Acta Cybernetica 12 (1996) 313-324.

Steiner Minimum Trees for Equidistant Points on Two Sides of an Angle*

Rainer E. Burkard † Tibor Dudás †

Abstract

In this paper we deal with the Steiner minimum tree problem for a special type of point sets. These sets consist of the vertex of an angle 2α and equidistant points lying on the two sides of this angle.

1 Introduction

Jarník and Kössler (1934) formulated the following problem: *Determine the shortest tree which connects n given points in the plane.* Seven years later, Courant and Robbins (1941) describe this problem in their classical book "What is Mathematics?" and contribute this problem for $n = 3$ to J. Steiner, though Torricelli and Cavalieri gave solutions for the triangle already in 1640. For an account on the history of this problem see Hwang, Richards and Winter (1992). Since Courant and Robbins this problem is called *Steiner Minimum Tree (SMT) Problem.*

For an arbitrary point set X in the plane with $|X| = n$ the problem is quite difficult. Until Melzak (1961) it was not even known that it is finitely solvable. Garey, Graham and Johnson (1977) proved that the Steiner minimum tree problem is $\mathcal{NP}$-hard. This means that unless $\mathcal{P} = \mathcal{NP}$ there does not exist a polynomial (and efficient) algorithm which solves this problem. Therefore a considerable interest arose in studying special point sets X for which an SMT can be found in polynomial time. The first special point sets considered were *ladders*, see Chung and Graham (1978) and the recent correction by Burkard, Dudás and Maier (1994). Other special point sets include *zigzag lines* (Du, Hwang and Weng, 1983), *checkerboards* (Chung, Gardner and Graham, 1989), *Chinese checkerboards* (Hwang and Du, 1991), *bar waves* (Du and Hwang, 1987), *sets of four points* (Du, Hwang, Song and Weng), *regular polygons* (Du, Hwang and Weng, 1987) and *points on a circle* (Du, Hwang and Chao, 1985).

In this article we contribute a new special case : *triangle ladders*, where the given points consist of the vertex of an angle 2α and further points lying equidistantly

*This research has been partially supported by the Spezialforschungsbereich F 003 "Optimierung und Kontrolle" / Projektbereich Diskrete Optimierung.

†TU Graz, Institut für Mathematik B, Steyrergasse 30, A-8010 Graz, Austria

on the two sides of this angle. We shall determine the structure of an SMT in dependence of the angle α.

2 Definitions and preliminaries

Let X be a point set in the plane consisting of n points ($|X| = n$). If we connect these points just by straight lines whose total length is minimum, we get a *minimum spanning tree (MST)*. For n points, such an MST can be determined in $O(n \log n)$ steps (see e. g. Edelsbrunner, 1987) by using arguments from computational geometry like Delaunay triangulations. Thus an MST problem is a well solvable problem. But in general a minimum spanning tree is not a Steiner minimum tree. Consider an equilateral triangle with side lengths a. A minimum spanning tree consists of any two sides and has length $2a$. If we introduce, however, a new point S in the center of this triangle and connect this new point with all three vertices, we get an SMT of length $a\sqrt{3}$, see Figure 1. So, by inserting new points, the so-called *Steiner points*, the total length of a connection of the given (*regular*) points can be decreased.

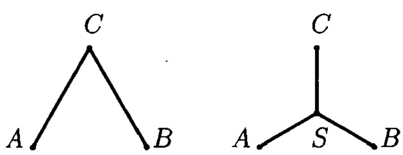

Figure 1. A minimum spanning tree a) and a Steiner minimum tree b) for the vertices A, B, C of an equilateral triangle

In the following we denote a minimum spanning tree for the point set X by $M^*(X)$ and a Steiner minimum tree by $S^*(X)$. An *edge* of a tree is denoted by $e = (A, B)$, where A and B are the *incident* vertices. If A and B are connected by an edge, we say, A and B are *adjacent*. The *length* of an edge $e = (A, B)$ is the Euclidean distance $d(A, B)$ of the points A and B. The *length of a tree* T is the sum of the lengths of the edges in T and it is denoted by $l(T)$. The next lemmas summarize some important facts about Steiner minimum trees, for the simple proofs see e. g. Hwang, Richards and Winter (1992).

Lemma 2.1 *(a) In any Steiner point of $S^*(X)$, exactly three edges meet in an angle of exactly* $120°$.

