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On graphs with perfect internal matchings*

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Abstract

Graphs with perfect internal matchings are studied as underlying objects of certain molecular switching devices called soliton automata. A perfect internal matching of a graph is a matching that covers all vertices of the graph, except possibly those with degree one. Such a matching is called a state of the graph. It is proved that for every two states there exists a so called mediator alternating network which can be used as a switch between those two states. As a consequence of this result it is shown how transitions of soliton automata can be decomposed into a sequence of simpler moves. Elementary graphs having a perfect internal matching are defined through an equivalence relation on their edges. Another equivalence relation on the set of vertices is introduced to characterize the well-known canonical partition of elementary graphs in the new generalized sense.

1 Introduction

The results of this paper were motivated by the developments of a research aiming to construct a computer based on molecular switching components. Molecules exhibiting a switching behavior have long been investigated by chemists, cf. [4], but it was not until recently that the first mathematical model of a switching molecular device was introduced in [5] under the name soliton automaton.

The underlying object of a soliton automaton is a so called soliton graph, which is a finite undirected graph modeling the topological structure of a molecule. Atoms are represented by vertices and chemical bonds by edges. The multiplicity of bonds (single or double) is set by a weight assignment to the edges of the corresponding soliton graph. It is assumed that the molecule consists of carbon and hydrogen atoms only, and that among the neighbors of each carbon atom there exists a unique one to which the atom is connected by a double bond. This latter property can nicely be captured by the concept of matching in graphs.

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The above topological model of molecules has already been used earlier to study some other properties of a chemical compound. The reader is referred to [10, Section 8.7] for a detailed discussion on Hückel graphs, which are models of molecules having the same alternating pattern of single and double bonds that we are concerned with in this paper. The only essential difference between Hückel graphs and soliton graphs in terms of matching theory is the following. Hückel graphs are generally supposed to have a perfect matching, whereas in soliton graphs only the internal vertices (i.e. those with degree greater than one) are required to be covered by an appropriate matching. Such a matching is called a perfect internal matching. Vertices with degree one are considered to be external in soliton graphs. The collection of such vertices is treated as an interface for the internal part of the graph, so that these vertices need not be covered by a perfect internal matching.

A state of a soliton automaton is a perfect internal matching of the underlying soliton graph. State transitions are induced by directing a particle (electron, soliton) from one external vertex of the graph to another or even the same external vertex along an alternating walk. Making the walk will then result in a new state by switching all the bonds to the opposite throughout the walk in a dynamic way. For more details, see e.g. [5].

The aim of this paper is to develop a suitable mathematical arsenal for the study of soliton automata. There has already been some previous work done towards this goal. Soliton automata with some special properties have been investigated in [6], [7] and [8]. In [1], an algebraic framework has been introduced to provide a new calculus for dealing with finite undirected multigraphs. Concerning matchings, the Gallai-Edmonds Structure Theorem has been proved for maximum internal matchings in [2]. This theorem plays a central role in the algebra of graphs having a perfect internal matching, which has been described in [3]. Although the present paper is self-contained, some familiarity with [10] will be helpful for the reader.

2 Review of basic concepts and notations

By a graph we mean, throughout the paper, a finite undirected non-empty graph with loops and multiple edges allowed. If G is a graph, then $V(G)$ and $E(G)$ will denote the set of vertices and the set of edges of G, respectively. An edge $e \in E(G)$ connects two vertices $v_1, v_2 \in V(G)$, which are called the *endpoints* of e, and e is said to be *incident* with v_1 and v_2. If $v_1 = v_2$, then e is called a loop around v_1. Two edges sharing at least one endpoint are said to be *adjacent* in G.

For a vertex v in graph G, we define the *degree* of v to be the number of occurrences of v as an endpoint of some edge in $E(G)$. By this definition, the endpoints of a loop are considered to be two different occurrences of the same vertex. The vertex v is called *isolated* if its degree $d(v)$ is zero, *external* if $d(v) = 1$ and *internal* if $d(v) \geq 2$. An edge $e \in E(G)$ is an *internal edge* if both endpoints of e are internal. *External edges* are those that are incident with at least one external vertex.

A *matching* of/in graph G is a subset $M \subseteq E(G)$ such that no vertex of G

occurs more than once as an endpoint of some edge in M. Again, it is understood that loops, having themselves two occurrences of the same endpoint, cannot be present in M. The endpoints of the edges contained in matching M are said to be *covered* by M. A matching M is called *perfect* if it covers all of $V(G)$. A *perfect internal matching* is one that covers all the internal vertices of G. An edge $e \in E(G)$ is *allowed* (*mandatory*) if e is contained in some (respectively, all) perfect internal matching(s) of G. *Forbidden* edges are those that are not allowed. A perfect internal matching in G will also be referred to as a *state* of G.

