Right group-type automata*

I. Babcsányi † A. Nagy†

Abstract

In this paper we deal with state-independent automata whose characteristic semigroups are right groups (left cancellative and right simple). These automata are called right group-type automata. We prove that an A-finite automaton is state-independent if and only if it is right group-type. We define the notion of the right zero decomposition of quasi-automata and show that the state-independent automaton **A** is right group-type if and only if the quasi-automaton $\mathbf{A}_S^*$ corresponding to **A** is a right zero decomposition of pairwise isomorphic group-type quasi-automata. We also prove that the state-independent automaton **A** is right group-type if and only if the quasi-automaton $\mathbf{A}_S^*$ corresponding to **A** is a direct sum of pairwise isomorphic strongly connected right group-type quasi-automata. We prove that if **A** is an A-finite state-independent automaton, then $|S(A)|$ is a divisor of $|AS(A)|$. Finally, we show that the quasi-automaton $\mathbf{A}_S^*$ corresponding to an A-finite state-independent automaton **A** is a right zero decomposition of pairwise isomorphic quasi-perfect quasi-automata if and only if $|AS(A)| = |S(A)|$.

In his paper [5], A. C. Fleck introduced the notion of the characteristic semigroup of automata. This notion is a very useful tool for the examination of automata from semigroup theoretical aspects. In particular, it seems to be successful for state-independent automata. In this case the characteristic semigroup is left cancellative (Lemma 2). If a state-independent (quasi-) automaton is also A-finite, then its characteristic semigroup is a right group (see [9] or Lemma 3).

In 1966, Ch. A. Trauth ([8]) introduced the notion of the group-type automaton (state-independent automaton whose characteristic semigroup is a group) and characterized the quasi-perfect (strongly connected and group-type) automata. He proved that if $\mathbf{A}_i$ $(i \in I)$ is a family of quasi-perfect (quasi-) automata and G_i $(i \in I)$ is the family of corresponding characteristic semigroups, then a quasi-perfect (quasi-) automaton **A** is decomposable into an A-direct product of automata $\mathbf{A}_i$ if and only if the characteristic semigroup of **A** is a direct product of the groups G_i . In 1975, I. Babcsányi ([2]) dealt with the decomposition of group-type generated automata. He proved that every generated group-type quasi-automaton is a direct

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†Department of Mathematics, Transport Engineering Faculty, Technical University of Budapest, 1111 Budapest, Műegyetem rkp. 9., HUNGARY

sum of pairwise isomorphic quasi-perfect quasi-automata. In 1976, Y. Masunaga, S. Noguchi and J. Oizumi ([7]) proved that every strongly connected state-independent A-finite (quasi-) automaton is isomorphic to a A-direct product of a quasi-perfect (quasi-) automaton and a strongly connected reset (quasi-) automaton.

In this paper we extend the investigations to the (not necessarily A-finite) state-independent automata whose characteristic semigroups are right groups.

For notations and notions not defined here, we refer to [4] and [6].

Let $\mathbf{A} = (A, X, \delta)$ be an arbitrary automaton. We suppose that the transition function δ is extended to $A \times X^+$ (X^+ denotes the free semigroup over X) as usually, that is, $\delta(a, px) = \delta(\delta(a,p), x)$ ($p \in X^+$, $x \in X$). For brevity, let $\delta(a,p)$ be denoted by ap. For an arbitrary automaton $\mathbf{A} = (A, X, \delta)$, we consider the following quasi-automata $\mathbf{A}^* = (A, S(A), \delta^*)$ and $\mathbf{A}^*_S = (AS(A), S(A), \delta^*)$, where $S(A)$ is the characteristic semigroup of $\mathbf{A}$, δ^* is defined by $\delta^*(a, \overline{p}) = \delta(a,p)$ $(a \in A,\ p \in X^+)$ and $AS(A) = \{\delta^*(a,s);\ a \in A, s \in S(A)\}$. $\mathbf{A}^*_S$ will be called the *quasi-automaton corresponding to the automaton* $\mathbf{A}$.

Definition 1. An automaton or a quasi-automaton $\mathbf{A}$ is called a *(right) group-type automaton* if it is state-independent and $S(A)$ is a (right) group.

It is clear that an automaton $\mathbf{A}$ is state-independent if and only if $\mathbf{A}^*$ is state-independent. As $S(A) \cong S(A^*)$, it follows that $\mathbf{A}$ is a (right) group-type automaton if and only if $\mathbf{A}^*$ is (right) group-type.