(b) The angle of two edges that meet in a common regular point of $S^*(X)$ is at least $120°$.

(c) $S^*(X)$ has at most $n-2$ Steiner points.

The last statement motivates the following definition:

Definition 2.2 *$S^*(X)$ is a* full Steiner minimum tree (FSMT) *if $S^*(X)$ contains exactly $n-2$ Steiner points.*

Moreover, Steiner points cannot lie anywhere:

Lemma 2.3 *For any X the SMT $S^*(X)$ lies within the convex hull of X.*

In the following we shall use the notation of a clockwise path of an edge e in $S^*(X)$. The *clockwise path* of e is the path starting with e and turning clockwise whenever possible. For example in the SMT of Figure 1b the clockwise path of the edge (A, S) continues with the edge (S, B) Similarly, we can define the *counter-clockwise path* of e. Note that both types of paths can end only in regular points.

3 A new special type of points sets

In order to facilitate the description of point sets we consider a fixed coordinate system in the plane. We describe the position of points with respect to this coordinate system.

Definition 3.1 *For any fixed $n \in \mathbb{N}$ and angle α with $0° < \alpha < 90°$ the point set $L_n^\alpha = \{I, A_1, ..., A_n, B_1, ..., B_n\}$ is called* triangle ladder *if $I = (0,0)$, $A_i = (-i\sin\alpha, -i\cos\alpha)$ and $B_i = (i\sin\alpha, -i\cos\alpha)$ $(i = 1, ..., n)$ (see Figure 2). Moreover, $T_n^\alpha = L_n^\alpha \setminus \{I\}$ is the triangle ladder without I.*

Note that a triangle ladder consists of the vertex I of an angle and points lying equidistantly on the two sides of the angle 2α. The points A_n and B_n of L_n^α are called *terminal points*.

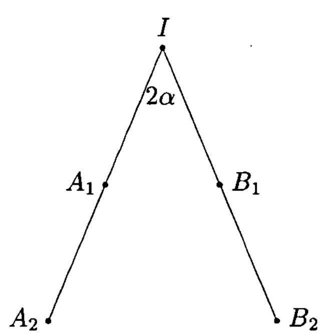

Figure 2. Triangle ladder L_2^α

Lemma 3.2 *For all n and α a terminal point in $S^*(L_n^\alpha)$ has exactly one incident edge.*

Proof. It is trivial that at least one edge is incident to A_n. Because of Lemma 2.1(b) at most three edges are incident to A_n. Suppose that two or three edges meet in A_n. Let β denote the angle between any two edges. It follows from Lemma 2.3 that $\beta \leq \angle B_n A_n I = 90° - \alpha < 90°$ because the convex hull of L_n^α is the triangle IA_nB_n. But this contradicts Lemma 2.1(b). So just one edge is incident to A_n and B_n. ■

Lemma 3.3 *In $S^*(L_n^\alpha)$ no edge has a length exceeding 1.*

Proof. Suppose that $S^*(L_n^\alpha)$ has an edge e whose length is greater than 1. Deleting the edge e from $S^*(L_n^\alpha)$ we get two connected components which can be reconnected by some edge on the sides of the angle, i. e. by an edge of length 1. The total length of $S^*(L_n^\alpha)$ decreases by this operation, a contradiction to the optimality of $S^*(L_n^\alpha)$. ■

Corollary 3.4 *If $n \geq 2$ and a terminal point is in $S^*(L_n^\alpha)$ adjacent to a Steiner point, then this Steiner point lies within the trapezoid $A_nB_nB_{n-1}A_{n-1}$.*

Lemma 3.5 *(A_i, B_i) $(i = 1, ..., n)$ cannot be an edge of $S^*(L_n^\alpha)$.*

Proof. Suppose that (A_i, B_i) is an edge of $S^*(L_n^\alpha)$ for any i. It follows from Lemma 2.3 that there is an other edge e of $S^*(L_n^\alpha)$ incident with A_i or B_i which lies within the triangle IA_iB_i. But the angle formed by e and (A_i, B_i) is less or equal than $90° - \alpha < 120°$, a contradiction to Lemma 2.1. ■

Let us next consider the path $P = A_nX_1...X_kB_n$ in $S^*(L_n^\alpha)$ which begins in the terminal point A_n and ends in the terminal point B_n. We get

Lemma 3.6 *P is a clockwise path of (A_n, X_1).*

Proof. If P is not a clockwise path of (A_n, X_1), then the clockwise path of (A_n, X_1) can nowhere end, a contradiction. ■

The following two lemmas describe simple conditions for Steiner minimum trees on nested sets.

