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On Binary Minimal Clones*

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Abstract

We determine all minimal clones which contain 3, 4, or 6 binary operations (including the two projections). Furthermore, we give examples of minimal clones containing $2k+2$ ($k \geq 1$), and $3k+2$ ($k \geq 2$) binary operations.

1 Introduction

Clones play a central role in universal algebra and in multiple valued logic (see e.g. [15]). A set of finitary operations is a *clone of operations* (or a *concrete clone*) if it is closed under composition and contains all projections. The clones on a fixed set form a complete lattice with respect to inclusion. If the set is finite, the lattice of clones is atomic and coatomic. The coatoms, i.e. the maximal clones, were classified in Ivo Rosenberg's profound paper [12]. On the contrary, quite little is known about the minimal clones (see Problem P12 in [10]). In a pioneering work Béla Csákány determined all minimal clones on the three-element set [1], [2]. Recently Bogdan Szczepara [14] has obtained all binary minimal clones on the four-element set.

As opposed to maximality, being a minimal clone is an inner property. It means that the clone is generated by any of its nontrivial members (i.e. non-projections). Therefore it is advantageous to consider clones abstractly, what we will do in Section 2.

In this paper we will consider only such minimal clones which are generated by a binary operation. So we investigate algebras with a single binary operation. Formerly such algebras were called *groupoids*, but the 1993 MSRI workshop on Universal Algebra and Category Theory reserved this word for describing certain categories. Hence we will use the newly coined word *binar* for such algebras. All binars we consider will be idempotent, i.e. satisfying $xx = x$. In Section 3 we describe a method for constructing free (relative to some variety) binars.

In Section 4 we give four types of examples of binary minimal clones. One of them yields a negative answer to a problem of Dudek [4].

In Csákány's list [1] all binary minimal clones contain one or two nontrivial binary operations. This observation motivated the investigation of binary minimal

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clones containing a given number of binary operations. We carry out this program in Section 5 for the cases where the number of binary operations is 3, 4, or 6 (including the two projections). The minimal clones with 5 or 7 binary operations have been determined by Dudek [3], [4].

Acknowledgments. We are deeply indebted to Béla Csákány: his pioneering work in the area inspired several ideas of the present paper, and our correspondence and discussions helped to develop these ideas. The method of Bernhard Ganter to construct free binars proved crucial to our computational approach. Discussions with Keith Kearnes paved the way to the discovery of minimal clones with an odd number of binary operations; such clones have been found independently by him at the same time.

2 Generalities

We consider *abstract clones*, i.e. heterogeneous algebras $\mathbf{C}$ on a series of base sets $C_1, C_2, \ldots$ equipped with composition operations $F_n^k : C_k \times C_n^k \longrightarrow C_n$ $(k, n = 1, 2, \ldots)$ and constants (that correspond to the projections) $p_i^j \in C_j$ $(i = 0, \ldots, j - 1; j = 1, 2, \ldots)$ satisfying the identities

$$F_n^k(x, F_n^m(y_0, z_0, \ldots, z_{m-1}), \ldots, F_n^m(y_{k-1}, z_0, \ldots, z_{m-1})) =$$

$$F_n^m(F_m^k(x, y_0, \ldots, y_{k-1}), z_0, \ldots, z_{m-1})$$

$$F_n^k(p_i^k, x_0, \ldots, x_{k-1}) = x_i$$

$$F_n^n(x, p_0^n, \ldots, p_{n-1}^n) = x,$$

where $k, m, n = 1, 2, \ldots$ and $i = 1, \ldots, k - 1$. See Taylor [16] (pp. 360–361), for more details consult [17].

Subclones, homomorphisms, etc. are defined in the natural way. A homomorphism of $\mathbf{C}$ into the clone of operations on a set A is called a *representation* of $\mathbf{C}$. If we single out a set of generators of $\mathbf{C}$ then representations of $\mathbf{C}$ give rise to algebras of the given type. All representations of $\mathbf{C}$ form a variety. Conversely, for any variety the *clone of the variety* is the clone of term functions over the free algebra with countably many generators. Its representations are exactly the algebras in the given variety. In virtue of this correspondence between varieties and (abstract) clones we will freely switch between the two viewpoints. In this respect C_n corresponds to the free algebra on n generators in the variety determined by $\mathbf{C}$.

An operation will be called *trivial* if it is a projection. A representation of $\mathbf{C}$ is *trivial* if its image consists of projections only. An algebra will be called *trivial* if its basic operations are projections.

A clone is *minimal*, if it is not trivial but the only proper subclone is the clone of trivial operations (i.e. projections). In other words, a clone is minimal if it is generated by any nontrivial member of it (and there are nontrivial members). The clone generated by f will be denoted by $[f]$. So a minimal clone can be generated by a single operation. It is convenient to choose one of minimum arity. According to a result of Rosenberg [13] a minimum arity operation f generating a minimal clone falls under the following five types:

(i) f is unary;

(ii) f is binary idempotent, i.e. satisfies $f(x,x) = x$;

(iii) f is ternary majority, i.e. satisfies $f(x,x,y) = f(x,y,x) = f(y,x,x) = x$;

(iv) f is k-ary semiprojection ($k \geq 3$), i.e. — up to renumbering the variables — $f(x_1,\ldots,x_k) = x_1$ for any identification of variables $x_i = x_j$ ($1 \leq i < j \leq k$);

(v) $f(x,y,z) = x + y + z$ for an elementary abelian 2-group with addition $+$.

In this paper we investigate case (ii). Although several results hold more generally, we do not attempt to formulate them here in full generality. In order to simplify notation we will write xy instead of $f(x,y)$. Moreover, to save parentheses we adopt the convention that $x_1x_2x_3\ldots x_n = (\ldots((x_1x_2)x_3)\ldots)x_n$, i.e. products are left-normed. In particular, we write xy^n for $(\ldots((xy)y)\ldots)y$.

