Acta Cybernetica 12 (1996) 249-260.

The Reflexive Domain of CPO's Ideals*

Gabriel Ciobanu†

Abstract

A *reflexive* structure is a triple (D, i, j), where D is an algebraic structure, and $i : [D \to D] \to D$, $j : D \to [D \to D]$ are mappings such that $i \circ j = id_{[D \to D]}$. We study reflexive structures in which the basic algebraic structure is a complete partially ordered set (cpo), and only continuous functions over cpo's are considered. We use the concepts, notations, results and techniques of domain theory [4, 7, 9]. We work with the ideals of some special cpo's. We present a reflexive domain of these ideals.

1 Introduction

The λ–calculus is considered as a "core" language which is able to capture the essential mechanisms of complex programming languages. The λ–calculus is also considered the prototype of purely functional programming languages. An excellent book on the λ–calculus is [1].

In its pure form, the λ–calculus has no numbers, arithmetic operations, records, loops, etc.; everything is a function. Its syntax comprises three forms of expressions: a variable x is a λ-expression, the application of a λ-expression to another λ-expression is a λ-expression, and the abstraction of a variable from a λ-expression is a λ-expression. The way by which these expressions "compute" is the application of functions to arguments, which is formally captured by a rule called β-reduction rule, or β-rule.

The λ–calculus is much more powerful than it might appear at first sight. It is proved that all other known models of computation (Turing Machines, general recursive functions) describe exactly the same class of functions. The λ–calculus can be developed as a syntactical theory, in which expressions are syntactically manipulated according to some reduction rules. However we already had an intuitive interpretation in mind: the λ-expressions are thought of being functions, and the β-reduction rule codifies a step in the evaluation of a function at a given argument. Therefore there exist two levels: the purely syntactical manipulation of the λ-expressions, and the intuitive interpretation. The relationship between the two

*The final version of this paper was written at Tohoku University, where the author was a visiting researcher supported by JSPS.

†Faculty of Computer Science"A.I.Cuza" University 6600 Iasi, Romania e-mail: GCiobanu@UAIC.Ro

levels is given by the formal semantics of the λ–calculus. Possible interpretations of λ-expressions as functions in the usual mathematical sense lead to possible semantics and models of the λ–calculus. If we consider a denotational view, then we need to choose a semantic domain D (D is usually a set of functions) and define a denotational function which maps each expression into an element of D. Since λ-expressions take λ-expressions as arguments and return λ-expressions as a result, each element of D is actually a function from D to D. Trying to construct naively such a set D leads to a mathematical paradox. Dana Scott solved the problem in 1969, and the paradox can be avoided by considering only some of the functions from D to D [8]. The same insight can be used to construct different domains with different properties. This paper follows this research line, presenting the reflexive domain of some cpo's ideals. The domain of the cpo's ideals has appeared in various forms and papers, for different purposes [e.g. 5, 6]. These various approaches are similar in many points. This paper describes a domain construction which is interesting on its own. Moreover, the interest for cpo's ideals comes also from the fact that we can look at ideals as being similar to computational trees.
This paper is a formal approach which gives basic results to define and develop formal semantics for different programming languages, and to prove properties of various kinds of computation - e.g. dataflow computation and nondeterministic computation [2, 3].

2 Scott domains and ideals

In the literature the term 'domain' generally refers to the class of structures involved in the denotational semantics of programming languages. It is common to locally 'bind' the term to a specific class of structures for the sake of precision. Domains are usually algebraic cpo's. However we should not take this name too rigidly; there have been different definitions of domains in the literature. When $(D, \sqsubseteq)$ is algebraic and $K(D)$ is denumerable, then D is called $\omega - algebraic$. Finally, a cpo is a *Scott domain* if it is ω-algebraic and consistently complete.
We give some useful definitions.

Definition 1.

i) A subset X of a partial order $(D, \sqsubseteq)$ is called bounded or consistent if there exists an upper bound for X. Otherwise it is called inconsistent.

ii) A directed set of a partial order $(D, \sqsubseteq)$ is a subset $S \subseteq D$ such that every pair of elements in S has an upper bound in S.

iii) A complete partial order (cpo) is a partial order $(D, \sqsubseteq)$ which has a least element $\perp$, and least upper bounds(lub) of all directed subsets of D.

Any set can be viewed as a cpo under the discrete order: $x \sqsubseteq y \ iff \ x = y$. Given a set S, the poset $S_\perp = S \cup \{\perp\}$, obtained by adding a new element $\perp$ to S and defining $x \sqsubseteq y$ iff either $x = y$ or $x = \perp$, yields a cpo called the lift of S, or the flat cpo determined by S.

Definition 2. Let D and E be cpo's and $f : D \to E$. The function f is monotone if $f(x) \sqsubseteq f(y)$ whenever $x \sqsubseteq y$; f is continuous if $f(\bigsqcup S) = \bigsqcup\{f(x) \mid x \in S\}$ for every directed $S \subseteq D$. f is said to be strict if $f(\perp_D) = \perp_E$. If D is a cpo, then $[D \to D]$ denotes the set of continuous functions from D to D.

Definition 3. Let D be a cpo. An element $x \in D$ is compact if, for every directed set $M \subseteq D$ such that $x \sqsubseteq \bigsqcup M$, there is some $y \in M$ such that $x \sqsubseteq y$. The set of compact elements in D is denoted by $K(D)$.
A cpo in which every element is a least upper bound of a directed collection of its compact approximations is said to be algebraic:

Definition 4. A cpo $(D, \sqsubseteq)$ is algebraic if for all $x \in D$, $\{e \sqsubseteq x \mid e \in K(D)\}$ is directed and $x = \bigsqcup\{e \sqsubseteq x \mid e \in K(D)\}$. When $(D, \sqsubseteq)$ is algebraic and $K(D)$ is denumerable, then D is called $\omega - algebraic$.