Lemma 3.7 *Let X and Y be two sets of points in the plane with $X \subset Y$. Then $l(S^*(X)) \leq l(S^*(Y))$.*

Proof. $S^*(Y)$ connects the point set X, because $X \subset Y$. So the above inequality follows from the definition of the SMT. ■

Lemma 3.8 *Let X and Y be two sets of points in the plane with $X \subset Y$. Suppose that all $y \in Y$ fulfills one of the following two conditions :*

(i) $y \in X$,

(ii) y lies on an edge of a fixed $S^*(X)$.

Then $S^*(X)$ is an SMT for Y.

Proof. Clearly, $S^*(X)$ connects the points of Y. Because of Lemma 3.7 $l(S^*(X)) \leq l(S^*(Y))$. But it means that $S^*(X)$ is an SMT for Y. ■

4 The case $\alpha \geq 60°$

In this section we will deal with triangle ladders L_n^α, where $\alpha \geq 60°$. The following theorems describe the simple solution in this case.

Theorem 4.1 *If $\alpha \geq 60°$, then the MST $M^*(L_n^\alpha)$ for L_n^α which consists of the edges $(I, A_1), (I, B_1), (A_i, A_{i+1}), (B_i, B_{i+1})$ $(i = 1, ..., n-1)$ is an SMT for L_n^α.*

Proof. Let us consider the triangle $T = IA_nB_n$. Clearly, $\angle A_nIB_n = 2\alpha \geq 120°$. It follows from Lemma 2.1 that the Steiner minimum tree for the triangle T cannot have any Steiner point. Thus the SMT for T is an MST with the edges (I, A_n) and (I, B_n). Hence, $l(M^*(T)) = d(I, A_n) + d(I, B_n) = d(I, A_1) + d(A_1, A_2) + ... + d(A_{n-1}, A_n) + d(I, B_1) + d(B_1, B_2) + ... + d(B_{n-1}, B_n) = l(M^*(L_n^\alpha))$. It is obvious that $T \subset L_n^\alpha$. So Lemma 3.8 yields that the SMT for T is an SMT for L_n^α. ∎

Theorem 4.2 *$M^*(T_n^\alpha)$ which consists of the edges $(A_1, B_1), (A_i, A_{i+1}), (B_i, B_{i+1})$ $(i = 1, ..., n-1)$ is an SMT for T_n^α.*

Proof. Let us consider the trapezoid $T = A_1A_nB_nB_1$. It can be proved by elementary geometry and by Lemma 2.1 that $S^*(T)$ does not contain any Steiner point, so $M^*(T)$ is an SMT for T. Clearly, $l(M^*(T)) = l(M^*(T_n^\alpha))$ and $T \subset T_n^\alpha$, so Lemma 3.8 implies the statement. ∎

5 The case $30° \leq \alpha < 60°$

In this section we examine the structure of the SMT for L_n^α, where $30° \leq \alpha < 60°$. We assume in the whole section that α fulfills the previous inequality.

Lemma 5.1 *If a Steiner point is adjacent to a terminal point in $S^*(L_n^\alpha)$, then there is a path between the terminal points which contains only Steiner points.*

Proof. Suppose that in $S^*(L_n^\alpha)$ the Steiner point S_1 is adjacent to A_n. It follows from Corollary 3.4 that S_1 lies within the trapezoid $A_nB_nB_{n-1}A_{n-1}$. If S_1 is adjacent to B_n, then we are ready. Therefore, let us suppose that S_1 is adjacent to an other Steiner point S_2. Because $\angle S_1A_nB_n < 90° - \alpha \leq 60°$ and $\angle A_nS_1S_2 = 120°$ the slope of the edge (S_1, S_2) is negative (see Figure 3).