In graph G, a *trail* is a sequence $\alpha = e_1, \ldots, e_n$ $(n \geq 0)$ of distinct adjacent edges $e_i \in E(G)$, $i \in [n] = \{1, \ldots, n\}$ such that no vertex of G occurs more than twice as an endpoint of some e_i. The integer n is said to be the *length* of the trail α. If, in addition, e_n is adjacent to e_1, or $n = 0$, then α is called a *cycle*, otherwise α is a *path*. In the latter case, if e_1 and e_n are both external edges, then α is said to be a *crossing*. Note that every trail α in G can be uniquely specified as an appropriate connected subgraph of G if we are not concerned about the way α is actually traversed. Moreover, if α is non-empty, then this subgraph can be uniquely identified with the set of edges contained in α. Two trails are said to be *disjoint* if they are such as subgraphs of G. Since in this paper, except for Section 5, we shall not be interested in the traversal of trails, we shall often refer to a non-empty trail $\alpha = e_1, \ldots, e_n$ as a set, i.e. as $\alpha = \{e_1, \ldots, e_n\}$, without causing any confusion. Note, however, that disjointness of two trails is ambiguous in general under this assumption.

Let M be a perfect internal matching of G. A trail $\alpha = e_1, \ldots, e_n$ is an *alternating trail* with respect to M (M-alternating trail, for short) if for every $i \in [n-1]$, $e_i \in M$ iff $e_{i+1} \notin M$. An M-alternating trail α is called *complete* if α is either a crossing or it is a non-empty even length cycle. An *alternating network* with respect to M (or M-alternating network) is a set of pairwise disjoint, complete M-alternating trails. Observe that if two complete alternating trails are running on disjoint sets of edges, then they must be disjoint as subgraphs of G, too. Thus, identifying complete alternating trails with the set of their edges does not cause ambiguity regarding the disjointness of such trails. Also note that, although an M-alternating network Γ consists of non-empty trails only, the network Γ itself can be empty.

Let M be a state (i.e., a perfect internal matching) of graph G and α be a complete alternating trail. By *making* α in state M we mean exchanging the status of the edges in α regarding their being present or not present in M. It is easy to see that this process results in another perfect internal matching of G, which will be denoted by $S_G(M, \alpha)$ or simply by $S(M, \alpha)$ if G is understood. Making an M-alternating network Γ in state M means making all the trails of Γ simultaneously in M. Since the trails contained in Γ do not intersect each other, the resulting state, denoted $S_G(M, \Gamma)$, is well-defined.

3 Characterizing state transitions by alternating networks

Our starting theorem relates two arbitrary states of a graph by means of a suitable alternating network that takes the one state into the other. This theorem also manifests the basic inductive proof technique applied in the paper: to obtain a simpler graph, cut an internal edge of the graph at hand and make a correspondence between the complete alternating trails of the original graph and those of the cut graph. Thus, the induction eventually goes by the number of internal edges.

Theorem 3.1 *For any two states M_1, M_2 of graph G, there exists a unique M_1-alternating network Γ for which $S_G(M_1, \Gamma) = M_2$.*

Proof. We prove the existence of Γ by induction on the number of internal edges of G. If G has no internal edges, then all of its components are either star graphs or single edges connecting two external vertices. In such components we can switch from one state to another by making a straightforward crossing.

Suppose now that G has at least one internal edge e, and assume that the assertion holds true for all graphs having fewer internal edges than G. Let v_1 and v_2 denote the two endpoints of e. We cut e by replacing it with two new external edges e_1 and e_2 that are incident with v_1 and v_2, respectively. Let G' denote the resulting graph. Obviously, G' has fewer internal edges than G and it has a perfect internal matching. Moreover, the perfect internal matchings of G are in a one-to-one correspondence with those perfect internal matchings M' of G' for which $e_1 \in M'$ iff $e_2 \in M'$. Applying the induction hypothesis for graph G' and states M_1', M_2' corresponding to the states M_1, M_2 of G, we obtain an M_1'-alternating network $\Gamma' = \{\alpha_1', \ldots, \alpha_k'\}$ in G' satisfying $S_{G'}(M_1', \Gamma') = M_2'$. If neither e_1 nor e_2 is present in any of the trails of Γ', then by putting $\Gamma = \Gamma'$ we are through. Otherwise one of the two cases below is met.

