots, n\}, E)$ in $\mathfrak{D}$ such that for any $m \in R$, $j \in \{1, \ldots, n\}$ and $((a_{11}, \ldots, a_{1n}), \ldots, (a_{m-11}, \ldots, a_{m-1n})) \in A^{m-1}$, the mapping Ψ_{mj} is independent of the elements a_{ts}, $t = 1, \ldots, m-1$, if $(s, j) \notin E$,

(iii) for any $m \in R$, $\sigma \in \Sigma_m$ and
$((a_{11}, \ldots, a_{1n}), \ldots, (a_{m-11}, \ldots, a_{m-1n})) \in A^{m-1}$,
$((a_{11}, \ldots, a_{1n}), \ldots, (a_{m-11}, \ldots, a_{m-1n}))\sigma^{\mathcal{A}} =$
$(a_{11}, \ldots, a_{m-11})\sigma_1^{\mathcal{A}_1} \times \ldots \times (a_{1n}, \ldots, a_{m-1n})\sigma_n^{\mathcal{A}_n}$
where

$$\sigma_j = \Psi_{mj}((a_{11}, \ldots, a_{1n}), \ldots, (a_{m-11}, \ldots, a_{m-1n}), \sigma), \qquad j = 1, \ldots, n.$$

We shall use the notation

$$\prod_{j=1}^{n} \mathcal{A}_j(\Sigma, \Psi, D)$$

for the product introduced above. In particular, if $\mathcal{A}_j$, $j = 1, \ldots, n$, are identical copies of some nondeterministic tree automaton $\mathcal{B}$, then we speak of a *general power* and we write $\mathcal{B}^n(\Sigma, \Psi, D)$ for $\prod_{j=1}^{n} \mathcal{A}_j(\Sigma, \Psi, D)$.

Let $\mathfrak{B}$ be a system of nondeterministic tree automata from $\mathfrak{U}_R$. It is said that $\mathfrak{B}$ is *isomorphically complete for* $\mathfrak{U}_R$ *with respect to the* $\mathfrak{D}$*-product* if any nondeterministic tree automaton from $\mathfrak{U}_R$ is isomorphic to a subautomaton of a $\mathfrak{D}$-product of nondeterministic tree automata from $\mathfrak{B}$.

In the sequel we shall need a special two-state nondeterministic tree automata. For every $m \in R$, let us assign a symbol to each m-ary relation on $\{0, 1\}$. Let $\bar{\Sigma}_m$ denote the set of these symbols and let $\bar{\Sigma} = \bigcup_{m \in R} \bar{\Sigma}_m$. Define the nondeterministic tree automaton $\mathcal{G} = (\{0, 1\}, \bar{\Sigma})$ such that, for every $m \in R$ and $\sigma \in \bar{\Sigma}_m$, $\sigma^{\mathcal{G}}$ is the corresponding m-ary relation.

Now let $\mathfrak{D}$ be the set of all finite directed complete graphs having as vertices the sets $\{1, \ldots, n\}$, $n = 1, 2, \ldots$. Then the $\mathfrak{D}$-product is equal to the Glushkov-type product which is also called *general product*. We note that in this case the finite directed complete graphs are considered as possible networks. Since n determines the corresponding complete graph uniquely, we omit the graph component from the notation of the general product.

Regarding the general product, the following statement can be proved easily.

Lemma. *Let $\mathcal{A} = (A, \Sigma) \in \mathfrak{U}_R$, $\mathcal{A}_j = (A_j, \Sigma^{(j)}) \in \mathfrak{U}_R$, $j = 1, \ldots, n$, and $\mathcal{B}_{jt} = (B_{jt}, \Sigma^{(jt)}) \in \mathcal{U}_R$, $t = 1, \ldots, i_j$, $j = 1, \ldots, n$. If $\mathcal{A}$ is isomorphic to a subautomaton of a general product $\prod_{j=1}^{n} \mathcal{A}_j(\Sigma, \Psi)$ and, for each $j \in \{1, \ldots, n\}$, $\mathcal{A}_j$ is isomorphic to a subautomaton of a general product $\prod_{t=1}^{i_j} \mathcal{B}_{jt}(\Sigma^{(j)}, \Psi^{(j)})$, then $\mathcal{A}$ is isomorphic to a subautomaton of a general product of the nondeterministic tree automata $\mathcal{B}_{jt}$, $t = 1, \ldots, i_j$, $j = 1, \ldots, n$.*

The following theorem provides necessary and sufficient conditions for a system of nondeterministic tree automata from $\mathfrak{U}_R$ to be isomorphically complete for $\mathfrak{U}_R$ with respect to the general product.

