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# Fairness in Grammar Systems

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**Abstract**

The paper deals with two fairness concepts in cooperating distributed grammar systems. The effect of this restriction on the protocol of cooperation among the components of a grammar system is investigated. In all modes of derivation, the fairness restrictions lead to an increase in the generative power. Surprinsingly, even in the regular case.

## 1 Introduction

In modern computer science such notions as distribution, cooperation, parallelism, communication, synchronization are more and more vividly investigated. As practical materializations one can mention computer networks, parallel computing, distributed data bases, etc. There are several approaches to these ideas. In this paper we deal with grammar systems which form a grammatical approach.

A grammar system is a construct consisting of several usual grammars, working together, in a specified way, for generating a language. If the grammars work together on the same sentential form, then the system is called *cooperating/distributed* (CD for short) grammar system. If the grammars work on their own sentential forms and, from time to time, send the result of their work to other components, then the system is called *parallel communicating* grammar system.

This paper concerns CD grammar systems. Intuitively, such systems and their work can be described as follows: Initially, the axiom is the common sentential form. At each moment, one grammar is active, that means it rewrites the common string, and the others are inactive. The conditions under which a component can become active or it is disabled and leave the sentential form to other components are specified by the cooperation protocol. The language of terminal strings generated in this way is the language generated by the system. As basic stop conditions which will also be considered in this paper we mention: each component, when active, has to work for exactly $k$, at least $k$, at most $k$, or the maximal number of steps (a

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step means the application of a rewriting rule). Many other starting and stopping conditions were considered in the literature (see [3]).

Such systems were introduced by different motivations:

1. The generalization of the two-level substitution grammars was the main purpose of the paper [5] where the sintagm "cooperating grammar system" was proposed.

2. Modular grammars as an alternative for the time-varying grammars were presented in [1].

3. In the architecture of a CD grammar system one can recognize the structure of a blackboard model, as used in problem-solving area [6]: the common sentential form is the blackboard (the common data structure containing the current state of the problem to be solved), the component grammars are the knowledge sources contributing to solving the problem, the protocol of cooperation encodes the control on the work of the knowledge sources ([4]). This was the explicit motivation in [2], the paper where CD grammar systems in the form we consider here were introduced.

4. The increase of the computational power of components by cooperation.

5. The decrease of the complexity of different tasks by distribution.

In some sense, the theory of grammar systems is the theory of cooperation protocols; the focus is not on the generative capacity, but on the functioning of the system, and on its influence on the generative capacity and on other specific properties.

The aim of this paper is the investigation of a quite natural feature of the strategy of cooperation: fairness. We require that all components of the system have approximately the same contribution to the common work, concerning the time spent by each of them during the derivation process. The first attempt in this direction, called weak fairness, asks for that each component has to be activated almost of the same number of times (the difference between the number of times for which any two components are activated is bounded). But this concept says nothing about the period of time in which a component is working. So, if we want to have more precisely a fair behaviour of the system, called strong fairness, then it is necessary to measure also this time, i.e. to count the number of applications of rules of a component during the whole derivation.

The requirement that a system has to be "fair" increases, even for systems with regular components (this situation contrasts the "unfair" case), the generative power of the system.

## 2 Definitions

For an alphabet $V$, we denote by $V^*$ the free monoid generated by $V$ under the operation of concatenation; the empty string is denoted by $\lambda$, and we set $V^+ =$

$V^* \setminus \{\lambda\}$. The length of $x \in V^*$ is denoted by $|x|$. If $x \in V^*$ and $U \subseteq V$, then $|x|_U$ is the number of occurrences of symbols of in $x$ (the length of the string obtained by erasing from $x$ all symbols in $V \setminus U$). By $REG$, $CF$ and $ETOL$ we denote the families of regular, context-free and ETOL languages, respectively (see [7], [8]).

A *CD grammar system of degree* $n, n \geq 1$, is a construct

$$\Gamma = (N, T, S, P_1, \ldots, P_n),$$

where $N, T$ are disjoint alphabets, $S \in N$, and $P_1, \ldots, P_n$ are finite sets of rewriting rules over $N \cup T$.