Case 1. There exists a unique $j \in [k]$ such that α_j' is a crossing which connects e_1 to e_2. See Fig. 1a. Since M_1' corresponds to state M_1 of G, we have $e_1 \in M_1'$ iff $e_2 \in M_1'$. This implies that the length of α_j' is odd. Remerging e_1 with e_2 then gives rise to an alternating cycle α_j in G with respect to M_1. Moreover, making α_j' in G' and remerging e_1 with e_2 after has the same effect as making α_j directly in G. Making the network $\Gamma = \Gamma' - \alpha_j' \cup \alpha_j$ in G will therefore take state M_1 to state M_2 as required.

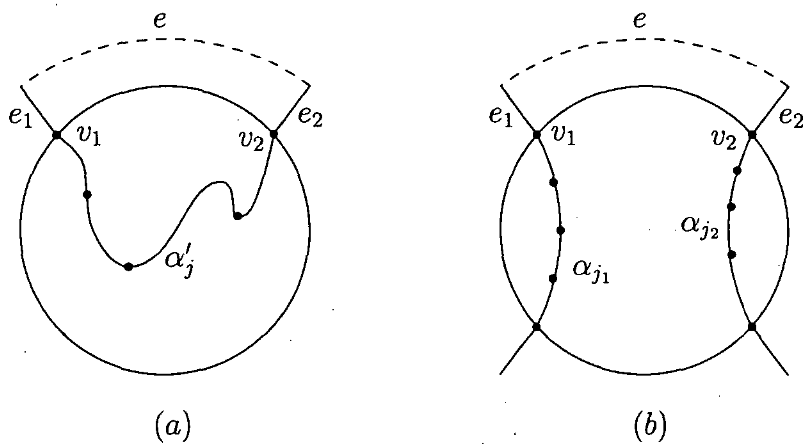

Figure 1.

Case 2. There exist two different crossings α'_{j_i}, $i = 1, 2$, $j_i \in [k]$ such that $e_i \in \alpha'_{j_i}$. See Fig. 1b. In this case the remerging of e_1 and e_2 results in an M_1-alternating crossing α in G. Using the same argument as in Case 1, the desired alternating network is obtained as $\Gamma = \Gamma' - \{\alpha'_{j_1}, \alpha'_{j_2}\} \cup \alpha$.

To prove the uniqueness of Γ we need the following lemma.

Lemma 3.2 *For a graph G, let $\mathcal{C} = \{C_1, \ldots, C_n\}$ and $\mathcal{D} = \{D_1, \ldots, D_m\}$ be two sets of pairwise disjoint connected subgraphs of G. If $\cup\mathcal{C} = \cup\mathcal{D}$, then $\mathcal{C} = \mathcal{D}$.*

Proof. By symmetry it is sufficient to prove that for every $j \in [m]$ there exists some $i \in [n]$ such that $D_j = C_i$. Since the subgraphs contained in $\mathcal{C}$ are pairwise disjoint and D_j is connected, D_j is a subgraph of some C_i, $i \in [n]$. But then D_j must be equal to C_i, otherwise C_i would be covered by more than one subgraph taken from $\mathcal{D}$, contradicting the fact that C_i is connected. □

Now we turn back to the proof of Theorem 3.1. Let us assume that Γ and Δ are complete alternating trails such that $S(M_1, \Gamma) = S(M_1, \Delta) = M_2$. Obviously, both $\cup\Gamma$ and $\cup\Delta$ consist of exactly those edges $e \in E(G)$ for which $e \in M_1$ iff $e \notin M_2$. Therefore $\cup\Gamma = \cup\Delta$, and by Lemma 3.2, $\Gamma = \Delta$.

□

Observe that Theorem 3.1 is symmetric in M_1 and M_2, for Γ is an alternating network with respect to M_1 iff it is one with respect to $M_2 = S_G(M_1, \Gamma)$. In other words,

$$M_1 = S_G(S_G(M_1, \Gamma), \Gamma).$$

The network Γ is called the *mediator* alternating network between states M_1 and M_2, and is denoted by $\mathrm{Med}(M_1, M_2)$.

Let us fix a graph G having a perfect internal matching for Sections 3 and 4. An edge $e \in E(G)$ is said to be *constant* in state M of G if no complete M-alternating trail passes through e.

Corollary 3.3 *An edge e is constant in some state of G iff e is either forbidden* *or mandatory.*

Proof. By Theorem 3.1, e is constant in some state of G iff it is such in all states of G. □

Now we recall the concept of impervious edge from [5]. Although our definition is different from [5, Definition 4.2], it is easy to see that the concepts captured by the two definitions are essentially the same.