By an *absorption identity* we mean an identity of the form $x = t(x, y_1, \ldots, y_n)$, i.e. an identity where one side is a variable.

Lemma 2.1 *Let $\mathcal{V}$ be a variety with minimal clone and $A \in \mathcal{V}$ a nontrivial algebra. Then $\mathcal{V}$ satisfies every absorption identity that holds in A.*

Proof: The clone of A is a nontrivial homomorphic image of the clone of $\mathcal{V}$, hence the inverse image of the trivial subclone on A is a proper subclone of the clone of $\mathcal{V}$. Since the latter is a minimal clone, the inverse image of the trivial clone on A must be the trivial subclone of the clone of $\mathcal{V}$, i.e. if a term is trivial on A then it is trivial in the whole variety $\mathcal{V}$. This is what had to be proved. □

Now let us restrict our attention to algebras with one binary operation, i.e. *binars*.

Lemma 2.2 *Let A be an idempotent binar with minimal clone and define a variety $\mathcal{V}$ of binars by all 2-variable identities and absorption identities that hold in A. Then $\mathcal{V}$ has minimal clone.*

Proof: Let t be a nontrivial term of $\mathcal{V}$. Then t^A is also nontrivial, since $\mathcal{V}$ satisfies all absorption identities of A. We have assumed that A has minimal clone, hence $f^A \in [t^A]$ where f denotes the basic operation. This containment is expressed by a 2-variable identity in A. By the definition of $\mathcal{V}$ this identity holds in $\mathcal{V}$ as well, hence $f \in [t]$, so $[t] = [f]$, indeed. □

Lemma 2.3 *Let the binar A satisfy an equation $xy^k = x$ ($k \geq 2$). Then every identity on A is equivalent to an absorption identity.*

Proof: Let $t = t'$ be an identity. We prove the statement by induction on the length of t. If this length is 1 then t is a variable and we have an absorption identity, so there is nothing to prove. Otherwise, write $t = uv$ with terms u, v shorter than t and observe that $t = t'$ implies $u = uv^k = (uv)v^{k-1} = t'v^{k-1}$, and vice versa: $u = t'v^{k-1}$ implies $t = uv = (t'v^{k-1})v = t'$. Hence $t = t'$ is equivalent to an identity with shorter left-hand side, which is in turn equivalent to an absorption identity by the induction hypothesis. □

Corollary 2.1 *Let the variety $\mathcal{V}$ of binars have minimal clone and assume that $xy^k = x$ $(k \geq 2)$ holds in $\mathcal{V}$. Then $\mathcal{V}$ is generated by any nontrivial algebra in $\mathcal{V}$.*

Proof: Let $A \in \mathcal{V}$ be a nontrivial algebra. Then by Lemma 2.1 $\mathcal{V}$ satisfies every absorption identity that holds in A. However, in this case every identity on A is equivalent to an absorption identity by Lemma 2.3, hence $\mathcal{V}$ satisfies every identity that holds in A. □

We introduce a technical notion. We will say that the clone of the binar A is *2-minimal* if every nontrivial binary term function of A generates the same clone as the basic operation.

Lemma 2.4 *Let A be an idempotent binar with 2-minimal clone. Assume that A contains an element 0 such that $ab = 0$ only if $a = 0 = b$ $(a, b \in A)$. Let $\mathcal{V}$ be the variety defined by all 2-variable identities that hold in A. Then the clone of $\mathcal{V}$ is minimal.*

Proof: Let $t(x_1, \ldots, x_n)$ be a term in which each variable $x_1, \ldots, x_n$ does occur as a factor. If $n = 1$, then idempotence of the operation yields that t is a projection. If $n \geq 2$ then let $\bar{t}(x, y) = t(x, y, \ldots, y)$. An easy induction argument yields that $\bar{t}^A(a, b) = 0$ only if $a = 0 = b$ $(a, b \in A)$. Hence $\bar{t}^A$ is nontrivial. By 2-minimality $f^A \in [\bar{t}^A]$. Now we can finish the proof as in Lemma 2.2. □

3 Free binars

In constructing free binars we will follow a method we have learned from Bernhard Ganter [6].

Let $\mathbf{C}$ be a clone and $\mathcal{V}$ the corresponding variety. Then C_2 can be identified with the 2-generator free algebra in $\mathcal{V}$. The elements of the free algebra can be viewed in two different ways: on one hand as elements of that algebra, on the other hand as binary terms. The two viewpoints are united in the composition operation F_2^2 of the clone: In $F_2^2(t, u_0, u_1)$ t behaves as a term operation and u_0, u_1 as elements to which t is applied. Now fix u_0, u_1 and consider the map $t \mapsto F_2^2(t, u_0, u_1)$. In this way we obtain the endomorphism of the algebra C_2 given by substituting u_i for x_i $(i = 0, 1)$ in the terms t. The set of the endomorphisms

$$\varepsilon_{u_0 u_1}(t) = F_2^2(t, u_0, u_1)$$

forms a *sharply* 2-*transitive* transformation monoid, in the sense that for every u_0, $u_1 \in C_2$ there is a unique endomorphism $\varepsilon_{u_0 u_1}$ such that $\varepsilon_{u_0 u_1}(x_i) = u_i$ $(i = 0, 1)$.

Conversely, this property can be used for constructing free algebras.