The elements of $N$ are nonterminals, those of $T$ are terminals; $P_1, \ldots, P_n$ are called *components* of the system. Here we work with CD grammar systems having only regular rules, i.e. rules of the form $A \to aB$ or $A \to a$ with $A, b \in N$, $a \in T$, or context-free rules, i.e. rules of the form $A \to w$ with $A \in N$, $w \in (N \cup T)^*$.

The domain of the $i^{th}$ component denoted by $dom(P_i)$ is defined as $dom(P_i) = \{A | A \to x \in P_i\}$.

On $(N \cup T)^*$ one can define the usual one step derivation with respect to $P_i$, denoted by $\Longrightarrow_{P_i}$. The derivations consisting of exactly $k$, at most $k$ (but at least one), at least $k$ such steps $\Longrightarrow_{P_i}$ are denoted by $\Longrightarrow_{P_i}^{=k}$, $\Longrightarrow_{P_i}^{\leq k}$, $\Longrightarrow_{P_i}^{\geq k}$, respectively. Furthermore, we write $x \Longrightarrow_{P_i}^{t} y$ iff $x \Longrightarrow_{P_i}^{\geq 1} y$ and there is no $z \in (N \cup T)^*$ such that $y \Longrightarrow_{P_i} z$.

Let

$$M = \{t\} \cup \bigcup_{k \geq 1} \{\leq k, = k, \geq k\}.$$

The language generated by the system $\Gamma$ in the derivation mode $f \in M$ is

$$\begin{aligned} L_f(\Gamma) \quad = \quad & \{w \mid w \in T^*,\ S \Longrightarrow_{P_{i_1}}^{f} w_1 \Longrightarrow_{P_{i_2}}^{f} \cdots \Longrightarrow_{P_{i_m}}^{f} w_m = w, \\ & m \geq 1, 1 \leq i_j \leq n, 1 \leq j \leq m\}. \end{aligned}$$

The respective classes of languages are denoted by $CDL_n(X, f)$, where $n$ is the degree of the grammar system, $X \in \{REG, CF\}$ indicates the type of the components (regular or context-free) and $f \in M$.

Let

$$D\ :\ S \Longrightarrow_{P_{i_1}}^{=m_1} w_1 \Longrightarrow_{P_{i_2}}^{=m_2} \cdots \Longrightarrow_{P_{i_t}}^{=m_t} w_k$$

be a derivation in the $f$-mode, $f \in M$ (i.e. $m_j$ gives the number of derivation steps performed by the component $P_{i_j}$ in $D$; especially, if $f \in \{= k\}$, then $m_i = k$ holds for $1 \leq j \leq t$, etc.) For any $1 \leq p \leq n$, we write

$$\psi_D(p) = \sum_{i_j = p} 1 \quad \text{and} \quad \varphi_D(p) = \sum_{i_j = p} m_j$$

(i.e by $\psi_D(p)$ we count the number of applications of $P_p$, and by $\varphi_D(p)$ we count the number of applications of rules of $P_p$). Conventionally, the empty sum delivers zero.

Let $\Gamma$ be a CD grammar system with at least two components. Then we set

$$dw(D) = max\{|\psi_D(i) - \psi_D(j)| \mid 1 \leq i, j \leq n\}$$

and

$$ds(D) = max\{|\varphi_D(i) - \varphi_D(j)| \mid 1 \leq i, j \leq n\}.$$

By these two numbers we measure the maximal difference between the contribution of components involved in the derivation $D$. The contribution of a component may be expressed as the number of its applications and the number of rules applications in the considered component, respectively. Moreover, for $u \in \{w, s\}$, $x \in (N \cup T)^*$ and $f \in M$, we define

$$du(x, f) = min\{du(D) \mid D \text{ is a derivation in the } f - \text{mode for } x\}.$$

In order to get a concept of fairness, we now restrict the numbers $du(x, f)$ for the words $x$ which belong to the language. If $u = w$, then we get a weaker notion since we only require that the components are used almost equally often, whereas $u = s$ gives a stronger notion where the times which the components work are approximately equal. Formally, this leads to the following definitions.