Definition 3.4 An edge $e \in E(G)$ is *viable* in state M if there exists an M-alternating path $e_1, \ldots, e_n$ from some external vertex of G to one of the endpoints of e such that

(i) $e \neq e_i$ for any $i \in [n]$;

(ii) e_n and e are M-alternating in the sense that $e_n \in M$ iff $e \notin M$.

The edge e is *impervious* if it is not viable (in state M).

Intuitively, e is viable in state M if there exists an M-alternating path that starts from an external vertex, reaches one endpoint of e without passing through e itself, and it can be continued on e in an alternating fashion. This continuation, however, need not be a path as shown by Fig. 2. In Fig. 2, double lines indicate edges belonging to the matching M rather than multiple edges in the graph G. The reader is referred to [5, Figure 11] for examples of impervious edges in graphs that are connected and have external vertices, too.

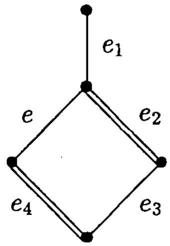

Figure 2.

Corollary 3.5 *An edge is impervious in some state of G iff it is impervious in all states of G.*

Proof. It is sufficient to prove that if $e \in E(G)$ is viable in some state M_1, then it is viable in any other state M_2. Assuming that e is viable in state M_1, let us cut e as described in the proof of Theorem 3.1 to obtain a graph G' with two new external edges e_1, e_2. Again, let M_1' and M_2' be the states of G' corresponding to M_1 and M_2, respectively. By assumption, there exists an M_1'-alternating crossing α' in G' containing exactly one of e_1 and e_2. It follows that exactly one of e_1 and e_2 will be present in some crossing of $\mathrm{Med}(M_2', S_{G'}(M_1', \alpha'))$. From this crossing, after remerging e_1 with e_2, we obtain a suitable M_2-alternating path in G that reaches one endpoint of e and can be continued on e in an alternating fashion. □

4 Elementary equivalence and canonical partition of elementary graphs

Recall from [10] that a graph G is elementary if it has a perfect matching and its allowed edges form a connected subgraph. If G has only a perfect internal matching, then consider the equivalence relation ϵ on $E(G)$ by which $e_1 \,\epsilon\, e_2$ iff either $e_1 = e_2$ or e_1 and e_2 are in the same connected component of the restriction of G to its allowed edges. Our aim is to characterize the relation ϵ in terms of complete alternating trails.

Definition 4.1 Two complete alternating trails α and β with respect to the same state M of G are *conjugated* if $\mathrm{Med}(S(M,\alpha), S(M,\beta))$ is a singleton.

It is immediate by the above definition that if α and β are conjugated, then they must intersect each other without being identical themselves. Indeed, if α and β are complete alternating trails, then $\alpha \cap \beta = \emptyset$ implies that $\mathrm{Med}(S(M,\alpha), S(M,\beta)) = \{\alpha, \beta\}$ and $\alpha = \beta$ implies that $\mathrm{Med}(S(M,\alpha), S(M,\beta)) = \emptyset$. (Remember that all complete alternating trails are non-empty, by definition.)

Lemma 4.2 *Let α and β be two complete alternating trails with respect to the same state M. Then, for every edge $e \in \beta - \alpha$ there exists a complete alternating trail γ with respect to some $\hat{M} \in \{M, S(M,\alpha)\}$ passing through e such that α and γ are either conjugated or disjoint.*

Proof. Let $n_\beta(\alpha)$ be the number of edges contained in $\beta - \alpha$. The proof is an inductive argument on $n_\beta(\alpha)$. The basis case $n_\beta(\alpha) = 0$ is trivial.

Let $n_\beta(\alpha) \geq 1$, and assume that the assertion of the lemma holds for all triples (α', β', M') such that α' and β' are complete alternating trails with respect to state M', and $n_{\beta'}(\alpha') < n_\beta(\alpha)$. Choose β' to be the complete alternating trail of the network $\Gamma = \mathrm{Med}(S(M,\alpha), S(M,\beta))$ containing e. Since the trails of Γ are running exclusively on those edges of α and β that are not contained in their intersection, $n_{\beta'}(\alpha) \leq n_\beta(\alpha)$. Moreover, $n_{\beta'}(\alpha) = n_\beta(\alpha)$ iff either $\beta' = \beta$ is disjoint from α or Γ is a singleton, in which cases there is nothing to prove. If, however, $n_{\beta'}(\alpha) < n_\beta(\alpha)$, then the induction hypothesis can be applied for $\alpha' = \alpha$ and β', which are complete alternating trails with respect to state $M' = S(M,\alpha)$. To complete the proof, one must take into account that $S(S(M,\alpha),\alpha) = M$.