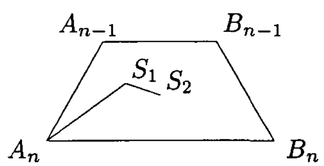

Figure 3

If S_2 is adjacent to B_n, then we are ready, otherwise we can continue the previous idea. So, we get Steiner points $S_1,...,S_k$ and S_k is finally adjacent to B_n. ∎

Lemma 5.2 *If a Steiner point S_1 is adjacent to a terminal point in $S^*(L_n^\alpha)$, then S_1 is adjacent to the other terminal point.*

Proof. Assume that S_1 is adjacent to A_n. It follows from Lemma 5.1 that there exist Steiner points $S_1,...,S_k$ such that the edges (S_i, S_{i+1}) $(i = 1,...,k-1)$ and (S_k, B_n) are edges of $S^*(L_n^\alpha)$. Let us consider the polygon $P = A_nS_1...S_kB_n$. It follows from Lemma 3.6 that P is a convex polygon. Let γ be the sum of the angles of P. Because P is a $(k+2)$-gon $\gamma = k180°$. On the other hand $\gamma = k120° + \angle B_nA_nS_1 + \angle S_kB_nA_n < (k+1)120°$. It follows that $k60° < 120°$, and so $k < 2$. This inequality implies $k = 1$. It means that on the path from A_n to B_n the only Steiner point is S_1. ∎

Lemma 5.3 *If $n \geq 2$, then in $S^*(L_n^\alpha)$ a terminal point is not adjacent to any Steiner point.*

Proof. Because of Corollary 3.4 it is to show that a terminal point is not adjacent to a Steiner point which lies within the trapezoid $A_nB_nB_{n-1}A_{n-1}$. Assume that A_n is adjacent to the Steiner point S_1 which lies in $A_nB_nB_{n-1}A_{n-1}$. Lemma 5.2 yields that S_1 is adjacent to B_n. It is easy to see that $d(A_n, B_n) \geq 2$, because $\alpha \geq 30°$ and $n > 1$. Clearly, $d(A_n, S_1) + d(S_1, B_n) > d(A_n, B_n) \geq 2$. But this means that at least one of the edges (A_n, S_1), (S_1, B_n) is longer than 1, a contradiction to Lemma 3.3. ∎

Theorem 5.4 *Let H denotes the triangle IA_1B_1. $S^*(L_n^\alpha)$ is the union of the tree $S^*(H)$ and the edges (A_i, A_{i+1}), (B_i, B_{i+1}) $(i = 1,...,n-1)$. For n=1,2,... the length of $S^*(L_n^\alpha)$ is given by $l(S^*(L_n^\alpha)) = 2(n-1) + \cos\alpha + \sqrt{3}\sin\alpha$.*

Proof. We prove the theorem by induction on n.

If $n = 1$, then clearly $S^*(L_n^\alpha)=S^*(H)$ and it can be proved by elementary geometry that $l(S^*(H)) = \cos\alpha + \sqrt{3}\sin\alpha$.

Suppose now that for $n-1$ the theorem is true. Let us consider L_n^α. Lemma 5.3 implies that the terminal point A_n is adjacent to A_{n-1}, because A_{n-1} is the nearest regular point to A_n. Similarly, B_n is adjacent to B_{n-1}. It follows from the induction assumption that $S^*(L_n^\alpha)$ is in fact the union of the tree $S^*(H)$ and the edges (A_i, A_{i+1}), (B_i, B_{i+1}) $(i = 1,...,n-1)$ and $l(S^*(L_n^\alpha))=l(S^*(L_{n-1}^\alpha))+d(A_n, A_{n-1})+d(B_n, B_{n-1}) = 2(n-2) + \cos\alpha + \sqrt{3}\sin\alpha + 2 = 2(n-1) + \cos\alpha + \sqrt{3}\sin\alpha$. ∎

In Figure 4 we present the SMT for $L_2^{30°}$.

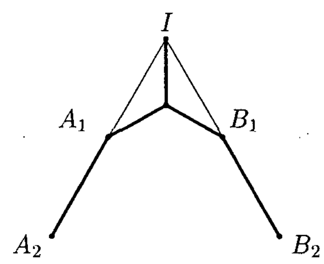

Figure 4. $L_2^{30^\circ}$ and its SMT

Let us now consider $L_n^{45^\circ}$. If we rotate all points by 135°, then we get the nonnegative, integer points $0, 1, 2, ..., n$ lying on the x and y axis. This is an interesting special case and the length of the SMT for this points set is given by $2(n-1) + \frac{\sqrt{2}}{2}(\sqrt{3}+1)$.