For a CD grammar system $\Gamma$ of degree $n \geq 2$, $f \in M$ and a natural number $q \geq 0$, we define the *weakly q-fair* language generated by $\Gamma$ in the $f$-mode as

$$L_f(\Gamma, w - q) = \{x \mid x \in L_f(\Gamma) \text{ and } dw(x, f) \leq q\}$$

and the *strongly q-fair* language of $\Gamma$ as

$$L_f(\Gamma, s - q) = \{x \mid x \in L_f(\Gamma) \text{ and } ds(x, f) \leq q\}.$$

For $X \in \{REG, CF\}$ and integers $n \geq 2$ and $q \geq 0$, by $CDL_n(X, f, w - q)$ and $CDL_n(X, f, s - q)$ we denote the families of weakly and strongly $q$-fair languages, respectively, generated by CD grammar systems with $n$ components.

Let us illustrate the concepts of fairness by two examples. We shall give just the components of the systems, the other components can easily be deduced under the assumption that $S$ is the axiom.

**Example 1** We consider the grammar system $\Gamma_1$ with the components

$$P_1 = \{S \to aA', A \to aA'\}, \quad P_2 = \{A' \to aA\},$$
$$P_3 = \{A \to bB', B \to bB'\}, \quad P_4 = \{B' \to bB, B' \to b\}.$$

Then, for $q \geq 0$ and $f \in \{t, = 1, \geq 1\} \cup \{\leq k \mid k \geq 1\}$, we obtain

$$L_f(\Gamma_1) = \{a^{2n}b^{2m} \mid n \geq 1, m \geq 1\}$$

and

$$L_f(\Gamma_1, w - q) = L_f(\Gamma_1, s - q) = \{a^{2n}b^{2m} \mid n \geq 1, m \geq 1, |n - m| \leq q\}.$$

Note that each component of $\Gamma_1$ is regular whereas the $q$-fair languages generated by $\Gamma_1$ are not regular.

**Example 2** Let $\Gamma_2$ be the grammar system having the components

$$\begin{aligned} P_1 &= \{S \to AB, A \to aAb, A \to ab\}, \\ P_2 &= \{B \to C, C \to cC, C \to c\}. \end{aligned}$$

Clearly, for all $f \in \{t, = 1, \geq 1\} \cup \{\leq k \mid k \geq 1\}$ and $q \geq 0$,

$$L_f(\Gamma_2) = L_f(\Gamma_2, w - q) = \{a^n b^n c^m \mid n, m \geq 1\}$$

holds (use each component exactly once) whereas

$$L_f(\Gamma_2, s - q) = \{a^n b^n c^m \mid n, m \geq 1, n = m + p \text{ or } m = n + p, 0 \leq p \leq q\}.$$

We mention that the languages $L_f(\Gamma_2, s - q)$ of the context-free grammar system $\Gamma_2$ are not context-free. Moreover, for all $k \geq 2$, we have

$$\begin{aligned} L_{=k}(\Gamma_2) &= \{a^n b^n c^m \mid n = r_1 \cdot k - 1, m = r_2 \cdot k - 1, r_1 \geq 1, r_2 \geq 1\}, \\ L_{=k}(\Gamma_2, w - q) &= \{a^{r_1 \cdot k - 1} b^{r_1 \cdot k - 1} c^{r_2 \cdot k - 1} \mid r_1,\ r_2 \geq 1,\ |r_1 - r_2| \leq q\}, \\ L_{=k}(\Gamma_2, s - q) &= \{a^{r_1 \cdot k - 1} b^{r_1 \cdot k - 1} c^{r_2 \cdot k - 1} \mid r_1,\ r_2 \geq 1,\ |(r_1 - r_2) \cdot k| \leq q\}. \end{aligned}$$

We add some remarks to the definitions.