Theorem 1. *A system $\mathfrak{B}$ of nondeterministic tree automata from $\mathfrak{U}_R$ is isomorphically complete for $\mathfrak{U}_R$ with respect to the general product if and only if, for all $m \in R$ and $\mathbf{i} = (i_1, \ldots, i_m) \in \{0,1\}^m$, $\mathfrak{B}$ contains a nondeterministic tree automata $\mathcal{A}^{(\mathbf{i})} = (A^{(\mathbf{i})}, \Sigma^{(\mathbf{i})})$ satisfying the following conditions:*

(1) *$A^{(\mathbf{i})}$ has two different elements $a_0^{(\mathbf{i})}$ and $a_1^{(\mathbf{i})}$,*

(2) *there exists a $\bar{\sigma}_{\mathbf{i}} \in \Sigma_m^{(\mathbf{i})}$ with $(a_{i_1}^{(\mathbf{i})}, \ldots, a_{i_{m-1}}^{(\mathbf{i})})\bar{\sigma}_{\mathbf{i}}^{\mathcal{A}^{(\mathbf{i})}} \bigcap \{a_0^{(\mathbf{i})}, a_1^{(\mathbf{i})}\} = \{a_{i_m}^{(\mathbf{i})}\}$,*

(3) *for all $u \in R$ and $\mathbf{s} = (s_1, \ldots, s_{u-1}) \in \{0,1\}^{u-1}$, there is a $\sigma_{\mathbf{i},\mathbf{s}} \in \Sigma_u^{(\mathbf{i})}$ for which $\{a_0^{(\mathbf{i})}, a_1^{(\mathbf{i})}\} \subseteq (a_{s_1}^{(\mathbf{i})}, \ldots, a_{s_{u-1}}^{(\mathbf{i})})\sigma_{\mathbf{i},\mathbf{s}}^{\mathcal{A}^{(\mathbf{i})}}$ provided that $u \neq 1$, and there is a $\sigma_{\mathbf{i}}^* \in \Sigma_1^{(\mathbf{i})}$ with $\{a_0^{(\mathbf{i})}, a_1^{(\mathbf{i})}\} \subseteq \sigma_{\mathbf{i}}^{*\mathcal{A}^{(\mathbf{i})}}$ if $1 \in R$ and $u = 1$.*

Proof. In order to prove the necessity, let us suppose that $\mathcal{B}$ is an isomorphically complete system of nondeterministic tree automata for $\mathfrak{U}_R$ with respect to the general product. Then there are $\mathcal{A}_j = (A_j, \Sigma^{(j)}) \in \mathfrak{U}_R$, $j = 1, \ldots, n$, such that $\mathcal{G}$ is isomorphic to a subautomaton $\mathcal{A} = (A, \Sigma)$ of a general product $\prod_{j=1}^{n} \mathcal{A}_j(\Sigma, \Psi)$. Let μ denote a suitable isomorphism and let

$$\mu(0) = (a_{01}, \ldots, a_{0n}) \text{ and } \mu(1) = (a_{11}, \ldots, a_{1n}).$$

Let us denote by K the set $\{k : 1 \leq k \leq n \ \& \ a_{0k} \neq a_{1k}\}$. Obviously, $K \neq \emptyset$. Now let $m \in R$ and $(i_1, \ldots, i_m) \in \{0,1\}^m$ be arbitrarily fixed elements. We distinguish two cases depending on m.

First let us suppose that $m \neq 1$. By the definition of $\mathcal{G}$, there is a $\bar{\sigma} \in \Sigma_m$ with $(i_1, \ldots, i_{m-1})\bar{\sigma}^{\mathcal{G}} = i_m$. Since μ is an isomorphism, this yields

$$(\mu(i_1), \ldots, \mu(i_{m-1}))\bar{\sigma}^{\mathcal{A}} = \mu(i_m).$$

Therefore, $a_{i_m k} \in (a_{i_1 k}, \ldots, a_{i_{m-1} k})\bar{\sigma}_k^{\mathcal{A}_k}$ holds, for all $k \in K$, where

$$\bar{\sigma}_k = \Psi_{mk}\big((a_{i_1 1}, \ldots, a_{i_1 n}), \ldots, (a_{i_{m-1} 1}, \ldots, a_{i_{m-1} n}), \bar{\sigma}\big).$$

But then there exists at least one index $l \in K$ such that

$$(a_{i_1 l}, \ldots, a_{i_{m-1} l})\bar{\sigma}_l^{\mathcal{A}_l} \bigcap \{a_{0l}, a_{1l}\} = \{a_{i_m l}\}.$$

Now let $1 \neq u \in R$ and $\mathbf{s} = (s_1, \ldots, s_{u-1}) \in \{0,1\}^{u-1}$ be arbitrary. By the definition of $\mathcal{G}$, there exists a $\sigma_{\mathbf{s}} \in \bar{\Sigma}_u$ with $(s_1, \ldots, s_{u-1})\sigma_{\mathbf{s}}^{\mathcal{G}} = \{0,1\}$. Since μ is an isomorphism, this implies

$$(\mu(s_1), \ldots, \mu(s_{u-1}))\sigma_{\mathbf{s}}^{\mathcal{A}} = \{\mu(0), \mu(1)\}.$$