Lemma 3.1 *Let F be a set with designated elements $0 \neq 1$ and let M be a sharply 2-transitive transformation monoid on F, i.e. $M \subseteq F^F$ such that for every pair of elements u_0, $u_1 \in F$ there is a unique $m_{u_0 u_1} \in M$ with $m_{u_0 u_1}(i) = u_i$ $(i = 0, 1)$. For every $f \in F$ define a binary operation by $\widehat{f}(u_0, u_1) = m_{u_0 u_1}(f)$. Then $\mathbf{F} = (F; \{\widehat{f} | f \in F\})$ is a free algebra over the set $\{0, 1\}$ in the variety generated by $\mathbf{F}$.*

Proof: We have to show that for every u_0, $u_1 \in F$ there is an endomorphism ε of $\mathbf{F}$ such that $\varepsilon(i) = u_i$ $(i = 0, 1)$ (see [7], p. 165, Corollary 1). We show that in fact $m = m_{u_0 u_1} \in \mathrm{End}(\mathbf{F})$. Indeed, since $\widehat{f}(m(a_0), m(a_1)) = m_{m(a_0)m(a_1)}(f)$ and $m(\widehat{f}(a_0, a_1)) = m(m_{a_0 a_1}(f))$ we have to check that $m \circ m_{a_0 a_1} = m_{m(a_0)m(a_1)}$. Both sides are members of M, so by sharp 2-transitivity of M it suffices to verify whether they agree on 0 and 1. Indeed, we have for $i = 0, 1$: $m(m_{a_0 a_1}(i)) = m(a_i) = m_{m(a_0)m(a_1)}(i)$. □

Note that each operation $\widehat{f}$ is idempotent if and only if all constant maps belong to M. Furthermore, observe that the clone is 2-minimal if and only if each nontrivial binary operation generates all others, i.e. if and only if 0 and 1 generate the binar $(F, \widehat{f})$ for each $f \neq 0, 1$.

As an illustration of this method we give a new proof of a result of J. Dudek [5], Theorem 2.3(a):

Proposition 3.1 *Assume that a binary minimal clone contains finitely many binary operations. If every nontrivial binary operation in the clone is commutative, then there is only one nontrivial binary operation.*

Proof: Let us consider the sharply 2-transitive monoid M associated with the clone. Let us denote by $\beta \in M$ the permutation interchanging the generators 0, 1 and fixing all other elements: $\beta(f) = F_2^2(f, 1, 0)$. Take an arbitrary $\alpha \in M$ with $\alpha(0) = 0$ and $\alpha(1) \notin \{0, 1\}$, and let k be such that $0, 1, \alpha(1), \ldots, \alpha^{k-1}(1)$ are all different but $\alpha^k(1) \in \{0, 1, \alpha(1), \ldots, \alpha^{k-1}(1)\}$. We distinguish three cases:

(a) $\alpha^k(1) = 0$. Set $\gamma = \alpha^{k-1} \in M$ and $\delta = (\gamma\beta)^2 \in M$. Then $\gamma(0) = 0$, $\delta(0) = \gamma(\beta(\gamma(\beta(0)))) = \gamma(\beta(\gamma(1))) = \gamma(\beta(\alpha^{k-1}(1))) = \gamma(\alpha^{k-1}(1)) = \alpha^{2(k-1)}(1) = 0$, $\delta(1) = \gamma(\beta(\gamma(\beta(1)))) = \gamma(\beta(0)) = \gamma(1)$, so γ and δ agree on 0 and 1. However, we have $\gamma(\alpha^{k-1}(1)) = 0$, but

$$\delta(\alpha^{k-1}(1)) = \gamma(\beta(\gamma(\beta(\alpha^{k-1}(1))))) = \gamma(\beta(\gamma(\alpha^{k-1}(1)))) =$$

$$\gamma(\beta(0)) = \gamma(1) = \alpha^{k-1}(1) \neq 0,$$

contradicting the properties of M. So this case cannot occur.

(b) $\alpha^k(1) = 1$. Then α^k fixes both 0 and 1 hence it is the identity, therefore α is a permutation. Restricting α and β to the set $S = \{0, 1, \alpha(1), \ldots, \alpha^{k-1}(1)\}$ we get a k-cycle and a transposition with one common point. These permutations generate the full symmetric group on S. If $k-1 \geq 2$, it would contain a nontrivial permutation fixing 0 and 1, which is impossible. Hence $S = \{0, 1, \alpha(1)\}$ and $\alpha^2(1) = 1$. The two transpositions $\alpha|_S$ and $\beta|_S$ generate the full symmetric group of order 6 on S, and that together with the three constants form a sharply 2-transitive monoid on S. This means that S is closed under the clone operation F_2^2, so by minimality of the clone $S = C_2$.

(c) $\alpha^k(1) = \alpha^j(1)$ for some $1 \leq j < k$. Set $\gamma = \alpha^{(k-j)j}$, $\gamma(1) = 2$ and $S = \{0, 1, 2\}$. Then $\alpha^{k-j}(2) = \alpha^{k-j}(\alpha^{(k-j)j}(1)) = \alpha^{(k-j-1)j}(\alpha^k(1)) = \alpha^{(k-j-1)j}(\alpha^j(1)) = \alpha^{(k-j)j}(1) = 2$ and $\gamma(2) = (\alpha^{k-j})^j(2) = 2$. Now β, γ and the constants restricted to S again generate a sharply 2-transitive monoid on S, and the minimality of the clone yields again that $S = C_2$. □

4 Examples

In this section we describe four series of minimal clones. Two of them, the affine spaces over GF(p) and the p-cyclic binars (p prime), are well-known, the other two are new. Further examples appear in Section 5, namely in Theorems 5.1(b), 5.2(b) — the rectangular bands, 5.2(c), 5.2(e), 5.4(b), 5.4(c), and 5.4(e).