Definition 5. D is consistently complete if every upper bound set $X \subseteq D$ has a least upper bound(lub).

Definition 6. A cpo is a Scott domain if it is ω-algebraic and consistently complete. *The domains used in this paper are Scott domains.*

Proposition 7. *If $(D, \sqsubseteq_D)$ and $(E, \sqsubseteq_E)$ are domains, then $([D \to E], \sqsubseteq_{D\to E})$ is also a domain.*

$$f \sqsubseteq_{D\to E} g \text{ iff } f(x) \sqsubseteq_E g(x) \text{ for any } x \in D$$
$$(\perp_{D\to E})(x) = \perp_E \text{ for any } x \in D, \text{ and}$$
$$(\sqcup_{i\geq 0} f_i)(x) = \sqcup_{i\geq 0}(f_i(x)) \text{ for all increasing sequence } \{f_i\}_{i\geq 0}$$

We define a notion of ideal for all these domains.

Definition 8. Let D be a domain. Then $I \subseteq D$ is an ideal if

i) whenever $y \sqsubseteq x$ and $x \in I$, then $y \in I$;

ii) for every increasing sequence $\{x_i\}_{i\geq 0} \subset I$ we have $\sqcup_{i\geq 0} x_i \in I$.

Remark. According to this definition the empty set forms an ideal with $\sqcup\emptyset = \perp$. Consequently all continuous functions are strict, since $f(\sqcup\emptyset) = \sqcup f(\emptyset)$ implies $f(\perp) = \perp$.
Let $\mathcal{I}(D)$ be the set of the ideals of D. If D is a domain, let $x \in D$ and set $[x] = \{y \in D | y \sqsubseteq x\}$. It is easy to verify that $[x]$ is an ideal, precisely because D is a domain. The ideal $[x]$ is called the principal ideal generated by x, and it is the smallest ideal containing x.
Let X be a subset of D, and $\overline{X} = \{\sqcup_{i\geq 0} x_i | \{x_i\}_{i\geq 0}$ is an increasing sequence in $X\}$. The function $X \mapsto \overline{X}$ is defined on those subsets of D which are downwards closed in the sense *i*) of the previous definition.

Remark. *The function $X \mapsto \overline{X}$ is a closure operator.*

Recall that, in general, a function $X \mapsto \overline{X}$ on subsets of some set is called a closure operator if *i*) $I \subseteq J$ implies $\overline{I} \subseteq \overline{J}$, *ii*) $I \subseteq \overline{I}$, and *iii*) $\overline{\overline{I}} = \overline{I}$.
In our case it is not difficult to prove *i*) and *ii*), therefore we only sketch the proof of $\overline{\overline{I}} = \overline{I}$. First we remark that I and $\overline{I}$ have the same compact elements. Suppose

now that $x \in \overline{I}$ is a compact element; then x is the lub of an increasing sequence $\{a_i\}$ from I. Since x is compact, there is an index $j \geq 0$ such that $x = a_j$, and so $x \in I$. Taking into account that any element of an ω-algebraic cpo is the lub of an increasing sequence of compact elements, it is easy to show that $\overline{I}$ and $\overline{\overline{I}}$ have the same elements, namely the lub's of the increasing sequences defined by the compact elements of I.
The set $\overline{I}$ is called the closure of I, and I is said to be closed if $I = \overline{I}$. Thus, by *iii*), the closure of any closed set is closed. Moreover, by *i*), if $I \subseteq J = \overline{J}$, then $\overline{I} \subseteq \overline{J} = J$, i.e. $\overline{I}$ is included in every downwards closed set which includes I. Since $\overline{I}$ is itself a downwards closed set which includes I, then the closure of I is the smallest downwards closed set which includes I.
We give the following results, which can be easily proved.

Proposition 9.

i) for any $I \in \mathcal{I}(D)$, $\overline{K(I)} = I$;

ii) for any increasing sequence $\{I_k\}_{k\geq 0} \subset \mathcal{I}(D)$, *we have* $\sqcup_{k\geq 0} I_k = \overline{\cup_{k\geq 0} K(I_k)}$;

iii) $\forall I_1, I_2 \in \mathcal{I}(D), \quad I_1 = I_2$ *iff* $K(I_1) = K(I_2)$;

iv) $K(\sqcup_{k\geq 0} I_k) = \cup_{k\geq 0} K(I_k)$.

As an intermediate step to the final result, we will show that we can solve the domain equation $D = [\mathcal{I}(D) \to D]$; this means that we are looking for a domain D which is isomorphic to the domain of functions from $\mathcal{I}(D)$ to D. The method we use to solve such a domain equation is called *inverse limit construction.*

3 Inverse limit construction for $D = [\mathcal{I}(D) \to D]$

The method of the inverse limit construction was developed by D. Scott. The result is that, for any recursive equations of form $D = T(D)$, if $T(D)$ is an expression composed out of D such that $T(D)$ is a domain when D is, then there is a domain D_∞ that is isomorphic to $T(D_\infty)$, i.e. there is a solution of these equations.
In order to prove that $\mathcal{I}(D)$ is a domain, we start by studying the set $K(\mathcal{I}(D))$ of compact elements.

Definition 10. A set X is maximal complete if for every $x \in X$ there exists a maximal element $m \in X$ such that $x \sqsubseteq m$.

Lemma 11. *Any ideal* $[M]$ *generated by a finite set* M *is maximal complete and the set of its maximal elements is a subset of* M. *Any maximal complete ideal is generated by the set of its maximal elements.*

Proposition 12. *If* D *is an* ω*-algebraic cpo, then an ideal* I *is a compact element in* $\mathcal{I}(D)$ *iff it is maximal complete and the set of its maximal elements is finite, containing only compact elements of* D.