Then $\{a_{0k}, a_{1k}\} \subseteq (a_{s_1 k}, \ldots, a_{s_{u-1} k})\sigma_{\mathbf{s},k}^{\mathcal{A}_k}$ holds, for all $k \in K$, where

$$\sigma_{\mathbf{s},k} = \Psi_{uk}((a_{s_1 1}, \ldots, a_{s_1 n}), \ldots, (a_{s_{u-1} 1}, \ldots, a_{s_{u-1} n}), \sigma_{\mathbf{s}}).$$

Therefore, $\{a_{0l}, a_{1l}\} \subseteq (a_{s_1 l}, \ldots, a_{s_{u-1} l})\sigma_{\mathbf{s},l}^{\mathcal{A}_l}$. If $1 \in R$ and $u = 1$, then, by the definition of $\mathcal{G}$, there is a $\sigma^* \in \bar{\Sigma}_1$ with $\sigma^* = \{0,1\}$. But then $\sigma^{*\mathcal{A}} = \{\mu(0), \mu(1)\}$, and so, $\{a_{0k}, a_{1k}\} \subseteq \sigma_k^{*\mathcal{A}_k}$, for all $k \in K$, where $\sigma_k^* = \Psi_{1k}(\sigma^*)$. Thus $\{a_{0l}, a_{1l}\} \subseteq \sigma_l^{*\mathcal{A}_l}$. This ends the proof of the necessity in the case $m \neq 1$.

Let us assume that $m = 1$. By the definition of $\mathcal{G}$, there is a $\bar{\sigma} \in \bar{\Sigma}_1$ with $\bar{\sigma}^{\mathcal{G}} = \mathbf{i}_1$. But then $\bar{\sigma}^{\mathcal{A}} = \mu(\mathbf{i}_1)$. Therefore, $a_{i_1 k} \in \bar{\sigma}_k^{\mathcal{A}_k}$ is valid, for all $k \in K$, where $\bar{\sigma}_k = \Psi_{1k}(\bar{\sigma})$. From this it follows that there exists at least one $l \in K$ such that

$$\bar{\sigma}_l^{\mathcal{A}_l} \bigcap \{a_{0l}, a_{1l}\} = \{a_{i_1 l}\}.$$

Now let $u \in R$ and $\mathbf{s} = (s_1, \ldots, s_{u-1}) \in \{0,1\}^{u-1}$ be fixed arbitrarily. In a similar way as above, it is easy to see that there is a $\sigma_{\mathbf{s},l} \in \Sigma_u^{(l)}$ such that $\{a_{0l}, a_{1l}\} \subseteq (a_{s_1 l}, \ldots, a_{s_{u-1} l})\sigma_{\mathbf{s},l}^{\mathcal{A}_l}$ if $u \neq 1$, and there is a $\sigma_l^* \in \Sigma_1^{(l)}$ with $\sigma_l^{*\mathcal{A}_l} = \{0,1\}$ if $u = 1$. This ends the proof of the necessity.

In order to prove the sufficiency, let us suppose that $\mathfrak{B}$ satisfies the conditions of Theorem 1. The isomorphic completeness of $\mathfrak{B}$ is proved in two steps.

First we show that $\mathcal{G}$ is isomorphic to a subautomaton of a general product of nondeterministic tree automata from $\mathfrak{B}$. For this reason let us denote by W the set $\bigcup_{m \in R}\{0,1\}^m$ and let $|W| = n$. Moreover, let γ denote a one-to-one mapping of the set $\{1, \ldots, n\}$ onto W. By our assumption on $\mathfrak{B}$, for any $j \in \{1, \ldots, n\}$, there exists an $\mathcal{A}^{(\gamma(j))} = (A^{(\gamma(j))}, \Sigma^{(\gamma(j))}) \in \mathfrak{B}$ satisfying conditions (1), (2) and (3) with $\mathbf{i} = \gamma(j)$. Form the general product $\prod_{j=1}^n \mathcal{A}^{(\gamma(j))}(\bar{\Sigma}, \Psi)$ in the following way.

Let $A = \{(a_0^{(\gamma(1))}, \ldots, a_0^{(\gamma(n))}), (a_1^{(\gamma(1))}, \ldots, a_1^{(\gamma(n))})\}$. Since $a_0^{(\gamma(j))} \neq a_1^{(\gamma(j))}$, $j = 1, \ldots, n$, we obtain that $|A| = 2$. Let us define the mapping μ of $\{0,1\}$ onto A by

$$\mu(0) = (a_0^{(\gamma(1))}, \ldots, a_0^{(\gamma(n))}) \text{ and } \mu(1) = (a_1^{(\gamma(1))}, \ldots, a_1^{(\gamma(n))}).$$