4.1 Affine spaces

Let $V \neq 0$ be a vector space over the field F. Then the affine space on the set V has the following basic operations: $x - y + z$ and $\lambda x + (1 - \lambda)y$ for each $\lambda \in F$. The clone of the affine space consists of the terms $\sum \lambda_i x_i$ where $\lambda_i \in F$, $\sum \lambda_i = 1$. This clone is minimal if and only if F is a p-element field for some prime p. If $p = 2$ then the clone is generated by the ternary minority function $x + y + z$. If $p > 2$ then the clone contains nontrivial binary operations, e.g. $2x - y$, so it is within the scope of our present interest. However, even then it is more convenient to use the ternary operation $f(x, y, z) = x - y + z$ to axiomatize the variety:

$$f(x, x, y) = f(y, x, x) = y$$

$$f(f(x_{11}, x_{12}, x_{13}), f(x_{21}, x_{22}, x_{23}), f(x_{31}, x_{32}, x_{33})) =$$
$$f(f(x_{11}, x_{21}, x_{31}), f(x_{12}, x_{22}, x_{32}), f(x_{13}, x_{23}, x_{33}))$$

$$f^p(x, y, z) = x$$

where $f^1(x, y, z) = f(x, y, z)$, $f^{j+1}(x, y, z) = f(f^j(x, y, z), y, z)$.

We will denote this clone by $\mathbf{A}(p)$. Note that the number of binary operations in $\mathbf{A}(p)$ is p.

4.2 p-cyclic binars

The term *p-cyclic binar* ("groupoid") has been introduced by Płonka [9] using the following axioms:

$$xx = x, \quad x(yz) = xy, \quad (xy)z = (xz)y, \quad xy^p = x$$

(Recall that $xy^p = (\ldots((xy)y)\ldots)y$ with p factors y.) He showed [8] that they have minimal clones, whenever p is a prime.

We will denote this clone by $\mathbf{C}(p)$. For representations of $\mathbf{C}(p)$ see [11]. The binary operations in $\mathbf{C}(p)$ are xy^j, yx^j ($j = 0, 1, \ldots, p-1$), their number is $2p$.

4.3 Binary minimal clones with $2k+2$ ($k \geq 1$) binary operations

Define a binar F on the set $\{0, 1, a_0, \ldots, a_{k-1}, b_0, \ldots, b_{k-1}\}$ with the operation

$$xy = \begin{cases} a_0 & \text{if } x = 0,\ y = 1 \\ a_{j+1} & \text{if } x = 0,\ y = b_j \\ b_0 & \text{if } x = 1,\ y = 0 \\ b_{j+1} & \text{if } x = 1,\ y = a_j \\ x & \text{otherwise} \end{cases} \tag{1}$$

where the indices are taken modulo k. Examples for $k = 1$ and 2 can be found in Section 5: F_6 and F_{13} respectively. (There $a_0 = 2$, $b_0 = 3$, $a_1 = 4$, $b_1 = 5$.)

Lemma 4.1 *F is a free binar generated by* 0 *and* 1*, and it has* 2*-minimal clone.*

Proof: Let $f_0(x, y) = xy$ and $f_{j+1}(x, y) = xf_j(y, x)$ for $j = 0, \ldots, k-1$. One can check by induction on j that

$$f_j(x, y) = \begin{cases} a_j & \text{if } x = 0,\ y = 1 \\ b_j & \text{if } x = 1,\ y = 0 \\ xy & \text{otherwise} \end{cases}$$

In particular, $f_k = f_0$. Now it is straightforward to verify that $x, y, f_0(x, y), \ldots, f_{k-1}(x, y), f_0(y, x), \ldots, f_{k-1}(y, x)$ is a complete list of the binary term functions of F and $x \mapsto 0$, $y \mapsto 1$, $f_j(x, y) \mapsto a_j$, $f_j(y, x) \mapsto b_j$ gives an isomorphism between the free binar of term functions over F and F. Hence F is free.

We also have $f_{j+1}(x, y) = f_j(x, f_j(y, x))$, hence every nontrivial binary term generates all other binary terms, i.e. the clone is 2-minimal. □

Lemma 4.2 *F satisfies the identities*

$$x(xy_1 \ldots y_m) = x \qquad (m = 0, 1, 2, \ldots) \tag{2}$$

Proof: If $x = a_j$ or $x = b_j$ then $xy = x$ for all y, so the equation holds. If $x = 0$ then we can prove by induction on m that for arbitrary $y_1, \ldots, y_m$ we have $xy_1 \ldots y_m \in \{0, a_0, \ldots, a_{k-1}\}$. Then $x(xy_1 \ldots y_m) = x$ holds again. The case $x = 1$ is symmetric. □

Proposition 4.1 *Let $\mathcal{V}$ be the variety defined by the 2-variable identities of F and by the identities (2). Then $\mathcal{V}$ has minimal clone.*

Proof: Let t be a term with first variable x. We prove by induction on the length of t that either t is a projection in $\mathcal{V}$, or identification of all variables different from x yields a nontrivial binary term in $\mathcal{V}$. Then it will follow that the clone of $\mathcal{V}$ is minimal, since $\mathcal{V}$ satisfies all 2-variable identities of F and the clone of F is 2-minimal by Lemma 4.1.