Proof :
$\Longleftarrow$: We have to prove that if $I \subseteq \sqcup_{j \geq 0} A_j$ then there exists a $k \geq 0$ such that $I \subseteq A_k$. Let $\{a_1, ..., a_n\}$ be the set of maximal elements of I. We have $I \subseteq \sqcup_{j \geq 0} A_j = \overline{\cup_{j \geq 0} K(A_j)}$; therefore, for any maximal elements a_p we have $a_p \in \overline{\cup_{j \geq 0} K(A_j)}$. Since a_p is a compact element, there exists $k_p \geq 0$ such that $a_p \in K(A_{k_p})$. There are a finite number of maximal elements; thus there exists $k = max\{k_1, ...k_n\}$ such that $a_p \in A_k$ for all $p = 1, ..., n$. Using the previous lemma, it is easy to see that $I \subseteq A_k$.
$\Longrightarrow$: Since D is a domain, $K(I)$ is denumerable, and so $K(I) = \{a_i \mid i \geq 0\}$. We consider now an increasing sequence of ideal generated by subsets of compact elements, namely $[a_0] \sqsubseteq [a_0,\ a_1] \sqsubseteq \ldots \sqsubseteq [a_0, \ldots, a_n] \sqsubseteq \ldots \sqsubseteq I$. $I = \overline{K(I)} = \overline{\cup_{i \geq 0} \{a_0, \ldots, a_i\}} = \overline{\cup_{i \geq 0} K([a_0, \ldots, a_i])} = \sqcup_{i \geq 0} [a_0, \ldots, a_i]$. Now, if D is ω-algebraic, then there exists $k \geq 0$ such that $I = [a_0, \ldots, a_k]$. According to the previous lemma, $[a_0, \ldots, a_k]$ is maximal complete, and the set of maximal elements is a subset of $\{a_0, \ldots, a_k\}$ which is finite and contains only compact elements of D.

Proposition 13. *If D is an ω-algebraic cpo, then $\mathcal{I}(D)$ is also an ω-algebraic cpo.*

Proof : There are countably many compact ideals, because they are characterized by a finite number of compact elements from D (and the compact elements of D are denumerable). In $\mathcal{I}(D)$, the set of compact ideals less than a given ideal I is directed. This fact is a consequence of the equality $[a_0, ..., a_n] \cup [b_0, ..., b_m] = [a_0, ..., a_n, b_0, ..., b_m]$. The lub of this directed set included in $K(\mathcal{I}(D))$ is I. If D is ω-algebraic, then $I = \overline{K(I)} = \overline{\cup_{i \geq 0} \{a_0, \ldots, a_i\}} = \overline{\cup_{i \geq 0} K([a_0, \ldots, a_i])} = \sqcup_{i \geq 0} [a_0, \ldots, a_i]$, where $K(I) = \{a_i \mid i \geq 0\}$. The ideals $[a_0, \ldots, a_i]$ are ω-algebraic; therefore I is the lub of the ω-algebraic ideals less than I.
We have to prove also that $\mathcal{I}(D)$ is consistently complete; we use the following result

Lemma 14. *Let $(\mathcal{I}(D), \sqsubseteq)$ be the cpo of ideals of D, where the order relation is the set inclusion.*

i) *If $X, Y \in \mathcal{I}(D)$, then $X \cup Y, X \cap Y \in \mathcal{I}(D)$.*

ii) *$(\mathcal{I}(D), \cap, \sqcup)$ is a complete lattice, where the meet operator is the usual intersection, and the joint operator is defined by $\sqcup_{j \in J} X_j = \cap\{I \in \mathcal{I}(D) \mid \forall j \in J,\ X_j \subseteq I\}$.*

Consequently, any bound subset of $\mathcal{I}(D)$ has a least upper bound given by the joint operator.

Remark. *The infinite union of ideals may not be an ideal.*
However, if D is ω-algebraic, then $\sqcup_{j \in J} X_j = \overline{\cup_{j \in J} K(X_j)}$.

Finally, as a consequence of the previous two results, we give now the following proposition:

Proposition 15. *If D is a domain, then $\mathcal{I}(D)$ is a domain.*

Definition 16. Let $f : D \to E$ be a continuous function between two domains. Then
$\mathcal{I}(f) : \mathcal{I}(D) \to \mathcal{I}(E)$ is defined by $\mathcal{I}(f) = \lambda I.\overline{\{y \in E \mid \exists x \in I : y \sqsubseteq f(x)\}}$
$\mathcal{I}(f)(I)$ is the minimum ideal containing $f(I)$.

Lemma 17.

$$\mathcal{I}(f)(I) = \overline{\{y \in E \mid \exists x \in I : y \sqsubseteq f(x)\}} = \overline{\{y \in K(E) \mid \exists x \in K(I) : y \sqsubseteq f(x)\}}.$$

Proof : Using $I = \overline{K(I)}$ from the Proposition 9 i), and since we work with a closure operator, then
$\overline{\{y \in E \mid \exists x \in I : y \sqsubseteq f(x)\}} = \overline{\{y \in K(E) \mid \exists x \in I : y \sqsubseteq f(x)\}}$. Now, if x is not a compact element, then there is an increasing sequence $\{x_i\}_{i\geq 0}$ of compact elements such that $x = \sqcup_{i\geq 0} x_i$. Using the continuity of f, $y \sqsubseteq f(x) = f(\sqcup_{i\geq 0} x_i) = \sqcup_{i\geq 0} f(x_i)$; but y is a compact element of E, and there exists $n \geq 0$ such that $y \sqsubseteq f(x_n)$ and $x_n \in K(I)$ – because $x_n \sqsubseteq x$ and I is an ideal. Therefore, if $\exists x \in I : y \sqsubseteq f(x)$, then $\exists x' \in K(I) : y \sqsubseteq f(x') \sqsubseteq f(x)$, i.e. $\overline{\{y \in K(E) \mid \exists x \in I : y \sqsubseteq f(x)\}} = \overline{\{y \in K(E) \mid \exists x \in K(I) : y \sqsubseteq f(x)\}}$.