Now let $1 \neq m \in R$, $\sigma \in \bar{\Sigma}_m$, $(a_{i_t}^{(\gamma(1))}, \ldots, a_{i_t}^{(\gamma(n))}) \in A$, $t = 1, \ldots, m-1$, be arbitrarily fixed elements and let $\mathbf{i}^*$ denote the vector $(i_1, \ldots, i_{m-1})$. Then, for any $j \in \{1, \ldots, n\}$, let

$$\Psi_{mj}\left((a_{i_1}^{(\gamma(1))},\ldots,a_{i_1}^{(\gamma(n))}),\ldots,(a_{i_{m-1}}^{(\gamma(1))},\ldots,a_{i_{m-1}}^{(\gamma(n))}),\sigma\right)=$$

$$=\begin{cases}\bar{\sigma}_{\gamma(j)} & \text{if } \mathbf{i}^*\sigma^{\mathcal{G}}=i_m \text{ and } \gamma(j)=(i_1,\ldots,i_m),\\ \sigma_{\gamma(j),\mathbf{i}^*} & \text{if } \mathbf{i}^*\sigma^{\mathcal{G}}=i_m \text{ and } \gamma(j)\neq(i_1,\ldots,i_m),\\ \sigma_{\gamma(j),\mathbf{i}^*} & \text{if } \mathbf{i}^*\sigma^{\mathcal{G}}=\{0,1\},\\ \bar{\sigma}_{\gamma(j)} & \text{if } \mathbf{i}^*\sigma^{\mathcal{G}}=\emptyset \text{ and } (\gamma(j)=(i_1,\ldots,i_{m-1},0)\\ & \text{or } \gamma(j)=(i_1,\ldots,i_{m-1},1)),\\ \sigma_{\gamma(j),\mathbf{i}^*} & \text{if } \mathbf{i}^*\sigma^{\mathcal{G}}=\emptyset \text{ and } \gamma(j)\neq(i_1,\ldots,i_{m-1},0)\\ & \text{and } \gamma(j)\neq(i_1,\ldots,i_{m-1},1).\end{cases}$$

In all other cases when $1\neq m\in R$, let the value of Ψ_{mj} be an arbitrarily fixed element of $\Sigma_m^{(\gamma(j))}$.

If $1\in R$ and $m=1$, then the mappings Ψ_{1j}, $j=1,\ldots,n$, are defined in the following way. For any $\sigma\in\bar{\Sigma}_1$, let

$$\Psi_{1j}(\sigma)=\begin{cases}\bar{\sigma}_{\gamma(j)} & \text{if } \sigma^{\mathcal{G}}=i_1 \text{ and } \gamma(j)=(i_1),\\ \sigma^*_{\gamma(j)} & \text{if } \sigma^{\mathcal{G}}=i_1 \text{ and } \gamma(j)\neq(i_1),\\ \bar{\sigma}_{\gamma(j)} & \text{if } \sigma^{\mathcal{G}}=\emptyset \text{ and } (\gamma(j)=(0) \text{ or } \gamma(j)=(1)),\\ \sigma^*_{\gamma(j)} & \text{otherwise.}\end{cases}$$

Now consider the subautomaton $\mathcal{A}=(A,\bar{\Sigma})$ of the general product $\prod_{j=1}^n \mathcal{A}^{(\gamma(j))}(\bar{\Sigma},\Psi)$ which is determined by the set A. It is easy to show that μ is an isomorphism of $\mathcal{G}$ onto the subautomaton $\mathcal{A}$.

As a second step, we prove that an arbitrary nondeterministic tree automaton from $\mathfrak{U}_R$ is isomorphic to a subautomaton of a general power of $\mathcal{G}$. For this reason let $\mathcal{C}=(C,\Sigma)\in\mathfrak{U}_R$ be arbitrary with $C=\{c_1,\ldots,c_r\}$. Let us take all the r-dimensional column vectors with components 0, 1, and order them in lexicographically increasing order. Let $\mathbf{Q}^{(r)}$ denote the matrix formed by these column vectors. Then $\mathbf{Q}^{(r)}$ is a matrix of type $r\times 2^r$ over $\{0,1\}$ and its row vectors are pairwise different. Moreover, let us observe that for any subset V of the set $\{1,\ldots,r\}$, there exists exactly one index $k\in\{1,\ldots,2^r\}$ such that for all $t\in\{1,\ldots,r\}$, $t\in V$ if and only if $q_{tk}^{(r)}=0$. Let us define the one-to-one mapping ν of $\{c_1,\ldots,c_r\}$ onto the set of the row vectors of $\mathbf{Q}^{(r)}$ by $\nu(c_i)=(q_{i1}^{(r)},\ldots,q_{i2^r}^{(r)})$, $i=1,\ldots,r$. Let $A=\{\nu(c_i):i=1,\ldots,r\}$. Then $A\subseteq\{0,1\}^{2^r}$. Now we define the general power $\mathcal{G}^{2^m}(\Sigma,\Psi)$ in the following way.