Theorem 5.5 *$M^*(T_n^\alpha)$ which consists of the edges $(A_1, B_1), (A_i, A_{i+1}), (B_i, B_{i+1})$ $(i = 1, ..., n-1)$ is an SMT for T_n^α. The length of $S^*(T_n^\alpha)$ is given by $l(S^*(T_n^\alpha)) =$ $2n - 1$.*

Proof. We prove the theorem by induction on n.

If $n = 1$, the statements are trivial.

Suppose that the theorem is true for $n - 1$ where $n > 1$. Because of Lemma 5.3 A_n is adjacent to A_{n-1} and B_n is adjacent to B_{n-1} in $S^*(T_n^\alpha)$. The rest of the proof is similar to the proof of Theorem 5.4. ∎

6 The case $\alpha < 30^\circ$

In this section we present results for the case $\alpha < 30^\circ$. We show all solutions for $\alpha > 14.478^\circ$ and outline the general stucture of solutions for smaller angles.

Suppose that $n \geq 2$ and (A_n, S_1) is an edge of $S^*(L_n^\alpha)$. Because of Corollary 3.4 S_1 lies within the trapezoid $A_nB_nB_{n-1}A_{n-1}$. Consider the clockwise path P of (A_n, S_1). Let $P = A_nS_1X_1...X_k$, where X_j denotes a regular point or a Steiner point $(1 \leq j \leq k)$.

Case 1. $\angle B_nA_nS_1 < 60^\circ$

The edge on P leaving S_1 has a negative slope, and it can be proved (see Lemma 5.1 and Lemma 5.2) that (S_1, B_n) is an edge of $S^*(L_n^\alpha)$.

Case 2. $\angle B_nA_nS_1 \geq 60^\circ$

a) Suppose that $X_i \neq B_{n-1}$ for $1 \leq i \leq k$. Then it follows from Lemma 3.3 that X_1 a Steiner point. It is easy to check that the edge on P leaving X_1 has a negative slope. But then $X_k = B_n$ and X_1,...,X_{k-1} are Steiner points. We show that $k = 2$. Consider the polygon $A_nS_1X_1...X_{k-1}B_n$ and let γ denote the sum of its angles.

Clearly, $\gamma = k180°$. On the other hand $\gamma = k120° + \angle B_nA_nS_1 + \angle X_{k-1}B_nA_n < k120° + 180°$. Hence, $k60° < 180°$, and so $k < 3$. This last inequality implies $k \leq 2$ and it is trivial that $k \neq 1$.

b) Suppose that there exists an index i with $1 \leq i \leq k$ such that $X_i = B_{n-1}$. In this case (B_n, B_{n-1}) is an edge of $S^*(L_n^\alpha)$: if B_n is adjacent to a Steiner point S_2, then the counterclockwise path of (B_n, S_2) can nowhere end, a contradiction. The fact that (B_n, B_{n-1}) is an edge of $S^*(L_n^\alpha)$ and Lemma 2.1 imply $i \geq 2$, so X_1 is a Steiner point. As above, the edge on P leaving X_1 has a negative slope. But then $i = k-1$ $(X_{k-1} = B_{n-1})$, $X_k = B_n$ and $X_1, ..., X_{k-2}$ are Steiner points. Similarly as above, it can be proved that $k = 3$, so $X_2 = B_{n-1}$ and $X_3 = B_n$.

Concluding these results, we show in Figure 5 and in Figure 6 the possibilities of subgraphs of $S^*(L_n^\alpha)$, which contain the terminal points. There are four cases. The subgraph T_1 contains the edges (A_n, A_{n-1}) and (B_n, B_{n-1}). In T_2 the terminal points are adjacent to a common Steiner point. The subgraph T_3 contains the edge (B_n, B_{n-1}) and the clockwise path from A_n to B_{n-1} with two Steiner points. In T_4 the terminal points are adjacent to different Steiner points S_1, S_2 and S_1 is adjacent to S_2. Note that there is a fifth case, namely if (A_n, A_{n-1}) is an edge and there is clockwise path from B_n to A_{n-1} with two Steiner points. But because of symmetry this case is similar to T_3.