Proposition 18. *If $f : D \to E$ is a continuous function, and $I \subseteq D$ is an ideal, then $\mathcal{I}(f)(I)$ is an ideal.*

Proof : We have to prove that
i) whenever $z \sqsubseteq y$ and $y \in \mathcal{I}(f)(I)$, then $z \in \mathcal{I}(f)(I)$, and
ii) for every increasing sequence $\{y_i\}_{i\geq 0}$ in $\mathcal{I}(f)(I)$ we have $\sqcup_{i\geq 0} y_i \in \mathcal{I}(f)(I)$.

- i) If $z \sqsubseteq y$ and $y \in \mathcal{I}(f)(I) = \overline{\{y \in E \mid \exists x \in I : y \sqsubseteq f(x)\}}$, then $z \sqsubseteq y$ and $\exists x \in I : y \sqsubseteq f(x)$; this means that $z \in \mathcal{I}(f)(I)$.
- ii) is clear from the definition of $\mathcal{I}(f)(I)$. We want to explain now our choice for the definition of $\mathcal{I}(f)(I)$. In general, the set $\{y \in E \mid \exists x \in I : y \sqsubseteq f(x)\}$ is not closed (it is not difficult to construct a counterexample). According to our definition, if we use elements of $\overline{I}$, we can define new increasing sequences; then we have to close the set again, and we obtain $\overline{\overline{I}}$. In order to have $\overline{I} = \overline{\overline{I}}$, the key property of domains is that of being ω-algebraic cpo's. The construction is now consistent, because even the function space does preserve this property, according to the fact that our domains are consistently complete(a result proved by G. Plotkin).

Proposition 19. *If $f : D \to E$ is continuous, then $\mathcal{I}(f)$ is also continuous.*

Proof : According to Lemma 17, $\mathcal{I}(f) : \mathcal{I}(D) \to \mathcal{I}(E)$ could be defined by
$$\mathcal{I}(f) = \lambda I.\overline{\{y \in K(E) \mid \exists x \in K(I) : y \sqsubseteq f(x)\}}$$
We have $\mathcal{I}(f)(\sqcup_{n\geq 0} I_n) = \overline{\{y \in K(E) \mid \exists x \in K(\sqcup_{n\geq 0} I_n) : y \sqsubseteq f(x)\}} =$ by Prop. 9 ii)
$\overline{\{y \in K(E) \mid \exists x \in \cup_{n\geq 0} K(I_n) : y \sqsubseteq f(x)\}} =$ by Prop. 9 iv)
$\overline{\cup_{n\geq 0}\{y \in K(E) \mid \exists x \in K(I_n) : y \sqsubseteq f(x)\}} = \overline{\cup_{n\geq 0} \mathcal{I}(f)(I_n)} = \sqcup_{n\geq 0} \mathcal{I}(f)(I_n)$.

Proposition 20. *If $f : D \to E, g : E \to F$ are continuous functions, then*

$$\mathcal{I}(g) \circ \mathcal{I}(f) = \mathcal{I}(g \circ f).$$

Proof : $(\mathcal{I}(g) \circ \mathcal{I}(f))(I) = \overline{\{z \in K(F) | \exists y \in K(\mathcal{I}(f)(I)) : z \sqsubseteq g(y)\}} =$
$\overline{\{z \in K(F) | \exists y \in \{y' \in K(E) | \exists x \in K(I) : y' \sqsubseteq f(x)\} : z \sqsubseteq g(y)\}} =$
$\overline{\{z \in K(F) | \exists y \in K(E) : \exists x \in K(I) : y \sqsubseteq f(x) \ \& \ z \sqsubseteq g(y)\}}$.
Since $z \sqsubseteq g(y)$ and $y \sqsubseteq f(x)$, we have $z \sqsubseteq g(f(x))$ by the monotonicity of g.
Then $(\mathcal{I}(g) \circ \mathcal{I}(f))(I) \subseteq \overline{\{z \in K(F) | \exists x \in K(I) : z \sqsubseteq g(f(x))\}} = \mathcal{I}(g \circ f)(I)$.
In the other direction, starting from $z \sqsubseteq g(f(x))$, we have to find a compact element y such that $z \sqsubseteq g(y)$ and $y \sqsubseteq f(x)$. We already know that x and z are compact elements. If $f(x)$ is compact, then we take $y = f(x)$. If $f(x)$ is not compact, then there is an increasing sequence $\{t_i\}_{i \geq 0}$ of compact elements such that $f(x) = \sqcup_{i \geq 0} t_i$. Using the continuity of g, we can say that $z \sqsubseteq g(f(x)) = g(\sqcup_{i \geq 0} t_i) = \sqcup_{i \geq 0} g(t_i)$. z is a compact element; then there exists $n \geq 0$ such that $z \sqsubseteq g(t_n)$. Now we consider $y = t_n$; hence $y \sqsubseteq f(x)$ and y is compact. Thus we can write now
$\mathcal{I}(g \circ f)(I) = \overline{\{z \in K(F) | \exists x \in K(I) : z \sqsubseteq g(f(x))\}} \subseteq$
$\overline{\{z \in K(F) | \exists y \in K(E) : \exists x \in K(I) : y \sqsubseteq f(x) \ \& \ z \sqsubseteq g(y)\}} = (\mathcal{I}(g) \circ \mathcal{I}(f))(I)$.