Let $1\neq m\in R$, $\sigma\in\Sigma_m$, $(q_{i_t1}^{(r)},\ldots,q_{i_t2^r}^{(r)})\in A$, $t=1,\ldots,m-1$, be arbitrary elements. In this case $\nu(c_{i_t})=(q_{i_t1}^{(r)},\ldots,q_{i_t2^r}^{(r)})$, $t=1,\ldots,m-1$. Let us suppose that $(c_{i_1},\ldots,c_{i_{m-1}})\sigma^{\mathcal{C}}=\{c_{s_1},\ldots,c_{s_l}\}$. Then $0\leq l\leq r$. For each $j\in\{1,\ldots,2^r\}$, let us denote by V_j the set $\{q_{s_1j}^{(r)},\ldots,q_{s_lj}^{(r)}\}$. Obviously, $V_j\subseteq\{0,1\}$, $j=1,\ldots,2^r$. Thus, by the definition of $\mathcal{G}$, there exists a $\sigma_j\in\Sigma_m$ with $(q_{i_1j}^{(r)},\ldots,q_{i_{m-1}j}^{(r)})\sigma_j^{\mathcal{G}}=V_j$. Let us define the mappig Ψ_{mj} by

$$\Psi_{mj}((q_{i_11}^{(r)},\ldots,q_{i_12^r}^{(r)}),\ldots,(q_{i_{m-1}1}^{(r)},\ldots,q_{i_{m-1}2^r}^{(r)}),\sigma)=\sigma_j.$$

In all other cases when $1 \neq m \in R$, let the value of Ψ_{mj} be an arbitrarily fixed symbol from Σ_m.

If $1 \in R$, $\sigma \in \Sigma_1$, $j \in \{1, \ldots, 2^r\}$, then the mappings Ψ_{1j}, $j = 1, \ldots, 2^r$, are defined as follows. Let us assume that $\sigma^C = \{c_{s_1}, \ldots, c_{s_t}\}$ and define the sets V_j, $j = 1, \ldots, 2^r$, in the same way as above. Again, by the definition of $\mathcal{G}$, there is a $\sigma_j^* \in \Sigma_1$ with $\sigma_j^{*\mathcal{G}} = V_j$. We put

$$\Psi_{1j}(\sigma) = \sigma_j^*.$$

Now let us consider the subautomaton $\mathcal{A} = (A, \Sigma)$ of the general power $\mathcal{G}^{2^r}(\Sigma, \Psi)$. Then it is easy to see that ν is an isomorphism of $\mathcal{C}$ onto $\mathcal{A}$. By our Lemma, the above isomorphic representations imply the sufficiency of the conditions which ends the proof of Theorem 1.

Remark. If $R = \{2\}$, then $\mathfrak{U}_{\{2\}}$ is the class of all nondeterministic automata. In this case our theorem gives a characterization of the isomorphically complete systems for the class of nondeterministic automata with respect to the general product. Therefore, Theorem 1 in [4] can be obtained as a corollary of our theorem.

In [8], n-dimensional hypercubes are used as possible networks. Now we define the product related to these networks for nondeterministic tree automata and characterise the isomorphically complete systems with respect to this product.

To introduce the formal definition of cube-product we need some preparation. Let $n \geq 2$ be an arbitrary integer and consider the n-dimensional hypercube. The set of vertices of this hypercube is $S_n = \{0, 1\}^n$. Define the mapping λ_n on this set as follows: for any vector $(s_1, \ldots, s_n) \in S_n$, let

$$\lambda_n(s_1, \ldots, s_n) = 1 + \sum_{t=1}^{n} s_t \cdot 2^{n-t}.$$

Then λ_n is a one-to-one mapping of S_n onto the set $\{1, \ldots, 2^n\}$.

Let us form the graph $D_n = (\{1, \ldots, 2^n\}, E_n)$, where for any $1 \leq i, j \leq 2^n$, $(i, j) \in E_n$ if and only if $\lambda_n^{-1}(i)$ is adjacent to $\lambda_n^{-1}(j)$. For any $j \in \{1, \ldots, 2^n\}$, let us denote by $J_j^{(n)}$ the set of all ancestors of j in D_n. Then $J_j^{(n)} \subseteq \{1, \ldots, 2^n\}$.

It is easy to see that for any $n \geq 2$ and integer $j \geq 1$,

$$|J_j^{(n)}| = n \text{ if } 1 \leq j \leq 2^n, \tag{4}$$

$$J_j^{(n+1)} = \begin{cases} J_j^{(n)} \cup \{j + 2^n\} & \text{if } 1 \leq j \leq 2^n, \\ \{l + 2^n : l \in J_{j-2^n}^{(n)}\} \cup \{j - 2^n\} & \text{if } 2^n < j \leq 2^{n+1}. \end{cases} \tag{5}$$

Now let $n \geq 2$ be an arbitrary integer and let $\mathcal{A} = (A, \Sigma) \in \mathfrak{U}_R$, $\mathcal{A}_j = (A_j, \Sigma^{(j)}) \in \mathcal{U}_R$, $j = 1, \ldots, 2^n$. In addition, take a family Ψ of mappings

$$\Psi_{mj} : (A_1 \times, \ldots, \times A_{2^n})^{m-1} \times \Sigma_m \to \Sigma_m^{(j)}, \quad m \in R, \ 1 \leq j \leq 2^n.$$

It is said that the nondeterministic tree automaton $\mathcal{A} = (A, \Sigma)$ is a *cube-product* of $\mathcal{A}_j$, $j = 1, \ldots, 2^n$, with respect to Ψ if the following conditions are satisfied:

(a) $A = \prod_{j=1}^{2^n} A_j$

(b) for any $m \in R$, $\sigma \in \Sigma_m$ and $(a_{i1}, \ldots, a_{i2^n}) \in \prod_{j=1}^{2^n} A_j$, $i = 1, \ldots, m-1$, the mapping Ψ_{mj} is independent of the elements a_{ts}, $t = 1, \ldots, m-1$, if $s \notin J_j^{(n)}$,

(c) for any $m \in R$, $\sigma \in \Sigma_m$ and $((a_{11}, \ldots, a_{12^n}), \ldots, (a_{m-11}, \ldots, a_{m-12^n})) \in A^{m-1}$,

$((a_{11}, \ldots, a_{12^n}), \ldots, (a_{m-11}, \ldots, a_{m-12^n}))\sigma^{\mathcal{A}} =$

$(a_{11}, \ldots, a_{m-11})\sigma_1^{\mathcal{A}_1} \times \ldots \times (a_{12^n}, \ldots, a_{m-12^n})\sigma_{2^n}^{\mathcal{A}_{2^n}}$,

where

$$\sigma_j = \Psi_{mj}((a_{11}, \ldots, a_{12^n}), \ldots, (a_{m-11}, \ldots, a_{m-12^n}), \sigma), \qquad j = 1, \ldots, 2^n.$$

Since n determines the hypercube uniquely, we use the notation $\prod_{j=1}^{2^n} \mathcal{A}_j(\Sigma, \Psi)$ for the cube-product just introduced.

Now we are ready to prove the following statement.

Theorem 2. *A system $\mathfrak{B}$ of nondeterministic tree automata from $\mathfrak{U}_R$ is isomorphically complete for $\mathfrak{U}_R$ with respect to the cube-product if and only if, for all $m \in R$ and $\mathbf{i} = (i_1, \ldots i_m) \in \{0,1\}^m$, $\mathfrak{B}$ contains a nondeterministic tree automata $\mathcal{A}^{(\mathbf{i})} = (A^{(\mathbf{i})}, \Sigma^{(\mathbf{i})})$ satisfying the following conditions:*

(6) *$A^{(\mathbf{i})}$ has two different elements $a_0^{(\mathbf{i})}$ and $a_1^{(\mathbf{i})}$,*

(7) *there exists a $\bar{\sigma}_{\mathbf{i}} \in \Sigma_m^{(\mathbf{i})}$ with $(a_{i_1}^{(\mathbf{i})}, \ldots, a_{i_{m-1}}^{(\mathbf{i})})\bar{\sigma}_{\mathbf{i}}^{\mathcal{A}^{(\mathbf{i})}} \bigcap \{a_0^{(\mathbf{i})}, a_1^{(\mathbf{i})}\} = \{a_{i_m}^{(\mathbf{i})}\}$,*

(8) *for all $u \in R$ and $\mathbf{s} = (s_1, \ldots, s_{u-1}) \in \{0,1\}^{u-1}$, there is a $\sigma_{\mathbf{i},\mathbf{s}} \in \Sigma_u^{(\mathbf{i})}$ for which $\{a_0^{(\mathbf{i})}, a_1^{(\mathbf{i})}\} \subseteq (a_{s_1}^{(\mathbf{i})}, \ldots, a_{s_{u-1}}^{(\mathbf{i})})\sigma_{\mathbf{i},\mathbf{s}}^{\mathcal{A}^{(\mathbf{i})}}$ provided that $u \neq 1$, and there is a $\sigma_{\mathbf{i}}^* \in \Sigma_1^{(\mathbf{i})}$ with $\{a_0^{(\mathbf{i})}, a_1^{(\mathbf{i})}\} \subseteq \sigma_{\mathbf{i}}^{*\mathcal{A}^{(\mathbf{i})}}$ if $1 \in R$ and $u = 1$.*

Proof. The necessity follows from the proof of Theorem 1. In order to prove the sufficiency, let us denote by W the set $\bigcup_{m \in R}\{0,1\}^m$ and let $W' = \{(i_1, \ldots, i_m) : (i_1, \ldots, i_m) \in W \ \&\ i_m = 0\}$. Let $|W'| = n$ and let γ denote a one-to-one mapping of the set $\{1, \ldots, n\}$ onto W'. Then, by our assumption on $\mathfrak{B}$, for any $p \in \{1, \ldots, n\}$, there exists a nondeterministic tree automaton $\mathcal{A}^{(\gamma(p))} = (A^{(\gamma(p))}, \Sigma^{(\gamma(p))}) \in \mathfrak{B}$ satisfying conditions (6), (7) and (8) with $\mathbf{i} = (i_1, \ldots, i_m) = \gamma(p)$, where $i_m = 0$. For the sake of simplicity, let us denote by 0, 1 the elements $a_0^{(\gamma(p))}$, $a_1^{(\gamma(p))}$, respectively, for all $p \in \{1, \ldots, n\}$.