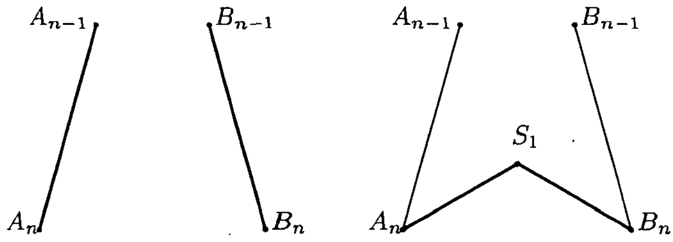

Figure 5. The subgraphs T_1 (left) and T_2 (right)

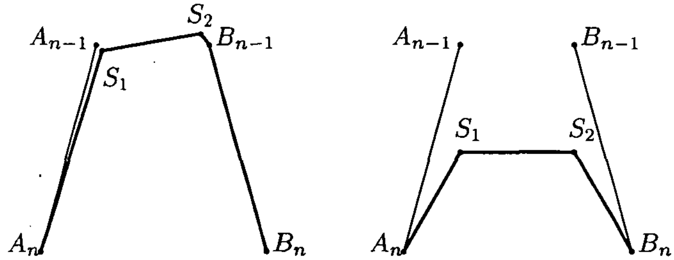

Figure 6. The subgraphs T_3 (left) and T_4 (right)

In the following we shall need two results which can easily be derived by elementary geometry.

(i) Consider any quadrangle $ABCD$ with $\angle CDA = \angle BCD = 120°$. If $d(A,D) = d(B,C) = d(C,D) = 1$, we get $d(A,B) = 2$. Therefore, if $d(A,B) > 2$, then at least one of the distances $d(A,D)$, $d(B,C)$, $d(C,D)$ is greater than 1.

(ii) Consider the trapezoid $A_nB_nB_{n-1}A_{n-1}$. It is easy to see that $d(A_{n-1}, B_{n-1}) = 2(n-1)\sin\alpha$. By the cosine law we get

$$d(A_n, B_{n-1}) = \sqrt{1 + 4(n-1)^2\sin^2\alpha - 4(n-1)\sin\alpha\cos(90° + \alpha)} =$$
$$\sqrt{1 + 4n(n-1)\sin^2\alpha}. \tag{1}$$

The next theorem tells us, when the terminal points in $S^*(L_\alpha^n)$ are adjacent to A_{n-1} and B_{n-1}.

Theorem 6.1 *Let*

$$N_\alpha = \max\{\frac{1}{\sin\alpha}, \frac{\sqrt{1 + \frac{3}{\sin^2\alpha}} + 1}{2}\}. \tag{2}$$

If $n > N_\alpha$, then (A_n, A_{n-1}) and (B_n, B_{n-1}) are edges of $S^*(L_n^\alpha)$.

Proof. Let $n > N_\alpha$. We will show that $S^*(L_n^\alpha)$ cannot contain the subtrees T_2, T_3 and T_4.

The condition $n > \frac{1}{\sin\alpha}$ guarantees that $d(A_n, B_n) = 2n\sin\alpha > 2$. If T_2 occurs, then due to the triangle inequality at least one of the edges (A_n, S_1) or (S_1, B_n) must be longer than 1. This is impossible due to Lemma 3.3. If T_4 occurs, we note first that the angles in S_1 and S_2 are 120°, since S_1 and S_2 are Steiner points. Therefore by (i), one of the edges (A_n, S_1), (S_1, S_2) or (S_2, B_n) is longer than 1, which again is impossible. The condition $n > \frac{\sqrt{1+\frac{3}{\sin^2\alpha}}+1}{2}$ guarantees that $d(A_n, B_{n-1}) > 2$ (insert this expression in (1)). An analogue argument as above using (i) and Lemma 3.3 shows that T_3 cannot occur. ∎

Corollary 6.2 *Let $n_\alpha = [N_\alpha]$. For any $n > n_\alpha$ the edges $(A_n, A_{n-1}), \ldots, (A_{n_\alpha+1}, A_{n_\alpha})$ and $(B_n, B_{n-1}), \ldots, (B_{n_\alpha+1}, B_{n_\alpha})$ are edges of $S^*(L_n^\alpha)$.*