Definition 2. Let $\{S_e :\ e \in E\}$ be an E right zero semigroup decomposition of a semigroup S, that is, E is a right zero semigroup and S is a disjoint union of its subsemigroups S_e, $e \in E$ such that $S_e S_f \subset S_{ef} = S_f$, for every $e, f \in E$. We say that a quasi- automaton $\mathbf{A} = (A, S, \delta)$ is a *right zero decomposition* of quasi-automata $\mathbf{A}_e = (A_e, S_e, \delta_e)$ $(e \in E)$ with $A_e \cap A_f = \emptyset$ for all $e \neq f \in E$, if $A = \cup_{e \in E} A_e$ and $AS_e = \{\delta(a,s) :\ a \in A, s \in S_e\} \subseteq A_e$.

Lemma 1. *A state-independent automaton* $\mathbf{A}$ *is right group-type if and only if the quasi-automaton* $\mathbf{A}^*_S$ *corresponding to* $\mathbf{A}$ *is right group-type.*

Proof. Let $\mathbf{A}$ be a state-independent automaton. Then $\mathbf{A}^*$ and so $\mathbf{A}^*_S$ is state-independent. Moreover, $S(A) \cong S(A^*) = S(A^*_S)$. If $\mathbf{A}$ is right group-type, then $\mathbf{A}^*_S$ is right group-type, too.

Conversely, let $\mathbf{A}^*_S$ be right group-type. As $\mathbf{A}^*$ is state-independent and $S(A^*) = S(A^*_S)$, we get that $S(A^*)$ is a right group. As $S(A) \cong S(A^*)$, the automaton $\mathbf{A}$ is right group-type. □

Theorem 1. *A state-independent automaton* $\mathbf{A}$ *is right group-type if and only if the quasi-automaton* $\mathbf{A}^*_S$ *corresponding to* $\mathbf{A}$ *is a right zero decomposition of pairwise isomorphic group-type quasi-automata.*

Proof. Let the state-independent automaton $\mathbf{A}$ be right group-type. Then, by Lemma 1, the quasi-automaton $\mathbf{A}^*_S$ corresponding to $\mathbf{A}$ is right group-type. Since $S(A^*_S)$ is a right group, it is a right•zero semigroup E of its subgroups G_e, where $G_e = Ge$ for some subgroup G of $S(A^*_S)$. Let $A_e = AG_e$, $e \in E$. It is evident that $\mathbf{A}_e = (A_e, G_e, \delta_e)$ are group-type qusi-automata. We show that $A_e \cap A_f = \emptyset$ if $e \neq f$. Let us suppose that $age = bhf \in A_e \cap A_f$ for some $a, b \in A$, $g, h \in G$ and $e, f \in E$. Then $agf = bhf$ from which it follows that $age = agf$. As $\mathbf{A}$ is state-independent we have $e = f$. Hence $A_e = A_f$. It is evident that $AG_e \subseteq A_e$ and $A_S^* = \cup_{e \in E} A_e$. Consequently, $\mathbf{A}_S^*$ is a right zero decomposition of the group-type quasi-automata $\mathbf{A}_e$, $e \in E$. To complete the proof we show that the quasi-automata $\mathbf{A}_e, e \in E$ are isomorphic with each other. Let $\alpha_{e,f} : A_e \to A_f$ and $\beta_{e,f} : G_e \to G_f$ defined by

$$\alpha_{e,f}(age) = agf, \quad \beta_{e,f}(ge) = gf, \quad a \in A, \ g \in G.$$

It is easy to check that $(\alpha_{e,f}, \beta_{e,f})$ is an isomorphism of $\mathbf{A}_e$ onto $\mathbf{A}_f$.

Conversely, assume that $\mathbf{A}_S^*$ is a right zero decomposition of pairwise isomorphic group-type quasi-automata $\mathbf{A}_e = (A_e, G_e, \delta_e)$, $e \in E$. Then it is easy to see that $S(A_S^*)$ is a right group, and so, $\mathbf{A}_S^*$ is right group-type. Therefore, by Lemma 1, we obtain that $\mathbf{A}$ is right group-type. □

The following example shows that if an automaton $\mathbf{A}$ is right group-type then it is not necessarily a right zero decomposition of pairwise isomorphic group-type automata.