Proposition 21. *If $\{f_i\}_{i \geq 0}$ is an increasing sequence of continuous functions $f_i : D \to E$, and $I \in \mathcal{I}(D)$, then $\mathcal{I}(\sqcup_{i \geq 0} f_i)(I) = \sqcup_{i \geq 0} \mathcal{I}(f_i)(I)$, i.e. $\mathcal{I}$ is continuous.*

Proof : By Lemma 17, $\mathcal{I}(\sqcup_{i \geq 0} f_i)(I) = \overline{\{y \in K(E) | \exists x \in K(I) : y \sqsubseteq \sqcup_{i \geq 0} f_i(x)\}}$.
Since y is a compact element, then there is $n \geq 0$ such that $y \sqsubseteq f_n(x)$. Using a previous remark, $\mathcal{I}(\sqcup_{i \geq 0} f_i)(I) = \overline{\{y \in K(E) | \exists x \in K(I) \ \exists n \geq 0 : y \sqsubseteq f_n(x)\}} = \overline{\cup_{i \geq 0} K(\mathcal{I}(f_i)(I))} = \sqcup_{i \geq 0} \mathcal{I}(f_i)(I)$.

The following definitions and results are included only to make the construction clear; the proofs can be found in different papers or books [e.g. 1, 4, 7, 8, 9].

Definition 22. Let D, D' be domains. A pair of continuous functions $(f : D \to D', g : D' \to D)$ is a retraction pair iff $g \circ f = id_D$ and $f \circ g \sqsubseteq id_{D'}$

i) f is called an embedding and g is called a projection.

ii) We will use the notation $(f, g) : D \leftrightarrow D'$.

Proposition 23.

i) An embedding has an unique corresponding projection.

ii) A projection has an unique corresponding embedding.

iii) The functions of a retraction pair are strict functions.

Definition 24. Let $(f, g) : D \leftrightarrow D'$ be a retraction pair. Then $(f, g)^R : D' \leftrightarrow D$ denotes the reversal of (f, g) and is defined as (g, f). The reversal of a retraction pair might not be a retraction pair.

Proposition 25. *Let $r = (f,g) : D \leftrightarrow D'$ and $s = (f',g') : D' \leftrightarrow D''$ be retraction pairs. Then*

i) $r \circ s = (f' \circ f, g \circ g') : D \leftrightarrow D''$ *is a retraction pair;*

ii) $(r \circ s)^R = s^R \circ r^R$;

iii) $(r^R)^R = r$.

Proposition 26. *The composition and reversal operations upon retraction pairs are continuous.*

Definition 27. Let $r = (f,g) : D \leftrightarrow E$ and $s = (f',g') : D' \leftrightarrow E'$ be retraction pairs. We define
$r \to s = ((\lambda h.f' \circ h \circ g), (\lambda k.g' \circ k \circ f)) : [D \to D'] \leftrightarrow [E \to E']$.

Proposition 28. *Let $(f,g) : D \leftrightarrow E$ be a retraction pair. Then $(\mathcal{I}(f), \mathcal{I}(g)) : \mathcal{I}(D) \leftrightarrow \mathcal{I}(E)$ is a retraction pair.*

Proof : $\mathcal{I}(f)$ and $\mathcal{I}(g)$ are continuous, by the Proposition 19. By the Proposition 20,

$(\mathcal{I}(g) \circ \mathcal{I}(f))(A) = \mathcal{I}(g \circ f)(A) = \{y \in D | \exists x \in A : y \sqsubseteq g(f(x)) = x\} = A, \forall A \in \mathcal{I}(D)$,

$(\mathcal{I}(f) \circ \mathcal{I}(g))(B) = \mathcal{I}(f \circ g)(B) = \{y \in E | \exists x \in B : y \sqsubseteq f(g(x)) \sqsubseteq x\} \subseteq B, \forall B \in \mathcal{I}(E)$.

Corollary 29. *Let $(f,g) : D \leftrightarrow E$ be a retraction pair, and $I \in \mathcal{I}(D)$.
For any $J \in \mathcal{I}(E)$ such that $\mathcal{I}(g)(J) = I$ we have $\mathcal{I}(f)(I) \subseteq J$.*

Proof : $\mathcal{I}(f)(I) = \mathcal{I}(f)(\mathcal{I}(g)(J)) = (\mathcal{I}(f) \circ \mathcal{I}(g))(J) \subseteq J$.

Remark. *If $J \in \mathcal{I}(E)$ such that $g(y) \in I$ for any $y \in J$, then $\mathcal{I}(f)(I) \subseteq J$.*

Corollary 30. *Let $r : D \leftrightarrow E$ be a retraction pair. Then $(\mathcal{I}(r))^R = \mathcal{I}(r^R)$.*

Proof : By the Proposition 28, $\mathcal{I}(r) = \mathcal{I}(f,g) = (\mathcal{I}(f), \mathcal{I}(g))$.
It is not difficult to see that $(\mathcal{I}(r)^R = (\mathcal{I}(f), \mathcal{I}(g))^R = (\mathcal{I}(g), \mathcal{I}(f)) = \mathcal{I}(g,f) = \mathcal{I}(r^R)$.

In our inverse limit construction we have the following domain expressions:

$$T(D) = [\mathcal{I}(D) \to D]\ , \text{ and } \ T(r) = (\mathcal{I}(r) \to r).$$

Lemma 31. *For all domain expressions F, and retraction pairs $r : D \leftrightarrow E, s : E \leftrightarrow F$,*

i) $F(id_{D \leftrightarrow D}) = id_{F(D) \leftrightarrow F(D)}$, where $id_{D \leftrightarrow D} = (id_D, id_D)$;

ii) $F(r)$ *is also a retraction pair;*

iii) $F(r)^R = F(r^R)$;

iv) $F(s \circ r) = F(s) \circ F(r)$.