Now consider the matrices $\mathbf{Q}^{(k)}$, $k = 2, 3, \ldots$, introduced in the proof of Theorem 1. In our argument we make use of some properties of these matrices. First let us observe that

$$\mathbf{Q}^{(k+1)} = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{Q}^{(k)} & \mathbf{Q}^{(k)} \end{pmatrix} \tag{9}$$

where $\mathbf{0}$ and $\mathbf{1}$ denote the costant vectors of sise 1×2^k with components 0 and 1, respectively. On the other hand, it can be seen (cf. [8] or [9]) that

(10) for any $k \geq 2$ and $1 \leq j \leq 2^k$, the $k+1$-tuples $(q_{tj}^{(k)}, q_{tj_1}^{(k)}, \ldots, q_{tj_k}^{(k)})$, $t = 1, \ldots, k$, are pairwise different where $\{j_1, \ldots, j_k\} = J_j^{(k)}$.

Now using (5), (9) and (10), it is easy to see that

(11) for any $k \geq 2$, $k > s \geq 1$, $1 \leq j \leq 2^k$, the k-tuples $(q_{tj_1}^{(k)}, \ldots, q_{tj_k}^{(k)})$, $t = s+1, \ldots, k$, are pairwise different where $\{j_1, \ldots, j_k\} = J_j^{(k)}$.

Now let $\mathcal{C} = (\{c_1, \ldots, c_r\}, \Sigma)$ be an arbitrary nondeterministic tree automaton from $\mathfrak{U}_R$. We prove that $\mathcal{C}$ is isomorphic to a subautomaton of a cube-product of nondeterministic tree automata from $\mathfrak{B}$.

For this purpose, let us denote by s the least positive integer with $n \leq 2^s$. Let $k = r + s$. Delete the first s rows of $\mathbf{Q}^{(k)}$. Then, by (9), the resulting matrix consists of 2^s copies of $\mathbf{Q}^{(r)}$ in its partitioned form. Let $\mathbf{Q}$ denote this matrix. For the sake of simplicity, let us denote by q_{tj}, $t = 1, \ldots, r$, $j = 1, \ldots, 2^k$, the elements of $\mathbf{Q}$. Then from (11) it follows that

(12) for any $1 \leq j \leq 2^k$, the k-tuples $(q_{tj_1}, \ldots, q_{tj_k})$, $t = 1, \ldots, r$, are pairwise different where $J_j^{(k)} = \{j_1, \ldots, j_k\}$.

Let us define the one-to-one mapping μ of $\{c_1, \ldots, c_r\}$ onto the set of the row vectors of $\mathbf{Q}$ by $\mu(c_i) = (q_{i1}, \ldots, q_{i2^k})$, $i = 1, \ldots, r$, and let $B = \{\mu(c_i) : i = 1, \ldots, r\}$.

Form the cube-product

$$\underbrace{\mathcal{A}^{(\gamma(1))}\times\ldots\times\mathcal{A}^{(\gamma(1))}}_{2^r\text{ times}}\times\ldots\times\underbrace{\mathcal{A}^{(\gamma(n))}\times\ldots\times\mathcal{A}^{(\gamma(n))}}_{2^r\text{ times}}\times$$
$$\times\underbrace{\mathcal{A}^{(\gamma(1))}\times\ldots\times\mathcal{A}^{(\gamma(1))}}_{2^k-n2^r\text{ times}}(\Sigma,\Psi)$$

in the following way. Observe that
$B\subseteq A=$

$$=\underbrace{A^{(\gamma(1))}\times\ldots\times A^{(\gamma(1))}}_{2^r\text{ times}}\times\ldots\times\underbrace{A^{(\gamma(n))}\times\ldots\times A^{(\gamma(n))}}_{2^r\text{ times}}\times\underbrace{A^{(\gamma(1))}\times\ldots\times A^{(\gamma(1))}}_{2^k-n2^r\text{ times}}.$$

Now let $1\neq m\in R$, $\sigma\in\Sigma_m$, $(q_{i_t1},\ldots,q_{i_t2^k})\in B$, $t=1,\ldots,m-1$, be arbitrary elements. Then $\mu(c_{i_t})=(q_{i_t1},\ldots,q_{i_t2^k})$, $t=1,\ldots,m-1$. Let us assume that $(c_{i_1},\ldots,c_{i_{m-1}})\sigma^C=\{c_{v_1},\ldots,c_{v_l}\}$. Then $0\leq l\leq r$. By the structure of $\mathbf{Q}$, there exists exactly one integer $d\in\{1,\ldots,2^r\}$ such that for each $p\in\{1,\ldots,2^s\}$, the following assertion is valid:

for all $t\in\{1,\ldots,r\}$, $q_{t,(p-1)2^r+d}=0$ if and only if $t\in\{v_1,\ldots,v_l\}$.