It is easy to see that

$$\frac{1}{\sin\alpha} = \frac{\sqrt{1 + \frac{3}{\sin^2\alpha}} + 1}{2}, \text{ if } \sin\alpha = \frac{1}{4} \tag{3}$$

and

$$N_\alpha = \begin{cases} \frac{1}{\sin\alpha}, & \text{if } \sin\alpha \le \frac{1}{4}\ (\alpha \le 14.478°), \\ \frac{\sqrt{1 + \frac{3}{\sin^2\alpha}} + 1}{2} & \text{otherwise.} \end{cases} \tag{4}$$

The previous corollary has the following consequence. If we fix α, we compute at first the number n_α. If $n > n_\alpha$, then the construction of the SMT $S^*(L_n^\alpha)$ is divided into two parts. First, we must construct the SMT $S^*(L_{n_\alpha}^\alpha)$ which we call the *upper component*. Then we have to connect the points $A_{n_\alpha+1}, \ldots, A_n, B_{n_\alpha+1}, \ldots, B_n$ by the edges described in Corollary 6.2 to the upper component $S^*(L_{n_\alpha}^\alpha)$.

Now we want to apply the above procedure for angles, which yield $n_\alpha = 2$. To do this, we have to determine $S^*(L_2^\alpha)$ for any α.

We know already $S^*(L_2^\alpha)$ for $\alpha \geq 30°$. Let us suppose $\alpha < 30°$. Then $M^*(L_2^\alpha)$ contains the edge (A_1, B_1). But Lemma 3.5 tells us that this $M^*(L_2^\alpha)$ cannot be an SMT. Thus $S^*(L_2^\alpha)$ has at least one Steiner point. Suppose that $S^*(L_2^\alpha)$ has one Steiner point. It can easily be seen that in this case the Steiner point S_1 is adjacent to the points I, A_1 and B_1 (use Lemma 2.1). Let T_2^1 denote the tree which consists of the edges (A_2, A_1), (B_2, B_1), (A_1, S_1), (B_1, S_1) and (S_1, I). The structure of T_2^1 can be seen in Figure 4. The length of T_2^1 is $2 + \sqrt{3}\sin\alpha + \cos\alpha$.

If $S^*(L_2^\alpha)$ has 3 Steiner points (it is a full tree), then it is unique up to reflection. This tree is denoted by T_2^2 and shown in Figure 7. The length of T_2^2 is given by $l(T_2^2) = \sqrt{4 + 6(4\sin^2\alpha + \sqrt{3}\sin 2\alpha)}$.

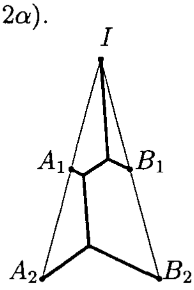

Figure 7. L_2^α and the tree T_2^2

As last case there remains the possibility that $S^*(L_2^\alpha)$ has two Steiner points (S_1 and S_2).

(a) S_1 is adjacent to I, A_1, B_1 and S_2 to A_1, A_2, B_2. This is impossible, because $\angle S_1 A_1 S_2 < 120°$.

(b) S_1 is adjacent to A_1, S_2, A_2 and S_2 to B_1, B_2, S_1. Then $S^*(L_2^\alpha)$ contains the edge (A_1, I) or (B_1, I). It can be verified that the length of this tree is greater than the length of T_2^1 or T_2^2, a contradiction.

(c) S_1 is adjacent to A_1, S_2, B_1 and S_2 to A_2, B_2, S_1. Then $S^*(L_2^\alpha)$ contains the edge (A_1, I) or (B_1, I) and in at least one of these cases we get an angle which is less than 120° , a contradiction. (If we choose the other case, we cannot get an SMT, because the two possible trees have the same length.)

Concluding these observations, we have

Theorem 6.3 *$S^*(L_2^\alpha)$ is of one of the following types (see Figure 8).*

Type a) is optimal for $a \geq 60°$, Type b) is optimal for $17.344° < \alpha < 60°$ and Type c) is optimal for $\alpha < 17.344°$.

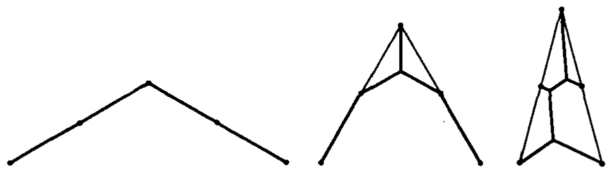

Figure 8. The three types (a,b,c) of SMTs for L_2^α

■

Note that Type c) is a full SMT.