Example 1. Let the state-independent automaton $\mathbf{A} = (\mathbf{A}, \mathbf{X}, \delta)$ be the direct sum of the automata $\mathbf{A_1} = (\mathbf{A_1}, \mathbf{X}, \delta_1)$ and $\mathbf{A_2} = (\mathbf{A_2}, \mathbf{X}, \delta_2)$ $((A_1 = \{1,2,3,4,5\}, (A_2 = \{6,7,8,9,10,11\}, X = \{x, y, z\})$ which are defined by the following transition tables:

$\mathbf{A_1}$	1	2	3	4	5
x	2	3	2	2	3
y	3	2	3	3	2
z	5	4	5	5	4

$\mathbf{A_2}$	6	7	8	9	10	11
x	7	8	7	7	8	7
y	8	7	8	8	7	8
z	10	9	10	10	9	10

The Cayley-table of the characteristic semigroup $S(A)$:

	$\overline{x}$	$\overline{y}$	$\overline{z}$	$\overline{z^2}$
$\overline{x}$	$\overline{y}$	$\overline{x}$	$\overline{z^2}$	$\overline{z}$
$\overline{y}$	$\overline{x}$	$\overline{y}$	$\overline{z}$	$\overline{z^2}$
$\overline{z}$	$\overline{y}$	$\overline{x}$	$\overline{z^2}$	$\overline{z}$
$\overline{z^2}$	$\overline{x}$	$\overline{y}$	$\overline{z}$	$\overline{z^2}$

The quasi-automaton $\mathbf{A}_S^*$ is a direct sum of the quasi-automata $\mathbf{A}_{1S}^*$ and $\mathbf{A}_{2S}^*$ given by the following transition tables:

$\mathbf{A}_{1S}^*$	2	3	4	5
$\overline{x}$	3	2	2	3
$\overline{y}$	2	3	3	2
$\overline{z}$	4	5	5	4
$\overline{z^2}$	5	4	4	5

$\mathbf{A}_{2S}^*$	7	8	9	10
$\overline{x}$	8	7	7	8
$\overline{y}$	7	8	8	7
$\overline{z}$	9	10	10	9
$\overline{z^2}$	10	9	9	10

It is easy to check that $\mathbf{A}_S^*$ is right group-type and is a right zero decomposition of the group-type quasi-automata $\mathbf{B}_1$ and $\mathbf{B}_2$ given by the following transtion tables. (We note that $\{S(B_1), S(B_2)\}$ is a right zero semigroup decomposition of $S(A)$.)

$\mathbf{B}_1$	2	3	7	8
$\overline{x}$	3	2	8	7
$\overline{y}$	2	3	7	8

$\mathbf{B}_2$	4	5	9	10
$\overline{z}$	5	4	10	9
$\overline{z^2}$	4	5	9	10

Lemma 2. ([3]) *The characteristic semigroup of a state-independent quasi-automaton is left cancellative.*

Lemma 3. *An A-finite automaton is state-independent if and only if it is right group-type.*

Proof. Let $\mathbf{A}$ be an A-finite state-independent automaton. Then, by Lemma 2, $S(A)$ is a (finite) left cancellative semigroup. It is easy to show that $S(A)$ is also right simple. Hence $S(A)$ is a right group, that is $\mathbf{A}$ is a right group-type automaton. The converse statement follows from the definition. □

The following example shows that the assertion of Lemma 3 is not true in infinite case.

Example 2. Let $\mathbf{A} = (A, X, \delta)$ be an automaton where A is the set of all positive integers, $X = \{x\}$ and δ is defined by $\delta(n, x) = n + 1$ $(n \in A)$. It is easy to see that $\mathbf{A}$ is state-independent whose characteristic semigroup is an infinite cyclic semigroup.

Lemma 4. *Every group-type quasi-automaton $\mathbf{A}_S^*$ corresponding to a state-independent automaton $\mathbf{A}$ is a direct sum of pairwise isomorphic quasi-perfect quasi-automata.*

Proof. See Lemma 2 and Lemma 4 of [2]. □

The following theorem is a generalization of Lemma 4 for right group-type (quasi-) automata.

Theorem 2. *A state-independent automaton $\mathbf{A}$ is right group-type if and only if the quasi-automaton $\mathbf{A}_S^*$ corresponding to $\mathbf{A}$ is a direct sum of pairwise isomorphic strongly connected right group-type quasi-automata.*

