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A note on connection between PNS and set covering problems

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Abstract

Process network synthesis (PNS) has enormous practical impact; however, its mixed integer programming model is tedious to solve because it usually involves a large number of binary variables. Using a combinatorial approach, a structural model of PNS can be given, and a branch-and-bound technique can be applied for searching an optimal solution. In some realistic examples of PNS, this method is efficient. Nevertheless, efficient methods are unavailable for solving these models generally. In this note, we describe a special class of PNS-problems as set-covering or set-partitioning problems. These problems are well-known to be NP-complete, thus, a PNS-problem is NP-hard.

1 Introduction

In a manufacturing system, materials of different properties are consumed through various mechanical, physical and chemical transformation to yield desired products. Devices in which these transformations are carried out are called operating units, e.g., a lathe or a chemical reactor. Thus, a manufacturing system can be considered as a network of operating units which is called process network. The importance of process network synthesis (PNS) arises from the fact that such networks are ubiquitous in the chemical and allied industries. Naturally, the cost minimization of a process network is indeed essential. Several papers have appeared in the literature on the application of global optimization in PNS (see, e.g., [1] and [5]). Another approach is a combinatorial one based on the feasible graphs of processes (cf.[2],[3] and [4]). This approach makes possible to show that the search of an optimal solution is difficult in general. Here, we prove this statement.

2 Definitions

Let M be a given set of objects which are materials capable of being converted or transformed by the processes under consideration. Transformation between two

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subsets of M occurs in an operating unit. It is necessary to link this operating unit to others through the elements of these two subsets of M. The resultant structure is a process graph which is a bipartite directed graph (see [3] and [4]). Its formal definition is given as follows:

Definition 1. Let M be a finite nonempty set, and let $\emptyset \neq O \subseteq \wp'(M) \times \wp'(M)$ with $M \bigcap O = \emptyset$, where $\wp'(M)$ denotes the set of all nonempty subsets of M. The elements of O are called *operating units* and for an operating unit $u = (\alpha, \beta) \in O$, α and β are called the *input-set* and *output-set* of u, respectively. Pair (M, O) is defined to be a *process graph*. The set of vertices of this graph is $M \bigcup O$, and the set of arcs is $A = A_1 \cup A_2$ with $A_1 = \{(X, Y) : Y = (\alpha, \beta) \in O \text{ and } X \in \alpha\}$ and $A_2 = \{(Y, X) : Y = (\alpha, \beta) \in O \text{ and } X \in \beta\}$. If there exist vertices $X_1, X_2, ..., X_n$, such that $(X_1, X_2), (X_2, X_3), \ldots, (X_{n-1}, X_n)$ are arcs of process graph (M, O), then $[X_1, X_n]$ is defined to be a *path* from vertex X_1 to vertex X_n.

Let process graphs (m, o) and (M, O) be given. (m, o) is defined to be a *subgraph* of (M, O), if $m \subseteq M$ and $o \subseteq O$.

Let us now consider a process design problem in which the nonempty set of desired products is $P \subseteq M$, the set of raw materials is $R \subseteq M$, and the nonempty set of available operating units is $O \subseteq \wp'(M) \times \wp'(M)$. Let us suppose that $P \bigcap R = \emptyset$ and $M \bigcap O = \emptyset$. Then, process graph (M, O) contains the interconnections among the operating units of O.

On the other hand, let us observe that each feasible process, producing the given set P of products from the given set R of raw materials using operating units from O, corresponds to a subgraph of (M, O). It can be visualised as the structure of the considered process. By appropriately examining the corresponding subgraphs of (M, O), therefore, we can determine an optimal process in principle. Since (P, R, O) describes the considered process design problem in structural point of view, this triplet is called the *structural model* of PNS.

If we do not consider further constraints such as material balance, then the subgraphs of (M, O) which can be assigned to a feasible process have common combinatorial properties. They are studied in [3] and their description is given by the following definition.

Definition 2. Subgraph (m, o) of (M, O) is called a *solution-structure* of (P, R, O) if the following properties are satisfied.

$(S1)$ $P \subseteq m$,

$(S2)$ $\forall X \in m$, $X \in R \Leftrightarrow$ no (Y, X) arc in the process graph (m, o),

$(S3)$ $\forall Y_0 \in o$, $\exists$ path $[Y_o, Y_n]$ with $Y_n \in P$,

$(S4)$ $\forall X \in m$, $\exists (\alpha, \beta) \in o$ such that $X \in \alpha \bigcup \beta$.