If t is a variable, the statement is obvious. So let $t = t_1t_2$. After the said identification of variables we obtain a binary term $t' = t'_1t'_2$. By the induction hypothesis either $t_1 = x$ or t'_1 is a nontrivial binary term. In the latter case (1) yields $t' = t'_1t'_2 = t'_1$, i.e. t' is nontrivial. So assume $t_1 = x$. If the first variable of t_2 is also x, then (2) implies $t = xt_2 = x$, a projection. If the first variable of t_2, say, y is different from x, then $t'_2 \in \{y, f_0(y,x), \ldots, f_{k-1}(y,x)\}$ and so $t' = xt'_2 \in \{f_0(x,y), \ldots, f_{k-1}(x,y)\}$ is a nontrivial binary term. □

4.4 Binary minimal clones with $3k+2$ $(k \geq 2)$ binary operations

We are going to construct very many free binars with minimal clone over the set $\{0, 1, a_0, \ldots, a_{k-1}, b_0, \ldots, b_{k-1}, c_0, \ldots, c_{k-1}\}$. Let τ be the permutation $(0\,1)(b_0\,c_0)\ldots(b_{k-1}\,c_{k-1})$. (The elements $a_0, \ldots, a_{k-1}$ are fixed by τ.) Let T be any binar on this set satisfying the following four conditions:

(i) for each $j = 0, \ldots, k-1$ we have the following part of the multiplication table of T:

	0	1	a_j	b_j	c_j
0	0	a_0	b_j	b_j	c_j
1	a_0	1	c_j	b_j	c_j
a_j	b_j	c_j	a_j	b_j	c_j
b_j	b_j	b_j	b_j	b_j	a_{j+1}
c_j	c_j	c_j	c_j	a_{j+1}	c_j

(The subscripts are taken modulo k, i.e. $a_k = a_0$.)

(ii) for each $j = 0, \ldots, k-1$ we have the following part of the multiplication table of T:

	b_j	c_j	a_{j+1}
b_j	b_j	a_{j+1}	a_{j+1}
c_j	a_{j+1}	c_j	a_{j+1}
a_{j+1}	a_{j+1}	a_{j+1}	a_{j+1}

(Again $a_k = a_0$.)

(iii) for every $0 \le i < j < k$, $u \in \{a_i, b_i, c_i\}$, $v \in \{a_j, b_j, c_j\}$ $\{u, v\}$ is a semilattice as a subalgebra of T.

(iv) τ is an automorphism of T.

Requirements (i) and (ii) uniquely determine the product of certain pairs of elements. For the remaining pairs $\{u, v\}$ (iii) leaves two choices: either $uv = vu = u$ or $uv = vu = v$. However, this choice determines also $\tau(u)\tau(v) = \tau(v)\tau(u) = \tau(uv)$, and $(\tau(u), \tau(v))$ and (u, v) are different pairs unless $u = a_i$ and $v = a_j$. Hence the number of binars satisfying (i)–(iv) is

$$2^{\binom{k}{2}+\frac{1}{2}[8\binom{k}{2}-2k]} = 2^{(5k^2-7k)/2}$$

Proposition 4.2 *Any T satisfying (i)–(iv) is a free binar generated by* 0 *and* 1 *and has minimal clone.*

Proof: Let $f_0(x, y) = xy$ and define for $j = 0, \ldots, k-1$ the terms $g_j(x, y) = xf_j(x, y)$ and $f_{j+1}(x, y) = g_j(x, y)g_j(y, x)$. One can check that

$$f_j(x,y) = \begin{cases} a_j & \text{if } x = 0,\ y = 1 \\ a_j & \text{if } x = 1,\ y = 0 \\ xy & \text{otherwise} \end{cases} \qquad g_j(x,y) = \begin{cases} b_j & \text{if } x = 0,\ y = 1 \\ c_j & \text{if } x = 1,\ y = 0 \\ xy & \text{otherwise.} \end{cases}$$

In particular, $f_k = f_0$. Now it is straightforward to verify that $x, y, f_0(x, y)$, $\ldots, f_{k-1}(x, y)$, $g_0(x, y)$, $\ldots, g_{k-1}(x, y)$, $g_0(y, x), \ldots, g_{k-1}(y, x)$ is a complete list of binary term functions of T and $x \mapsto 0$, $y \mapsto 1$, $f_j(x, y) \mapsto a_j$, $g_j(x, y) \mapsto b_j$, $g_j(y, x) \mapsto c_j$ gives an isomorphism between the free binar of term functions over T and T. Hence T is free.

We also have $g_j(x, y) = f_j(x, f_j(x, y))$ and $f_{j+1}(x, y) = g_j(g_j(x, y), g_j(y, x))$, hence every nontrivial binary term generates all other binary terms, i.e. the clone is 2-minimal.

Since $uv = 0$ only if $u = v = 0$, the clone of T is minimal Lemma 2.4. □

Corollary 4.1 *Let $\mathcal{V}$ be the variety defined the 2-variable identities of T. Then $\mathcal{V}$ has minimal clone.*

Our construction disproves a conjecture of Dudek [4], Problem 2: There are binary minimal clones other than the affine clones which contain a prime number of binary operations. Indeed, for any prime number $p > 5$, $p \equiv 5 \pmod 6$ our construction yields (a lot of) binary minimal clones with $p = 3\frac{p-2}{3} + 2$ binary operations.

5 Minimal clones with few binary operations

We are going to determine all binary minimal clones $\mathbf{C}$ with $|C_2| = 3$, 4, or 6. For the sake of completeness we will quote results of Dudek concerning the cases $|C_2| = 5$ and 7.

First of all we need a complete list — up to term equivalence — of 2-generator free binars with n elements ($n = 3$, 4, or 6). Then we check whether the clones of these free binars are 2-minimal, i.e. if every nontrivial binary term generates the basic operation. These two steps can be done automatically. Though they require tedious calculations, they pose no theoretical difficulties. For $n = 3, 4$ we did these calculations by hand, for $n = 6$ we used a computer. In fact we enumerated the sharply 2-transitive monoids on the set $\{0, 1, 2, 3, 4, 5\}$. Monoids were represented by 6×36 arrays whose entries were elements of the set $\{u, 0, 1, 2, 3, 4, 5\}$ (u stands for an undefined entry). Rows corresponded to transformations, columns to binary operations. The enumeration process was a backtrack search and it consisted of the following seven steps. In Step 1 a yet undefined entry of the array was chosen, in Steps 2 through 7 the chosen entry was defined to be $0, \ldots, 5$, respectively, and consequences of this definition were recorded in the array (consequences arise from composition of rows). Of course, Step 1 is the critical point of the procedure, one wishes to choose an entry yielding as many consequences as possible. Our strategy was to choose the topmost undefined entry in the column of the most frequently appearing symbol.