Proof. Let the state-independent automaton $\mathbf{A}$ be right group-type. Then, by Lemma 1, the quasi-automaton $\mathbf{A}_S^*$ corresponding to $\mathbf{A}$ be right group-type. For an arbitrary $a \in AS(A)$, we consider the following A-subautomaton $\mathbf{A}(a) = (A(a), S(A), \delta_a)$ of $\mathbf{A}_S^*$, where $A(a) = \{as : \ s \in S(A)\}$. As $S(A)$ is a right group, therefore $\mathbf{A}(a)$ is strongly connected. As every A-subautomaton of a state-independent (quasi-) automaton $\mathbf{A}$ is also state-independent such that its characteristic semigroup is $S(A)$, we get that $\mathbf{A}(a)$ is a right group-type automaton. It is easy to see that $A(a) \cap A(b) \neq \emptyset$ implies $A(a) = A(b)$ for every $a, b \in AS(A)$. Moreover $as \to bs$ $(a, b \in AS(A),\ s \in S(A))$ is an isomorphism of $\mathbf{A}(a)$ onto $\mathbf{A}(b)$.

Thus $\mathbf{A}_S^*$ is a direct sum of the pairwise isomorphic different A-subautomata $\mathbf{A}(a)$. The converse statement of the theorem is evident. □

We note that the quasi-automaton $\mathbf{A}_S^*$ considered in Example 1 is a direct sum of isomorphic strongly connected right group-type quasi-automata $\mathbf{A}_{1S}$ and $\mathbf{A}_{2S}$. It shows that the components of the direct sum are different from the components of the right zero decomposition.

Lemma 5. *If a quasi-automaton* $\mathbf{A} = (A, S, \delta)$ *is quasi-perfect, then* $|A| = |S(A)|$ (see Lemma 6 and Theorem 3 of [1]).

Corollary 1. *If* $\mathbf{A}$ *is an A-finite state-independent automaton, then* $|S(A)|$ *is a divisor of* $|AS(A)|$.

Proof. Let $\mathbf{A}$ be an A-finite state-independent automaton. Then, by Lemma 3, $\mathbf{A}$ is right group-type. By Lemma 1 and Theorem 1, $\mathbf{A}_S^*$ is a right zero decomposition of pairwise isomorphic group-type quasi-automata $\mathbf{A}_e = (A_e, G_e, \delta_e)$, $e \in E$. Then $|AS(A)| = |A_e||E|$ for arbitrary $e \in E$. By Lemma 4 and Lemma 5, $|A_e| = n|G_e|$ for some positive integer n. Hence $|AS(A)| = n|G_e||E| = n|S(A)|$. □

Corollary 2. *The quasi-automaton* $\mathbf{A}_S^*$ *corresponding to an A-finite state-independent automaton* $\mathbf{A}$ *is a right zero decomposition of pairwise isomorphic quasi-perfect quasi-automata if and only if* $|AS(A)| = |S(A)|$.

Proof. Let the quasi-automaton $\mathbf{A}_S^*$ corresponding to an A-finite state-independent automaton $\mathbf{A}$ be a right zero decomposition of pairwise isomorphic quasi-perfect quasi-automata $\mathbf{A}_e = (A_e, G_e, \delta_e)$, $e \in E$. By Lemma 5, $|A_e| = |G_e|$. Hence $|AS(A)| = |S(A)|$.

Conversely, let $\mathbf{A}$ be an A-finite state-independent automaton such that $|AS(A)| = |S(A)|$. By Lemma 3, $\mathbf{A}$ is right group-type. Then, by Lemma 1 and Theorem 1, $\mathbf{A}_S^*$ is a right zero decomposition of pairwise isomorphic group-type quasi-automata $\mathbf{A}_e = (A_e, G_e, \delta_e)$, $e \in E$. (Here $G_e = Ge$, for some subgroup G of $S(A)$, and $A_e = AG_e$.) It is sufficient to show that $\mathbf{A}_e$ are strongly connected. It is evident that $|S(A)| = |G||E|$ and $|AS(A)| = |A_e||E|$, for every $e \in E$. Then $|A_e| = |G|$, for every $e \in E$. As $\mathbf{A}$ is state-independent, we have $|aG_e| = |G| = |A_e|$, for every $e \in E$ and $a \in A_e$. From this it follows that $\mathbf{A}_e$ is strongly connected, for every $e \in E$. □

We note that the quasi-automata $\mathbf{A}_{1S}^*$ and $\mathbf{A}_{2S}^*$ in Example 1 satisfy the conditions of Corollary 2. For example, the quasi-perfect components of the right zero decomposition of $\mathbf{A}_{1S}^*$ are:

$\mathbf{A}_3$	2	3
$\overline{x}$	3	2
$\overline{y}$	2	3

$\mathbf{A}_4$	4	5
$\overline{z}$	5	4
$\overline{z^2}$	4	5

It is easy to check that these components are isomorphic.

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