□

Corollary 4.3 *Let α be a complete alternating trail in G with respect to state M, and let $e \in E(G)$ be an allowed edge adjacent to some edge in α. Then for every $e' \in \alpha$ there exists a complete alternating trail δ with respect to some $\hat{M} \in \{M, S(M,\alpha)\}$ which contains both e and e'.*

Proof. We can assume that $e \notin \alpha$. If e were mandatory, then every edge adjacent to e would be forbidden, contradicting the fact that all the edges of α are allowed. Thus, by Corollary 3.2, there exists a complete alternating trail β with $e \in \beta - \alpha$. Applying Lemma 4.2 we obtain a complete alternating trail γ with respect to some $\hat{M} \in \{M, S(M,\alpha)\}$ which also contains e and, moreover, α and γ are conjugated.

It is now clear that the required complete alternating trail δ can be chosen either as $\delta = \gamma$ or as $\delta = \mathrm{Med}(S(\hat{M},\alpha), S(\hat{M},\gamma))$, depending on whether γ passes through e or not.

□

Now we redefine the relation of elementary equivalence introduced at the beginning of this section.

Definition 4.4 Two edges e_1, e_2 are *elementary equivalent* if either $e_1 = e_2$ or there exists a complete alternating trail with respect to some state of G containing both e_1 and e_2.

The relation of elementary equivalence will be denoted by ϵ.

Theorem 4.5 *Elementary equivalence is an equivalence relation on $E(G)$.*

Proof. We only have to address transitivity of ϵ. Let e_1, e_2, e_3 be such that $e_1 \,\epsilon\, e_2$ and $e_2 \,\epsilon\, e_3$. Then there exists a complete M_1-alternating trail α joining e_1 to e_2 and a complete M_2-alternating trail joining e_2 to e_3, where M_1 and M_2 are some states of G. It follows that e_3 can be reached from some vertex lying on α by a path τ consisting of allowable edges only. Using Corollary 4.3, a straightforward induction on the length of τ shows that there exists a complete alternating trail δ with respect to some state M_3 which contains both e_1 and e_3. Thus, $e_1 \,\epsilon\, e_3$, which was to be proved.

□

It turns out from the proof of Theorem 4.5 that if e_1 and e_2 are adjacent edges in G, then either $e_1 \,\epsilon\, e_2$ or one of e_1 and e_2 is forbidden. This means that the relation ϵ coincides with the one that we intended to characterize at the beginning of this section. Spelling this out, the equivalence classes of ϵ that are different from a single forbidden edge are exactly the connected components of the restriction of G to its allowed edges.

Consider the relation ϵ_V on $V(G)$ by which $v_1 \,\epsilon_V\, v_2$ iff either $v_1 = v_2$ or v_1 and v_2 can be connected by a complete alternating trail with respect to some state of G. By virtue of Theorem 4.5, ϵ_V is also an equivalence relation. Slightly modifying the original definition given in [10], we call G *elementary* if ϵ_V is the universal relation on $V(G)$. Note that if G is elementary, then the relation ϵ is not necessarily universal on $E(G)$, for G might contain some forbidden edges as well.

For the rest of this section we shall assume that G is elementary. Our goal is to find the analog of the canonical partition $\mathcal{P}(G)$ of $V(G)$, where G is a graph with a perfect matching, for the case when G has just a perfect internal matching. The partition $\mathcal{P}(G)$ has been described in [10, Theorem 5.2.2] in many different ways, based on the concepts of extreme set and barrier. Unfortunately, we have not been able to generalize these concepts for graphs with perfect internal matchings yet, but we can still characterize $\mathcal{P}(G)$ by the following very simple relation $\sim$.

Definition 4.6 For two vertices $v_1, v_2 \in V(G)$, $v_1 \sim v_2$ if an extra edge e connecting v_1 and v_2 becomes forbidden in $G + e$ (i.e. in the extension of G by e).

According to part (b) of [10, Theorem 5.2.2], if G does not contain loops and external vertices, then $\sim$ is an equivalence relation and $\mathcal{P}(G)$ is the partition induced by $\sim$. Here we prove that $\sim$ is an equivalence anyhow.

Theorem 4.7 *The relation $\sim$ is an equivalence on $V(G)$.*

Proof. Since loops are forbidden edges, $\sim$ is reflexive. It remains to show the transitivity of $\sim$. Let $v_1 \sim v_2$ and $v_2 \sim v_3$ for distinct vertices $v_1, v_2, v_3 \in V(G)$. We have to prove that an extra edge e connecting v_1 with v_3 becomes forbidden in $G+e$. Assume, on the contrary, that e is allowed. It is clear that e is not mandatory, hence $G + e$ is still elementary. Moreover, due to the elementary property, there is an allowed edge e' incident with v_2 such that $e \,\epsilon\, e'$ holds in $G + e$. Therefore, by Theorem 4.5, there exists a complete alternating trail α with respect to some state M of $G + e$ containing e and reaching v_2 on the way. We distinguish two cases.