Let us denote the set of solution-structures of (P, R, O) by $S(P, R, O)$. Now we are ready to define a simple subclass of PNS-problems.

3 PNS-problems with weights

Let us consider PNS-problems in which each operating unit has a weight. We are to find a feasible process with the minimal weight where by weight of a process we mean the sum of the weights of the operating units belonging to the process under consideration. Each feasible process in such a class of PNS-problems is determined uniquely from the corresponding solution- structure and vice versa. Thus, the above problem can be formalized as follows:

PNS-problem with weigths

Let a structural model of PNS-problem (P, R, O) be given. Moreover, let w be a real-valued function defined on O, the weight function. The basic model is then

$$\min\{\sum_{u \in o} w(u) : (m, o) \in S(P, R, O)\}.$$

We refer to the problem defined above as a PNS$_w$*-problem*; we denote the class of such problems by PNS$_w$. In what follows, we define a subclass of PNS$_w$, which is equivalent to the class of the classical set covering problems.

Let us denote by PNS$_{w1}$ the subclass of PNS$_w$ for which a problem from PNS$_w$ given by (P, R, O) and w is contained in PNS$_{w1}$ if and only if $O \subseteq \wp'(R) \times \wp'(P)$. The visual meaning of this subclass can be given as follows:

It contains such process design problems in which the operating units use only the raw materials as inputs and yield only the desired materials as outputs; moreover, they perform in parallel. (We note that the performance in paralell way yields a difficult scheduling in the general case.)

Now let us consider an arbitrary PNS$_{w1}$-problem, given by (P, R, O) and w. Let $O = \{u_1, \ldots, u_n\}$ and $u_j = (\alpha_j, \beta_j) \in \wp'(R) \times \wp'(P)$, $j = 1, \ldots, n$. Then, it is easy to see that this PNS$_{w1}$-problem is equivalent to the set covering problem determined by the set P, the system of its subsets, β_j, $j = 1, \ldots, n$, and the weights, $w'(\beta_j) = w(u_j)$, $j = 1, \ldots, n$.

Conversely, let us consider an arbitrary set covering problem. Let it be given by the set P, the system of its subsets, β_j, $j = 1, \ldots, n$, and the weights, $w'(\beta_j)$, $j = 1, \ldots, n$. Now let R be an arbitrary nonempty finite set with $P \bigcap R = \emptyset$. Furthermore, let $u_j = (R, \beta_j)$, $j = 1, \ldots, n$, $O = \{u_1, \ldots, u_n\}$. If we define the function w on O by $w(u_j) = w'(\beta_j)$, $j = 1, \ldots, n$, then the PNS$_{w1}$-problem determined by (P, R, O) and w is equivalent to the set covering problem under consideration.

The above observations gives rise to the following statement.

Theorem. *The class* PNS$_{w1}$ *is equivalent to the class of the set covering problems.*

Obviously, if a PNS$_{w1}$-problem is equipped with the condition that each desired product can be produced by at most one operating unit in each solution-structure, then we can construct an equivalent set partitioning problem and vice versa. Since both the set covering and set partitioning problems are well-known to be NP-complete, the PNS_{w1}-problem must be NP-complete. This leads immediately to the next corollary.

Corollary. *The* PNS-*problem is* NP-*hard.*

Let us observe that in a PNS_{w1}-problem the set of materials is divided into two disjoint sets P and R, and each operating unit has a nonempty subset of R as inputs and a nonempty subset of P as outputs. Generalizing this feature, we can define a further subclass of PNS_w-problems. Specifically, let $k \geq 1$ be an arbitrary fixed integer in order for us to consider the problems in which $M = M_1 \bigcup \ldots \bigcup M_{k+1}$ where the sets $M_1, \ldots, M_{k+1}$ are pairwise disjoint nonempty sets. Furthermore, let $O = O_1 \bigcup \ldots \bigcup O_k$ with $O_i \subseteq \wp'(M_1 \bigcup \ldots \bigcup M_i) \times \wp'(M_{i+1})$, $i = 1, \ldots, k$. Let us call such a PNS_w-problem a PNS_{wk}*-problem.* It is of interest to know if there an equivalent known optimization problem exists.

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