Definition 32. A retraction sequence is a pair $(\{D_i\}_{i\geq 0},\ \{r_i : D_i \leftrightarrow D_{i+1}\}_{i\geq 0})$, such that for all $i \geq 0$ D_i is a domain, and r_i is a retraction pair.
Let $t_{mn} : D_m \leftrightarrow D_n$ be defined as

$$t_{mn} = \begin{cases} r_{n-1} \circ \ldots \circ r_m & \text{if } m < n \\ id_{D_m \leftrightarrow D_m} & \text{if } m = n \\ r_n^R \circ \ldots \circ r_{m-1}^R & \text{if } m > n \end{cases}.$$

If we consider $\phi_0 = id_{\{\perp\}}$, $\phi_{n+1} = (\mathcal{I}(\phi_n) \to \phi_n)$, and each r_i is a retraction pair $(\phi_i, \psi_i) : D_i \leftrightarrow D_{i+1}$,
then for $m < n$,

$$t_{mn} = (\theta_{mn}, \theta_{nm}) = (\phi_{n-1}, \psi_{n-1}) \circ \ldots \circ (\phi_{m+1}, \psi_{m+1}) \cdot (\phi_m, \psi_m) = $$
$$= (\phi_{n-1} \circ \ldots \circ \phi_{m+1} \circ \phi_m, \psi_m \circ \psi_{m+1} \circ \ldots \circ \psi_{n-1}).$$

We can represent this by

$$D_m \underset{\psi_m}{\overset{\phi_m}{\rightleftarrows}} D_{m+1} \underset{\psi_{m+1}}{\overset{\phi_{m+1}}{\rightleftarrows}} D_{m+2} \quad \ldots \quad D_{n-1} \underset{\psi_{n-1}}{\overset{\phi_{n-1}}{\rightleftarrows}} D_n$$

For the case when $m > n$,

$$t_{mn} = (\theta_{mn}, \theta_{nm}) = (\theta_{nm}, \theta_{mn})^R = t_{nm}^R.$$

Definition 33. The inverse limit of a retraction sequence

$$(\{D_i\}_{i\geq 0}, \{(\phi_i, \psi_i) : D_i \leftrightarrow D_{i+1} | i \geq 0\})$$

is the set:

$$D_\infty = \{\{x_0, x_1, ..., x_i, ...\} | \text{ for all } n > 0,\ x_n \in D_n \text{ and } x_n = \psi_n(x_{n+1})\}.$$

This set has a domain structure with $\perp_{D_\infty} = \{\perp_{D_n}\}_{n\geq 0}$ and the order $\{x_n\}_{n\geq 0} \sqsubseteq_{D_\infty} \{y_n\}_{n\geq 0}$ iff for any $n \in N$ we have $x_n \sqsubseteq_{D_n} y_n$.

Proposition 34. *If F maps a domain D to a domain $F(D)$, then the pair:*
$(\{D_i | D_0 = \{\perp\},\ D_{i+1} = F(D_i)\}_{i\geq 0},$
$\{(\phi_i, \psi_i) : D_i \leftrightarrow D_{i+1} | \phi_0 = (\lambda x.\ \perp), \psi_0 = (\lambda x.\ \perp), (\phi_{i+1}, \psi_{i+1}) = F(\phi_i, \psi_i)\}_{i\geq 0})$
is a retraction sequence.

To show that the inverse limit for the retraction sequence generated by F satisfies $D_\infty = F(D_\infty)$, we define the functions

$$\Phi : D_\infty \to F(D_\infty)\ ,\ \Psi : F(D_\infty) \to D_\infty,$$

using the retraction pairs (ϕ_i, ψ_i).
For $m \geq 0$,
$t_{m\infty} : D_m \leftrightarrow D_\infty$ is $(\theta_{m\infty}, \theta_{\infty m}) = (\lambda x.\{\theta_{m0}(x), \theta_{m1}(x), ...\}, \lambda\{x_1, ..., x_m, ...\}.x_m)$
$t_{\infty m} : D_\infty \leftrightarrow D_m$ is $(\theta_{\infty m}, \theta_{m\infty}) = {t_{m\infty}}^R$
$t_{\infty\infty} : D_\infty \leftrightarrow D_\infty$ is $(\theta_{\infty\infty}, \theta_{\infty\infty}) = (id_{D_\infty}, id_{D_\infty})$.

Lemma 35. *For any retraction sequence and $m, n, k \in N \cup \{\infty\}$*

(*i*) $t_{mn} \circ t_{km} \sqsubseteq t_{kn}$

(*ii*) *If $m \geq k$ or $m \geq n$, then $t_{mn} \circ t_{km} = t_{kn}$.*

(*iii*) *If $m \leq n$, then t_{mn} is a retraction pair.*

The isomorphism maps are defined as a retraction pair (Φ, Ψ),

$$(\Phi, \Psi) : D_\infty \leftrightarrow F(D_\infty) = \sqcup_{m=0}^{\infty} F(t_{m\infty}) \circ t_{\infty(m+1)}.$$

Theorem 36.

(*i*) $(\Phi, \Psi)^R \circ (\Phi, \Psi) = id_{D_\infty \leftrightarrow D_\infty}$,

(*ii*) $(\Phi, \Psi) \circ (\Phi, \Psi)^R = id_{F(D_\infty) \leftrightarrow F(D_\infty)}$.

Our domain expressions T satisfies all the requirements imposed on F. The necessary properties required for the new constructor $\mathcal{I}$ have been proved by previous results of this section. As a consequence, we have a domain V which is a solution of the domain equation $D = [\mathcal{I}(D) \to D]$.

4 The reflexive domain $\mathcal{I}(V)$

Recall that an algebraic structure is *reflexive* if there are two functions $i : [D \to D] \to D$ and $j : D \to [D \to D]$ such that $i \circ j = id_{[D \to D]}$. In our case we prove the existence of two *continuous* functions $i : [D \to D] \to D$ and $j : D \to [D \to D]$ such that, for all $f \in [D \to D]$ and $x \in D$, $j(i(f)) = f$, and $i(j(x)) \sqsubseteq_D x$.