Here we just present the results. The particular free binars F_i ($i = 1, \ldots, 15$) will be dealt with separately below, where we give their multiplication tables.

Lemma 5.1 *Up to term equivalence the following is a complete list of* 2*-generator free idempotent binars with* n *elements having a* 2*-minimal clone:*

(a) F_1 *and* F_2, *if* $n = 3$;

(b) $F_3, \ldots, F_9$, *if* $n = 4$;

(c) $F_{10}, \ldots, F_{15}$, *if* $n = 6$.

Now we determine case-by-case which of these 15 binars have minimal clone. Either we exhibit a nontrivial ternary term operation that turns into a projection by every identification of the variables — and hence the clone is not minimal, or we give some absorption identities which together with the 2-variable identities of F_i determine a variety with minimal clone. In some cases we will be able to reduce the number of 2-variable identities needed to define the variety by making use of certain implications, some of which are taken from Szczepara's thesis [14].

Lemma 5.2 *The following implications hold:*

(a) $x(yx) = xy$ *implies* $(xy)(yx) = xy^2$, $x(yx^2) = xy$, $(xy^2)(yx) = xy^3$, $(xy)(yx^2) = xy^2$, $(xy^2)(yx^2) = xy^3$;

(b) $xx = x$ *and* $x(yxz) = x$ *imply* $x(yxz_1 \ldots z_m) = x$ *for all* $m \geq 0$, *and* $x(xy) = x(yx) = x$, $(xy)x = (xy)y = (xy)(yx) = xy$;

(c) (cf. [14], Lemma 121) $xx = x$, $x(xy) = xy$ *and* $x(yx) = xy$ *imply* $(xy)x = xy$;

(d) (cf. [14], Lemma 125) $(xy)y = xy$ *and* $x(y(xy)) = xy$ *imply* $(xy)(y(xy)) = xy$.

Proof: (a) Substituting xy for x we get $(xy)(y(xy)) = (xy)y$. Since $y(xy) = yx$, we obtain the first identity. Substituting yx for y yields $x((yx)x) = x(yx) = xy$. Now let us substitute xy for x in $(xy)(yx) = xy^2$, then it follows that $(xy^2)(y(xy)) = xy^3$. Here $y(xy) = yx$, hence we get the third identity. Furthermore, $(xy)(yx^2) = (xy)((yx)(xy)) = (xy)(yx) = xy^2$ and $(xy^2)(yx^2) = ((xy)(yx))((yx)(xy)) = ((xy)(yx))(yx) = (xy^2)(yx) = xy^3$.

(b) First we derive the 2-variable identities: $x(xy) = x((xx)y) = x$, $x(yx) = x(yx(yx)) = x$, $(xy)x = (xy)(x(xy)) = xy$, $(xy)y = (xy)(y(xy)) = xy$, and $(xy)(yx) = (xy)((y(xy))x) = xy$. Next we show by induction on m that $x(yxz_1 \ldots z_m) = x$ holds. Let $t = yxz_1 \ldots z_{m-1}$. Then $xt = x$ by the induction hypothesis. Now $tx = t(xt) = t$, hence $x(yxz_1 \ldots z_m) = x(tz_m) = x((tx)z_m) = x$. □

Theorem 5.1 *Let* **C** *be a binary minimal clone with* 3 *binary operations. Then either*

(a) $\mathbf{C} = \mathbf{A}(3)$, *the clone of affine spaces over the* 3*-element field; or*

(b) *the nontrivial binary operation in* **C** *satisfies* $xx = x$, $xy = yx$ *and* $x(xy) = xy$. *The variety definied by these identities has minimal clone.*

Proof: By Lemma 5.1(a) there are two possibilities for the 2-generator free binar C_2:

F_1 :

	0	1	2
0	0	2	1
1	2	1	0
2	1	0	2

F_2 :

	0	1	2
0	0	2	2
1	2	1	2
2	2	2	2

Clearly, F_1 is the affine line over GF(3) [$xy = 2x+2y \pmod 3$]. In F_1 $(xy)y = x$ holds, hence by Corollary 2.1 **C** is equal to the clone of operations of F_1, i.e. $\mathbf{C} = \mathbf{A}(3)$.

By Lemma 5.1(a) the clone of F_2 is 2-minimal. It is easy to check that $xx = x$, $xy = yx$, $x(xy) = xy$ is a basis for 2-variable identities of F_2. By Lemma 2.4 the variety defined by the 2-variable identities of F_2 has minimal clone. □

Theorem 5.2 *Let* $\mathbf{C}$ *be a binary minimal clone with* 4 *binary operations. Then* $\mathbf{C}$ *contains a nontrivial binary operation for which one of the following holds:*

(a) $\mathbf{C} = \mathbf{C}(2)$, *the clone of* 2*-cyclic binars;*

(b) $\mathbf{C} = \mathbf{RB}$, *the clone of rectangular bands, defined by* $xx = x$, $x(yz) = (xy)z = xz$;

(c) $\mathbf{C}$ *satisfies* $xx = x$, $x(xy) = x(yx) = (xy)y = xy$;

(d) $\mathbf{C}$ *satisfies* $x(xy_1 \dots y_m) = x$ *for* $m = 0, 1, 2, \dots$ *and* $x(yx) = (xy)x = (xy)y = xy$;

(e) $\mathbf{C}$ *satisfies* $xx = x$ *and* $x((yx)z) = x$.