Case 1: α is an even length cycle, see Fig. 3a.

Since each of the edges (v_1, v_2) and (v_2, v_3) would become forbidden when adding them to G, the subpaths of α connecting v_1 with v_2 and v_2 with v_3 are of even length. Thus, together with e, the length of α turns odd, which is a contradiction.

Case 2: α is a crossing that connects two external vertices x, y, see Fig. 3b.

Without loss of generality we may assume that v_1 lies between v_2 and v_3 on α and that $e \notin M$. Again, the length of the subpath of α connecting v_1 and v_2 is even, for $v_1 \sim v_2$.

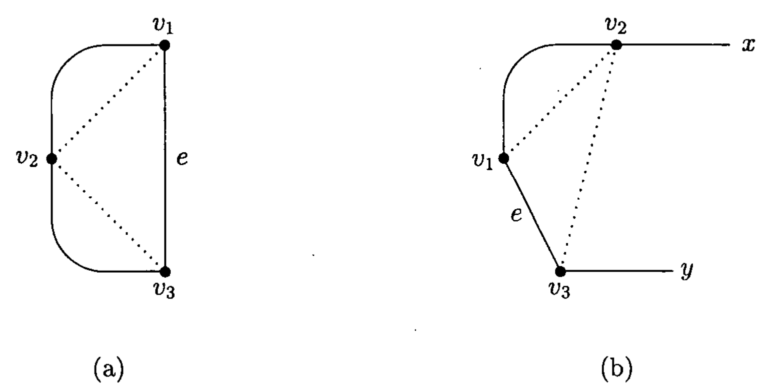

Figure 3.

Consequently, the crossing $x, \ldots, v_2, v_3, \ldots, y$ is M-alternating in $G + (v_2, v_3)$, contradicting the assumption that $v_2 \sim v_3$.

□

5 Connection to soliton automata

According to [5], a soliton graph is a pair (G, w), where G is an undirected graph and w is a weight function from $E(G)$ into the set of positive integers such that these data satisfy the following conditions:

(a) G has no loops or multiple edges;

(b) every connected component of G has at least one external node;

(c) for every $v \in V(G)$, $d(v) \leq 3$;

(d) for every internal vertex v, $w(v) = d(v) + 1$, where $w(v)$ stands for the sum of the weights of all edges incident with v;

(e) if v is an external vertex, then $w(v) \in \{1, 2\}$.

Conditions (d) and (e) imply that the weight of every edge in G is either 1 or 2, and for every internal vertex v there exists exactly one edge e incident with v such that $w(e) = 2$. Let $M \subseteq E(G)$ consist of those edges which have weight 2. Clearly, M is a perfect internal matching of G. Conversely, every perfect internal matching of G corresponds to a suitable weight function w satisfying (d) and (e) above. Hence our approach to soliton automata based on matching theory. Conditions (a), (b) and (c) impose restrictions on the graph structure only, so that they are irrelevant as far as matchings are concerned. We believe that the concept "soliton graph" should be independent of the particular weight function (perfect internal matching) chosen for it, that is why we would rather define a soliton graph simply to be a graph having a perfect internal matching.

Now we quote the definition of soliton path from [5]. Note that the terminology of the authors of [5] differs from ours in that they call a path what we defined as a walk in Section 2. Moreover, since their discussion excludes graphs with loops and multiple edges, it was sufficient for them to specify a path as a sequence of vertices rather than a sequence of edges.

Thus, according to [5], a *partial soliton path* in a soliton graph (G, w) is a path $v_0, v_1, \ldots, v_k$ satisfying the following conditions:

(a) v_0 is an external vertex;

(b) $v_1, v_2, \ldots, v_{k-1}$ are internal vertices;

(c) there is a sequence $(G, w_0), \ldots, (G, w_k)$ of weighted (not necessarily soliton) graphs

that are constructed as follows:

(c1) $w_0 = w$;

(c2) for $i = 0, 1, \ldots, k-2$, the function w_{i+1} is defined iff w_i is defined and $|w_i(v_i, v_{i+1}) - w_i(v_{i+1}, v_{i+2})| = 1$. In this case, for all edges $(v, v') \in E(G)$,

$$w_{i+1}(v, v') = \begin{cases} w_i(v, v') & \text{if } (v, v') \neq (v_i, v_{i+1}) \\ 3 - w_i(v_i, v_{i+1}) & \text{if } (v, v') = (v_i, v_{i+1}); \end{cases}$$

(c3) w_k is defined iff w_{k-1} is defined. In this case, for all $(v, v') \in E(G)$,

$$w_k(v, v') = \begin{cases} w_{k-1}(v, v') & \text{if } (v, v') \neq (v_{k-1}, v_k) \\ 3 - w_{k-1}(v_{k-1}, v_k) & \text{if } (v, v') = (v_{k-1}, v_k). \end{cases}$$

A partial soliton path is called a *(total) soliton path* if v_k above is an external vertex.