Let V be a domain such that $V \cong [\mathcal{I}(V) \to V]$. In order to show that $\mathcal{I}(V)$ is a reflexive domain, we consider the following functions:

Definition 37. Let $f : \mathcal{I}(V) \to \mathcal{I}(V)$ be a function, and $I, J \in \mathcal{I}(V)$ be ideals.
$F : [\mathcal{I}(V) \to \mathcal{I}(V)] \to \mathcal{I}(V)$
$G : \mathcal{I}(V) \to [\mathcal{I}(V) \to \mathcal{I}(V)]$
are defined by
$F(f) = \{\sigma \in [\mathcal{I}(V) \to V] | \sigma(I) \in f(I), \forall I \in \mathcal{I}(V)\}$, and
$G(I)(J) = \{\sigma(J) | \sigma \in [\mathcal{I}(V) \to V] \cap I\}$.

Proposition 38. *F and G are well defined.*

Proof :

i) If $f : \mathcal{I}(V) \to \mathcal{I}(V)$, then $F(f) \in \mathcal{I}(V)$.
Let σ be in $F(f)$, and $\rho \sqsubseteq \sigma$. $\rho(I) \sqsubseteq \sigma(I)$ for all $I \in \mathcal{I}(V)$, and by the definition of $F(f)$, $\sigma(I) \in f(I)$. $f(I)$ is an ideal, hence $\rho(I) \in f(I)$, i.e. $\rho \in F(f)$.
Now we consider an increasing sequence $\{\sigma_i\}_{i \geq 0}$ in $F(f)$. For any $i \geq 0$, $\sigma_i \in [\mathcal{I}(V) \to V]$, and $\sigma_i(I) \in f(I), \forall I \in \mathcal{I}(V)$. For an arbitrary $I \in \mathcal{I}(V), (\sqcup_{i \geq 0} \sigma_i)(I) = \sqcup_{i \geq 0} \sigma_i(I)$, by the continuity of σ_i. Since $\sigma_i(I) \in f(I)$, and $f(I)$ is an ideal, then $\{\sigma_i(I)\}_{i \geq 0}$ is an increasing sequence in $f(I)$ and $\sqcup_{i \geq 0} \sigma_i(I) \in f(I)$. Therefore $\sqcup_{i \geq 0} \sigma_i \in F(f)$.

ii) If $I \in \mathcal{I}(V)$, then $G(I) \in [\mathcal{I}(V) \to \mathcal{I}(V)]$.
We have to prove that $G(I)(J) \in \mathcal{I}(V)$, $\forall J \in \mathcal{I}(V)$, and $G(I)$ is continuous.
Let $J \in \mathcal{I}(V)$, and $\sigma(J) \in G(I)(J), \sigma \in [\mathcal{I}(V) \to V]$. If $a \in G(I)(J)$, then $\exists \sigma \in [\mathcal{I}(V) \to V] \cap I$ such that $\sigma(J) = a$. If we consider $b \sqsubseteq a$, then it is necessary to prove that $\exists \rho \in [\mathcal{I}(V) \to V] \cap I$ such that $\rho(J) = b$. We define
$\rho(x) = \begin{cases} max_{i \geq 0}\{c_i \sqsubseteq b | c_i \sqsubseteq \sigma(x)\} & \text{if } b \not\sqsubseteq \sigma(x); \\ b, & \text{if } b \sqsubseteq \sigma(x). \end{cases}$

Thus we have $\rho(J) = b$, because $b \sqsubseteq a = \sigma(J)$, and ρ is continuous. We consider now an increasing sequence $\{a_i\}_{i\geq 0} \subset G(I)(J) \in \mathcal{I}(V)$. This sequence is given by an increasing sequence $\{\sigma_i\}_{i\geq 0} \subset [\mathcal{I}(V) \to V] \cap I$. If we consider $\{\sigma_i\}_{i\geq 0}$ as functions, then the values of $G(I)(J)$ are determined by applying these functions to J, and $\sqcup_{i\geq 0} a_i = \sqcup_{i\geq 0}\sigma_i(J)$. We consider $\{\sigma_i\}_{i\geq 0}$ as elements of I, and we have $\sqcup_{i\geq 0} a_i = \sqcup_{i\geq 0}\sigma_i$, with $\sigma_i \in I$. I is an ideal, and therefore $\sqcup_{i\geq 0} a_i \in I$. Thus $\sqcup_{i\geq 0} a_i \in G(I)(J)$.

iii) We need to prove also that $G(I)$ is continuous. We consider an arbitrary increasing sequence $\{J_k\}_{k\geq 0} \subset \mathcal{I}(V)$. We prove that $G(I)(\sqcup_{k\geq 0} J_k) \subseteq \sqcup_{k\geq 0} G(I)(J_k)$: If $a \in G(I)(\sqcup_{k\geq 0} J_k)$, then $\exists \rho \in [\mathcal{I}(V) \to V] \cap I$ such that $\rho(\sqcup_{k\geq 0} J_k) = a$. Since ρ is continuous, then $a = \sqcup_{k\geq 0}\rho(J_k)$. But $\rho(J_k) \in G(I)(J_k) \subseteq \cup_{k\geq 0} G(I)(J_k)$. Therefore $a \in \sqcup_{k\geq 0} G(I)(J_k)$, by the definition of $\mathcal{I}(V)$. Now, $\cup_{k\geq 0} G(I)(J_k) = \cup_{k\geq 0}\{\sigma(J_k) | \sigma \in [\mathcal{I}(V) \to V] \cap I\} = \{\sigma(\cup_{k\geq 0} J_k) | \sigma \in [\mathcal{I}(V) \to V] \cap I\} \subseteq \{\sigma(\sqcup_{k\geq 0} J_k) | \sigma \in [\mathcal{I}(V) \to V] \cap I\}$. $\sqcup_{k\geq 0} G(I)(J_k) = \{\sqcup_{i\geq 0} a_i | \{a_i\}_{i\geq 0}$ is an increasing sequence in $\cup_{k\geq 0} G(I)(J_k)\} \subseteq \{\sqcup_{i\geq 0} a_i | \{a_i\}_{i\geq 0} \subset \{\sigma(\sqcup_{k\geq 0} J_k)\} = \{\sqcup_{k\geq 0}\sigma(J_k)\}\} = G(I)(\sqcup_{k\geq 0} J_k)$.