All clones defined by the equations in (a)–(e) are minimal.

Proof: By Lemma 5.1(b) we get seven possibilities for the 2-generator free binar C_2.

F_3 :

	0	1	2	3
0	0	2	0	2
1	3	1	3	1
2	2	0	2	0
3	1	3	1	3

F_4 :

	0	1	2	3
0	0	2	2	0
1	3	1	1	3
2	0	2	2	0
3	3	1	1	3

F_5 :

	0	1	2	3
0	0	2	2	2
1	3	1	3	3
2	2	2	2	2
3	3	3	3	3

Clearly, F_3 is the 2-generator free 2-cyclic binar. In F_3 $(xy)y = x$ holds, hence by Corollary 2.1 $\mathbf{C}$ is equal to the clone of operations of F_3, i.e. $\mathbf{C} = \mathbf{C}(2)$.

F_4 is a rectangular band. Hence it satisfies the absorption identities $xx = x$, $((xy)z)x = x$, $x(y(zx)) = x$ and $(xy)(zx) = x$. In fact, these imply the usual defining identities of $\mathbf{RB}$: $xz = [((xy)z)x][z((xy)z)] = (xy)z$, and similarly $xz = [(x(yz))x][z(x(yz))] = x(yz)$. In virtue of Lemma 2.1 $\mathbf{C} = \mathbf{RB}$, as the clone of rectangular bands is obviously minimal.

By Lemma 5.1 the clone of F_5 is 2-minimal. By Lemma 2.4 the variety defined by the 2-variable identities of F_5 has minimal clone. Idempotence, Lemma 5.2(a) and (c), and the interchanging of x and y yield that all 2-variable identities of F_5 follow from the ones listed in (c).

F_6 :

	0	1	2	3
0	0	2	0	2
1	3	1	3	1
2	2	2	2	2
3	3	3	3	3

F_7 :

	0	1	2	3
0	0	2	0	0
1	3	1	1	1
2	2	2	2	2
3	3	3	3	3

F_6 is the free binar with $k = 1$ constructed in Section 4.3. The identities given in (d) imply all the remaining 2-variable identities of F_6 using idempotence, Lemma 5.2(a) and the interchanging of x and y. Hence the results in Section 4.3 yield (d).

For F_7 we can proceed similarly. We show that $x(yxz_1 \dots z_m) = x$ for $m = 0, 1, 2, \dots$ holds in F_7. If $x = 2$ or $x = 3$, it is obvious. Let $x = 0$. Then $yx \in \{0, 2, 3\}$, and it follows by induction that $yxz_1 \dots z_m \in \{0, 2, 3\}$. Hence $x(yxz_1 \dots z_m) = x$ in this case. The case $x = 1$ is symmetric. Conversely, we show that any nontrivial operation satisfying the identities in (e) generates a minimal clone. Lemma 5.2(b) gives $x(yxz_1 \dots z_m) = x$ for all $m \geq 0$ and also some 2-variable identities. Now we can proceed similarly as in the proof of Proposition 4.1. Let t be an arbitrary term with first variable x, and identify all other variables with y. Then t turns into either x or xy. In the latter case we are done. In the first case we write $t = t_1(t_2t_3 \dots t_r)$, where t_2 is a variable. After the said identification we get $x = t'_1(t'_2t'_3 \dots t'_r)$. From the 2-variable identities it follows that $t'_1 = x$ and $t'_2 \dots t'_r = x$, xy, or yx. In the first two cases $t_2 = x$. By the induction hypothesis $t_1 = x$ and so $t = t_1(t_2t_3 \dots t_r) = x(xt_3 \dots t_r) = x$. In the third case $t_2 = y$. Let s be such that $t'_2 \dots t'_j = y$ for $j = 2, \dots, s-1$, but $t'_2 \dots t'_s \neq y$. Then $t'_2 \dots t'_s = yx$ and $t'_s = x$. By the induction hypothesis we have $t_1 = x$ and $t_s = x$, so we infer $t = t_1(t_2 \dots t_s \dots t_r) = x((t_2 \dots t_{s-1})xt_{s+1} \dots t_r) = x$.

F_8:

	0	1	2	3
0	0	2	0	0
1	3	1	1	1
2	2	2	2	0
3	3	3	1	3

F_9:

	0	1	2	3
0	0	2	3	1
1	3	1	0	2
2	1	3	2	0
3	2	0	1	3

In the last two cases the clones are not minimal. For F_8 we construct a nontrivial ternary semiprojection $t = (x(yz))(zx)$. Indeed, $t(0,0,1) = t(0,1,0) = t(0,1,1) = 0$, hence $t(x,x,z) = t(x,y,x) = t(x,y,y) = x$; but $t(0,1,2) = 2$, so t is not the projection onto the first variable. Hence t generates a nontrivial proper subclone.

Finally, for F_9 we have that $t = (xy)(zx)$ is a ternary minority operation, as $t(0,0,1) = t(0,1,0) = t(1,0,0) = 1$. Thus the clone of F_9 is not minimal. (In fact, F_9 is the affine line over the 4-element field.) □

Remark 5.1 *The binary minimal clones on the three-element set found by Csákány [1] fall into the following cases:*

Theorem 5.1(a): [624]; (b): [0], [10], [178]; Theorem 5.2(a): [68]; (c): [8], [11], [16], [17], [26]; (d): [35]; (e): [33].