On the other hand, let us observe that the column vectors of $\mathbf{Q}$ with indices $(p-1)2^r+d$, $p=1,\ldots,2^s$, are identical copies of some r-dimensional vector over $\{0,1\}$. Therefore, the vectors $(q_{i_1,(p-1)2^r+d},\ldots,q_{i_{m-1},(p-1)2^r+d})$, $p=1,\ldots,2^s$, are the copies of an $(m-1)$-dimensional vector $(i'_1,\ldots,i'_{m-1})$ over $\{0,1\}$. Now let $\mathbf{i}=(i'_1,\ldots,i'_{m-1},0)$. Since $\mathbf{i}\in W'$, there exists one and only one $p_0\in\{1,\ldots,n\}$ with $\gamma(p_0)=\mathbf{i}$. Let $j_0=(p_0-1)2^r+d$. Then for each $j\in\{1,\ldots,2^k\}$, the mapping Ψ_{mj} is defined by

$$\Psi_{mj}((q_{i_11},\ldots,q_{i_12^k}),\ldots,(q_{i_{m-1}1},\ldots,q_{i_{m-1}2^k}),\sigma)=$$
$$\begin{cases}\bar{\sigma}_{\mathbf{i}} & \text{if } j=j_0,\\ \sigma_{\gamma(p),(i'_1,\ldots,i'_{m-1})} & \text{if } j\neq j_0 \text{ and } (p-1)2^r<j\leq p2^r\\ & \text{for some } p\in\{1,\ldots,2^s\}.\end{cases}$$

In all other cases when $1\neq m\in R$, Ψ_{mj} can be defined arbitrarily in accordance with the definition of the cube-product.

Now let us suppose that $1\in R$. Let $\sigma\in\Sigma_1$ be arbitrary and $\sigma^C=\{c_{v_1},\ldots,c_{v_l}\}$. Then there exists again exactly one integer $d\in\{1,\ldots,2^r\}$ such that the following statement holds for each $p\in\{1,\ldots,2^s\}$:

for all $t\in\{1,\ldots,r\}$, $q_{t,(p-1)2^r+d}=0$ if and only if $t\in\{v_1,\ldots,v_l\}$.

In this case $(0)\in W'$, and so, there is one and only one $p_0\in\{1,\ldots,n\}$ with $\gamma(p_0)=(0)$. Let $j_0=(p_0-1)2^r+d$. For each $j\in\{1,\ldots,2^k\}$, let us define the mapping Ψ_{1j} as follows.

$$\Psi_{1j}(\sigma)=\begin{cases}\bar{\sigma}_{(0)} & \text{if } j=j_0,\\ \sigma^*_{\gamma(p)} & \text{if } j\neq j_0 \text{ and } (p-1)2^r<j\leq p2^r \text{ for some } p\in\{1,\ldots,2^s\}.\end{cases}$$

By (12), the mappings Ψ_{mj}, $m \in R$, $1 \leq j \leq 2^k$, are well-defined. On the other hand, it is easy to see that the mapping μ is an isomorphism of C onto that subautomaton of the defined cube-product which is determined by the set B. Therefore, $\mathfrak{B}$ is isomorphically complete for $\mathfrak{U}_R$ with respect to the cube-product. This ends the proof of Theorem 2.

Remark. In the case $R = \{2\}$ we obtain a characterization of the isomorphically complete systems for the class of nondeterministic automata with respect to the cube-product. Therefore, the main result of [5] can be obtain as a corollary of Theorem 2.

Notice that the necessary and sufficient conditions stated by Theorem 1 and Theorem 2 are the same which gives us the following corollary.

Corollary. *A system of nondeterministic tree automata from $\mathfrak{U}_R$ is isomorphically complete for $\mathfrak{U}_R$ with respect to the general product if and only if it is isomorphically complete for $\mathfrak{U}_R$ with respect to the cube-product.*

References

[1] Z. Ésik, Homomorphically complete classes of automata with respect to the α_2-product, *Acta Sci. Math.*, **48** (1985), 135-141.

[2] F. Gécseg, Composition of automata, *Automata, Languages and Programming,2nd Colloquium*, Saarbrücken, 1974, Lecture Notes in Computer Science (Springer-Verlag, Berlin Heidelberg New York Tokyo) **14** (1974), 351-363.

[3] F. Gécseg, *Products of Automata*, Springer-Verlag, Berlin Heidelberg New York Tokyo (1986).

[4] F. Gécseg and B. Imreh, On completeness of nondeterministic automata, *Acta Math. Hungarica*, **68** (1995), 151-159.

[5] F. Gécseg and B. Imreh, On the cube-product of nondeterministic automata *Acta Sci. Math.*, to appear.

[6] F. Gécseg and M. Steinby, *Tree automata,* Akadémiai Kiadó, Budapest, 1984.

[7] V. M. Glushkov, Abstract theory of automata, *Uspekhi Mat. Nauk*, **16:5 101** (1961), 3-62 (in Russian).

[8] B. Imreh, On complete systems of automata, in: *Proc. of the 2nd International Colloquium on Words, Languages and Combinatorics*, Kyoto, 1992, World Scientific (Singapore-New Jersey-London-Hong Kong), 1994, 207-215.

[9] B. Imreh, On a special composition of tree automata, *Acta Cybernetica* **10** (1992), 237-242.

[10] M. Steinby, On the structure and realizations of tree automata, in *Second Coll. sur les Arbres an Algèbre et en Programmation* Lille, 1979, 235-248.

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