Moreover, one can prove by a rather time consuming distinction of cases that for all α with $\sin\alpha > \frac{1}{4}$, $S^*(L_3^\alpha)$ contains the edges (A_3, A_2) and (B_3, B_2). On the other hand if α fulfills the above inequality, $N_\alpha < 4$ holds. Hence, $n_\alpha = 3$. This means that we know the SMTs for L_n^α for all n and for all α with $\sin\alpha > \frac{1}{4}$ (see Corollary 6.2).

For smaller angles an explicit analysis is complicated due to the many possible cases. We have seen that the tree T_2^2 is an FSMT. But for example if $\alpha = 10°$, the SMT $S^*(L_3^\alpha)$ is not an FSMT, it has only 4 Steiner points. The structure of the SMT for $L_3^{10°}$ can be seen in Figure 9.

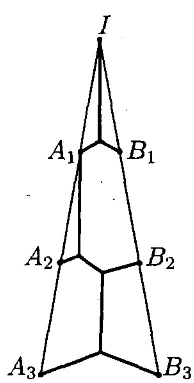

Figure 9. The structure of the SMT for $L_3^{10°}$

A principal solution method for smaller angles consists in determining the corresponding n_α, then the determination of an SMT for $L_{n_\alpha}^\alpha$ which is finally complemented by edges on the sides of the angle. The determination of $S^*(L_{n_\alpha}^\alpha)$ can be done by a general purpose Steiner tree algorithm, see e. g. Cockayne and Hewgill (1992). This method is well applicable for sets of up to 100 regular points. In our case it means that by this method problems with $\alpha \geq 1.17°$ can be tackled.

References

[1] R. E. Burkard, T. Dudás and T. Maier, Cut and Patch Steiner Trees for Ladders, *SFB Report* **12**, Institut für Mathematik, TU Graz (1994), in *Discrete Mathematics* **162** (1996).

[2] F. R. K. Chung, M. Gardner and R.L. Graham, Steiner trees on a checkerboard, *Mathematics Magazine* **62** (1989), 83–96.

[3] F. R. K. Chung and R. L. Graham, Steiner Trees for Ladders, *Annals of Discrete Mathematics* **2** (1978), 173–200.

[4] E. J. Cockayne and D. E. Hewgill, Leaf paths of full Steiner trees and computation of planar Steiner minimal trees, *Algorithmica* **7** (1992), 219–229.

[5] R. Courant and H. Robbins, *What is Mathematics?*, Oxford Univ. Press, New York (1941).

[6] D. Z. Du and F. K. Hwang, Steiner minimal trees for bar waves, *Acta Mathematicae Applicatae Sinica* **3** (1987), 246–256.

[7] D. Z. Du, F. K. Hwang and S. C. Chao, Steiner minimal trees for points on a circle, *Proceedings of the American Mathematical Society* **95** (1985), 613–618.

[8] D. Z. Du, F. K. Hwang, G. D. Song and J. F. Weng, Steiner minimal trees on sets of four points, *Discrete and Computational Geometry* **2** (1987), 401–414.

[9] D. Z. Du, F. K. Hwang and J. F. Weng, Steiner minimal trees on zig-zag lines, *Transactions of the American Mathematical Society* **278** (1983), 149–156.

[10] D. Z. Du, F. K. Hwang and J. F. Weng, Steiner minimal trees on regular polygons, *Discrete and Computational Geometry* **2** (1987), 65–87.

[11] H. Edelsbrunner, *Algorithms in Combinatorial Geometry*, Berlin, Springer (1987).

[12] M. R. Garey, R. L. Graham and D. S. Johnson, The complexity of computing Steiner minimal trees, *SIAM J. Appl. Math.* **32** (1977), 835–859.

[13] F. K. Hwang and D. Z. Du, Steiner minimal trees on Chinese checkerboards, *Mathematics Magazine* **64** (1991), 332–339.

[14] F. K. Hwang, D. S. Richards and P. Winter, The Steiner Tree Problem, *Annals of Discrete Mathematics* **53** (1992), Amsterdam, North-Holland, 3–10.

[15] V. Jarník and O. Kössler, O minimálních grafech obsahujících u daných bodu, *Ĉas. Pêstováni Mat.* **63** (1934) 223–235.

[16] Z. A. Melzak, On the problem of Steiner, *Canad. Math. Bull.* **4** (1961), 143–148.

Received March, 1996