Intuitively, a soliton path (walk) is an alternating walk with respect to some state M of the graph G that starts and ends at an external vertex. However, the status of each edge in the walk regarding its presence in M changes dynamically step by step while making the walk. More precisely, this status changes to the opposite right after having traversed the edge. Thus, by the time the walk is finished, a new state M' of G is reached. See [5, Lemma 3.3] for a proof of this last statement.

Here we provide a somewhat simpler definition of soliton walks using our own terminology. For the sake of convenience and unambiguity, we shall specify a walk of length n in graph G as a sequence $\alpha = v_0, e_1, \ldots, e_n, v_n$ of alternating vertices and edges, indicating also the starting point $v_0 \in V(G)$ of α and the vertex v_j, $j \in [n]$, that the walk has reached after traversing the j-th edge e_j. For every $j \in [n]$, $n_\alpha(j)$ will denote the number of occurrences of the edge e_j in the prefix $v_0, e_1, \ldots, e_j$. By a *backtrack* in a walk we mean the traversal of the same edge twice in a consecutive way.

Let us again fix a graph G having a perfect internal matching for the rest of this section.

Definition 5.1 A *soliton walk* in G with respect to state M is a walk $\alpha = v_0, e_1, \ldots, e_n, v_n$ subject to the following two conditions:

(a) v_0 and v_n are external vertices with $n \geq 1$;

(b) for every $j \in [n-1]$, $n_\alpha(j)$ and $n_\alpha(j+1)$ have the same parity iff e_j and e_{j+1} are

M-alternating, i.e., $e_j \in M$ iff $e_{j+1} \notin M$.

It is left to the reader to check that, for soliton graphs in the sense of [5], Definition 5.1 is equivalent to the above definition of soliton path with the only difference that we allow soliton walks to make a backtrack on external edges, too. Any backtrack in a soliton walk is, however, a redundant move as shown by Proposition 5.2 below. *Making* the walk α in state M means creating a new state $M' = S(M, \alpha)$ by setting for every $e \in E(G)$

$$e \in M' \text{ iff} \begin{cases} e \in M & \text{and } e \text{ occurs an even number of times in } \alpha \\ \text{or} & \\ e \notin M & \text{and } e \text{ occurs an odd number of times in } \alpha. \end{cases}$$

In the light of [5, Lemma 3.3] it should be clear that $S(M, \alpha)$ is indeed a state.

Proposition 5.2 *For every soliton walk α with respect to some state M there exists a backtrack-free soliton walk β with respect to M such that $S(M, \alpha) = S(M, \beta)$.*

Proof. Obvious induction on the number of backtracks contained in α, omitted. □

Now we reformalize the definition of soliton automata [5] in our matching theoretic framework.

Definition 5.3 A *soliton automaton* with underlying graph G is a finite state nondeterministic automaton

$$\mathcal{A}(G) = (S(G), X \times X, \delta)$$

subject to the following conditions:

(a) G has a perfect internal matching and has at least one external vertex;

(b) $S(G)$, the set of states of $\mathcal{A}(G)$, is the set of all states of G;

(c) $X \times X$ is the input alphabet, where $X \subseteq V(G)$ denotes the subset of all external vertices.

(d) $\delta : S(G) \times (X \times X) \rightarrow 2^{S(G)}$ is the transition function defined as follows. For every state M and external vertices $v_1, v_2 \in X(G)$, $M' \in \delta(M, (v_1, v_2))$ if there exists a soliton walk α from v_1 to v_2 with respect to M such that $M' = S(M, \alpha)$.

Let α be a soliton walk in G with respect to state M. From Theorem 3.1 we know that making α is equivalent to making an appropriate alternating network Γ with respect to M. The network Γ will consist of a number of alternating cycles $\beta_1, \ldots, \beta_n$, and, in the case when the two endpoints of α are distinct, an additional crossing γ. We are going to prove that the cycles $\beta_1, \ldots, \beta_n$ can be made separately one after the other as suitable soliton walks from the starting point of α back to the same vertex in such a way that making these walks and then finishing up with γ has the same effect as making α directly in state M. This result will admit a decomposition of the transitions of the automaton $\mathcal{A}(G)$ into simpler ones.