Proposition 39. *For any* $I \in \mathcal{I}(V)$, $\quad F(G(I)) \subseteq I$.

Proof: Indeed, by the definition of $G(I)(J)$,

$$F(G(I)) = \{\sigma \in [\mathcal{I}(V) \to V] | \sigma(J) \in G(I)(J), \forall J \in \mathcal{I}(V)\} =$$
$$= \{\sigma \in [\mathcal{I}(V) \to V] | \sigma \in [\mathcal{I}(V) \to V] \cap I\} \subseteq I$$

Lemma 40. *For any* $J \in \mathcal{I}(V)$, $G(F(f))(J) \subseteq f(J)$, $\forall f : \mathcal{I}(V) \to \mathcal{I}(V)$.

Proof: $G(F(f))(J) = \{\sigma(J) | \sigma \in [\mathcal{I}(V) \to V] \cap F(f)\} =$

$$= \{\sigma(J) | \sigma \in \{\rho \in [\mathcal{I}(V) \to V] | \rho(I) \in f(I), \forall I \in \mathcal{I}(V)\}\} =$$
$$= \{\sigma(J) | \sigma(I) \in f(I), \forall I \in \mathcal{I}(V)\} = \{\sigma(J) | \sigma(J) \in f(J)\} \subseteq f(J).$$

Theorem 41. *Let be* $f : \mathcal{I}(V) \to \mathcal{I}(V)$. *Then* $G(F(f)) = f$ *iff* f *is continuous.*

Proof:

i) We have already proved that $G(I) \in [\mathcal{I}(V) \to \mathcal{I}(V)]$ for all $I \in \mathcal{I}(V)$, and $F(f) \in \mathcal{I}(V)$ for all $f \in \mathcal{I}(V) \to \mathcal{I}(V)$.

ii) We have to prove only $f \subseteq G(F(f))$, because the inclusion $G(F(f)) \subseteq f$ has been proved in Lemma 40. We consider an arbitrary $J \in \mathcal{I}(V)$, and an arbitrary $a \in f(J)$. We can see a as the lub of an increasing sequence $\{a_i\}_{i\geq 0} \subset f(J)$, i.e. $a = \sqcup_{i\geq 0} a_i$.
We define $\sigma : \mathcal{I}(V) \to V$ by

$$\sigma(I) = \begin{cases} max_{i\geq 0}\{a_i | a_i \in f(I)\}, & \text{if } a \notin f(I), \\ a, & \text{if } a \in f(I). \end{cases}$$

It is clear that $\sigma(J) \in f(J)$.
To prove that $\sigma \in [\mathcal{I}(V) \to V]$, we consider an increasing sequence $\{I_k\}_{k\geq 0} \subset \mathcal{I}(V)$. σ is monotone, and $\sqcup_{k\geq 0}\sigma(I_k) \sqsubseteq \sigma(\sqcup_{k\geq 0} I_k)$. $\sqcup_{k\geq 0} I_k = \overline{\cup_{k\geq 0} I_k}$ implies

that if $b \in \sigma(\sqcup_{k\geq 0} I_k)$ then either $b \in \cup_{k\geq 0}\sigma(I_k)$, or there is an increasing sequence $\{b_k\}_{k\geq 0} \subset V$ such that $\sigma(I_k) = b_k$ and $b = \sqcup_{k\geq 0} b_k \in V$. Therefore $\sigma(\sqcup_{k\geq 0} I_k) \sqsubseteq \sqcup_{k\geq 0}\sigma(I_k)$. And $f(J) \subseteq G(F(f))(J)$.

Acknowledgements: Thanks to the referee for the helpful comments on the drafts of this paper.

References

[1] H.P. Barendregt. *The Lambda Calculus: its syntax and semantics*, North Holland, 1984.

[2] G. Ciobanu. A model for a λ-calculus extended for dataflow computation. *An. st. Univ. "A.I.Cuza" Iasi, Informatica* **1**: 57–67, 1992.

[3] G. Ciobanu. Domain equations for internal nondeterminism. Accepted to the 2nd Ershov Conference PSI'96, Novosibirsk, 1996 (to appear in Springer Verlag).

[4] C.A. Gunter, D.S. Scott. Semantic domains. In J.van Leeuwen(ed.) *Handbook of Theoretical Computer Science vol.***B**, 633–674, 1990.

[5] J. Levy. *The Calculus of Refinements, a Formal Specification Model Based on Inclusions.* PhD thesis, Universitat Politecnica de Catalunya, Barcelona, 1994.

[6] D. MacQueen, G. Plotkin, R. Sethi. An ideal model for recursive polymorphic types. *Information and Control* **71**: 95–130, 1986.

[7] D.A. Schmidt. *Denotational semantics: A methodology for language development*, Wm.C. Brown Publishers, 1988.

[8] D.S. Scott. Data types as lattices. *SIAM Journal of Computing* **5**: 522–587, 1976.

[9] M.B. Smyth, G. Plotkin. The category-theoretic solution of recursive domain equations. *SIAM Journal of Computing* **11**: 761–783, 1982.

Received January, 1994
Revised version received December, 1995