Theorem 5.3 (Dudek [3]) *Let* **C** *be a binary minimal clone with* 5 *binary operations. Then* $\mathbf{C} = \mathbf{A}(5)$, *the clone of affine spaces over the* 5*-element field.*

Theorem 5.4 *Let* $\mathbf{C}$ *be a binary minimal clone with* 6 *binary operations. Then* $\mathbf{C}$ *contains a nontrivial binary operation for which one of the following holds:*

(a) $\mathbf{C} = \mathbf{C}(3)$, *the clone of* 3*-cyclic binars;*

(b) $\mathbf{C}$ *satisfies* $xx = x$, $x(xy) = x(y(xy)) = (xy)x = (xy)y = (xy)(yx) = (xy)(x(yx)) = xy$, $(x(yx))y = (x(yx))(xy) = (x(yx))(y(xy)) = x(yx)$;

(c) $\mathbf{C}$ *satisfies* $xx = x$, $x(xy) = x(y(xy)) = (xy)x = (xy)y = (xy)(yx) = (xy)(x(yx)) = (x(yx))(xy) = xy$, $(x(yx))y = (x(yx))(y(xy)) = x(yx)$;

(d) $\mathbf{C}$ *satisfies* $x(xz_1 \dots z_m) = x$ *for* $m = 0, 1, 2, \dots$, $x(y(xy)) = (xy)x = (xy)y = (xy)(yx) = (xy)(x(yx)) = (xy)(y(xy)) = xy$, $(x(yx))y = (x(yx))(xy) = (x(yx))(y(xy)) = x(yx)$;

(e) $\mathbf{C}$ *satisfies* $x(xz_1 \dots z_m) = x$ *for* $m = 0, 1, 2, \dots$, $x(yx) = (xy)x = ((xy)y)x = xy$, $((xy)y)y = (xy)y$;

All clones defined by the equations in (a)–(e) are minimal.

Proof: By Lemma 5.1(c) we get 6 possibilities (up to term equivalence) for the 2-generator free binar C_2.

F_{10} :

	0	1	2	3	4	5
0	0	2	0	2	0	2
1	3	1	3	1	3	1
2	2	4	2	4	2	4
3	5	3	5	3	5	3
4	4	0	4	0	4	0
5	1	5	1	5	1	5

Clearly, F_{10} is the 2-generator free 3-cyclic binar. In F_{10} $((xy)y)y = x$ holds, hence by Corollary 2.1 $\mathbf{C}$ is equal to the clone of operations of F_{10}, i.e. $\mathbf{C} = \mathbf{C}(3)$.

F_{11} :

	0	1	2	3	4	5
0	0	2	2	4	4	2
1	3	1	5	3	3	5
2	2	2	2	2	2	2
3	3	3	3	3	3	3
4	4	4	4	4	4	4
5	5	5	5	5	5	5

F_{12} :

	0	1	2	3	4	5
0	0	2	2	4	4	2
1	3	1	5	3	3	5
2	2	2	2	2	2	2
3	3	3	3	3	3	3
4	4	4	2	4	4	4
5	5	5	5	3	5	5

By Lemma 5.1(c) the clones of F_{11} and F_{12} are 2-minimal. By Lemma 2.4 the varieties defined by the 2-variable identities of F_{11}, resp. F_{12} have minimal clones. Using Lemma 5.2(d) and obvious substitutions one can check that the identities given in (b) and (c) imply all 2-variable identities of F_{11} and F_{12}, respectively.

F_{13} :

	0	1	2	3	4	5
0	0	2	0	4	0	2
1	3	1	5	1	3	1
2	2	2	2	2	2	2
3	3	3	3	3	3	3
4	4	4	4	4	4	4
5	5	5	5	5	5	5

F_{14} :

	0	1	2	3	4	5
0	0	2	0	2	0	2
1	3	1	3	1	3	1
2	2	4	2	4	2	4
3	5	3	5	3	5	3
4	2	4	4	4	4	4
5	5	3	5	5	5	5

F_{13} is the free binar with $k = 2$ constructed in Section 4.3. The results there yield (d).

For F_{14} one can proceed similarly, leading to (e). Here one should apply Lemma 5.2(a). We leave proving the analog of Proposition 4.1 to the reader.

F_{15} :

	0	1	2	3	4	5
0	0	2	0	0	0	0
1	3	1	1	1	1	1
2	2	2	2	4	2	2
3	3	3	5	3	3	3
4	4	4	4	4	4	0
5	5	5	5	5	1	5

Here we define a ternary operation $t(x,y,z) = [(x(yz))(zx)][(yx)(xy)]$. Now $t(0,1,1) = t(0,0,1) = t(0,1,0) = 0$, so t is a semiprojection onto the first variable. As $t(0,1,2) = 2$, we see that t is nontrivial. Hence t generates a nontrivial proper subclone. □

Remark 5.2 *Let W_{ix} denote the variety defined by the equations in Theorem 5.i (x). (So the number of different nontrivial binary terms is i.) For a variety V and a term t let $V[t]$ denote the variety of algebras in V with basic operation t. Then we have the following relationship between our results and the six minimal clone varieties $V_1, \dots, V_6$ and their subvarieties V_3', V_3'', V_6', V_6'' found by Szczepara [14], pp. 205–206: W_{1a} has no four-element model, $W_{1b} = V_6'$, $W_{2a} = V_1[yx]$, $W_{2b} = V_5$, $W_{2c} = V_6''$, $W_{2d} = V_3''$, $W_{2e} = V_4$, $W_{4a} = V_2[yx]$, $W_{4b} = V_6$, every four-element binar in W_{4c} belongs to W_{2c}, every four-element binar in W_{4d} belongs to W_{2d}, $W_{4e} = V_3[(xy)x]$. Furthermore, $W_{4d} \cap W_{4e} = W_{2d} \supset V_3'$ and $W_{4b} \cap W_{4c} = W_{2c} \supset W_{1b}$.*

Theorem 5.5 (Dudek [4]) *Let* $\mathbf{C}$ *be a binary minimal clone with 7 binary operations. Then* $\mathbf{C} = \mathbf{A}(7)$, *the clone of affine spaces over the 7-element field.*

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