Lemma 5.4 *For any state M of G let Γ be an alternating network consisting of a number of cycles. Furthermore, let v_0 be an external vertex and $v \in V(G)$ be arbitrary. If there is an alternating path from v_0 to v with respect to M, then there is also one with respect to $S(M, \Gamma)$.*

Proof. This lemma is in fact a consequence of Corollary 3.4. Let α be an M-alternating path from v_0 to v. Without loss of generality we can assume that α is non-empty, i.e., $v_0 \neq v$. Then the last edge e of α is incident with v and it is viable in state M. Therefore, by Corollary 3.4, e is viable in state $S(M, \Gamma)$, too. Let v' denote the other endpoint of e. By checking the proof of Corollary 3.4 the reader can verify that the alternating path α' demonstrating that e is viable in state $S(M, \Gamma)$ will consist of those edges only that are either in α or in $\cup\Gamma$. Consequently, since $\cup\Gamma$ does not contain any external edges, the path α' will connect v_0 with either

v or v', and it will have an alternating continuation on e. From this point the proof is obvious. □

Let v be an external vertex of G. A soliton walk α is called a *v-saucepan* if it can be decomposed in the form $\alpha\beta\alpha^{-1}$, where α is an alternating path from v to some internal vertex u, β is an alternating cycle starting and ending at u such that β does not go through any vertices covered by α, and α^{-1} is the reverse of α. See Fig. 2 for an example of a saucepan.

Theorem 5.5 *Let $\beta_1, \ldots, \beta_n$ be disjoint alternating cycles with respect to state M that are reachable from an external vertex v_0 of G by a suitable M-alternating path. Then for every $i \in [n]$ there exists a v_0-saucepan δ with respect to state $S(M, \{\beta_1, \ldots, \beta_{i-1}\})$ such that*

$$S(S(M, \{\beta_1, \ldots, \beta_{i-1}\}), \delta_i) = S(M, \{\beta_1, \ldots, \beta_i\}).$$

Proof. Induction on n. The basis case $n = 0$ is vacuously true. Assuming that the statement holds for some $n \geq 0$, let $\beta_1, \ldots, \beta_{n+1}$ be alternating cycles satisfying the conditions of the theorem. By assumption, there exists an M-alternating path α to some vertex v lying on β_{n+1} which starts from v_0 and does not go through any vertices lying on β_{n+1}. Lemma 5.4 then implies that there is another alternating path α' with respect to $S(M, \{\beta_1, \ldots, \beta_n\})$ having the same properties. Therefore we can compose the required v_0-saucepan δ_{i+1} by going down to v from v_0 on α', making the cycle β_{i+1}, and returning to v_0 on the reverse of α'. □

Corollary 5.6 *Every transition of $A(G)$ on input (v_1, v_2) can be decomposed into a sequence of simpler transitions induced by suitable soliton walks $\beta_1, \ldots, \beta_n, \beta_{n+1}$ such that β_i is a v_1-saucepan for every $i \in [n]$, and, in the case of $v_1 \neq v_2$, β_{n+1} is a crossing from v_1 to v_2 .*

Proof. Immediate by Theorems 3.1 and 5.5. □

6 Conclusion

We have made a few simple observations on graphs having a perfect internal matching and on soliton automata. The heart of our results is Theorem 3.1, which specifies the relationship between two perfect internal matchings in a graph in terms of alternating paths and cycles. We have also introduced two equivalence relations ϵ and $\sim$. For a graph G, the relation ϵ can be used to isolate elementary components within the restriction of G to its allowed edges, while the equivalence $\sim$ tells which of the vertices contained in the same elementary component are or could be connected by a forbidden edge in G. Finally, we devoted a section to specify the connection between our work and the study of soliton automata. We would like to use the main result of this section, Theorem 5.5, to provide a decomposition of soliton automata in the spirit of [9].

It is not only the mere fact that we are dealing with perfect internal matchings instead of perfect matchings which makes our results different from the corresponding existing or nonexisting ones in classical matching theory. The difference also appears in the technique by which we prove these results. Rather than using the trick of deleting an appropriate vertex or several vertices in a graph, which seems to be dominant in the classical approach, we almost exclusively rely on the operations of cutting and remerging the edges of graphs. This technique makes our approach edge-oriented as opposed to the vertex-oriented classical approach. Our way of thinking about matchings is based entirely on the method of dealing with alternating paths and cycles, and it fits into the algebraic framework outlined in [1] and [3].

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