1. The above definitions assume that the grammar system has at least two components.

2. If $f$ is the mode $= k$, then the weak and strong concepts of fairness are nearly related to each other because

   $$L_{=k}(\Gamma, w - q) = L_{=k}(\Gamma, s - q')$$

   holds for $k \cdot q \leq q' < k \cdot (q + 1)$. Particularly,

   $$L_{=k}(\Gamma, w - 0) = L_{=k}(\Gamma, s - 0).$$

3. It is also possible to allow the value $\infty$ for $q$. Thus, we get the equalities

   $$CDL_n(X, f) = CDL_n(X, F, w - \infty) = CDL_n(X, f, s - \infty).$$

In the sequel, we are going to investigate the influence of the fairness limitation on the generative power.

# 3 The regular case

Let

$$M_1 = \{t, = 1, \geq 1\} \cup \{\leq k \mid k \geq 1\}.$$

First we recall that

$$CDL_n(REG, f) = REG$$

for all $n \geq 1, f \in M$. We now show that the situation is very different if we require a fair behaviour of the systems.

**Theorem 1** *(i) REG and $CDL_n(REG, f, s-0)$ are incomparable, for all $f \in M$ and $n \geq 2$.*

*(ii) $REG \subset CDL_n(REG, f, s-q)$, for all $f \in M_1$, $q \geq 1$, $n \geq 2$.*

*(iii) REG and $CDL_n(REG, f, s-q)$ are incomparable, for all $f \in M \setminus M_1$, $q \geq 1$, and $n \geq 2$.*

*Proof.* It is easy to observe that any language in $CDL_n(REG, f, s-0)$, $f \in M$, contains only words of length divisible by $n$. Thus the regular language $L = \{a^m | m \geq 1\}$ does not belong to the class $CDL_n(REG, f, s-0)$.

On the other hand, the grammar system $\Gamma$ consisting of the following components:

$$\begin{aligned}
P_1 &= \{S \to aS, S \to aS_2\}, \\
P_i &= \begin{cases} \{S_i \to bS_i, S_i \to bS_{i+1}\}, \ 2 \leq i \leq n-1, \ i \text{ is even} \\ \{S_i \to aS_i, S_i \to aS_{i+1}\}, \ 2 \leq i \leq n-1, \ i \text{ is odd} \end{cases} \\
P_n &= \begin{cases} \{S_n \to bS_n, S_n \to b\}, \ \text{if } n \text{ is even} \\ \{S_n \to aS_n, S_n \to a\}, \ \text{if } n \text{ is odd} \end{cases}
\end{aligned}$$

generates the 0-fair languages

$$L_{f_1}(\Gamma, s-0) = \begin{cases} \{(a^m b^m)^{\frac{n}{2}} \mid m \geq 1\}, \ \text{if } n \text{ is even} \\ \{(a^m b^m)^{\frac{n-1}{2}} a^m \mid m \geq 1\}, \ \text{if } n \text{ is odd} \end{cases}$$
$$\text{for } f_1 \in M_1,$$

$$L_{=k}(\Gamma, s-0) = \begin{cases} \{(a^{mk} b^{mk})^{\frac{n}{2}} \mid m \geq 1\}, \ \text{if } n \text{ is even} \\ \{(a^{mk} b^{mk})^{\frac{n-1}{2}} a^{mk} \mid m \geq 1\}, \ \text{if } n \text{ is odd} \end{cases}$$
$$\text{for } k \geq 1,$$

$$L_{\geq k}(\Gamma, s-0) = \begin{cases} \{(a^m b^m)^{\frac{n}{2}} \mid m \geq k\}, \ \text{if } n \text{ is even} \\ \{(a^m b^m)^{\frac{n-1}{2}} a^m \mid m \geq k\}, \ \text{if } n \text{ is odd} \end{cases}$$
$$\text{for } k \geq 1,$$

which are not regular.

We prove now that the family of regular languages is contained in the families of $q$-fair languages generated in the $f$-mode, $f \in M_1$, by regular grammar systems as soon as $q \geq 1$.

For a regular grammar $G = (N, T, S, P)$ we construct the grammar system with regular components

$$\Gamma = (N', T, S_1, P_1, P_2, \ldots, P_n),$$

with

$$N' = \{A_i \mid A \in N, 1 \leq i \leq n\}$$

and the sets of productions

$$\begin{aligned} P_i &= \{A_i \to aB_{i+1} \mid A \to aB \in P\} \cup \{A_i \to a \mid A \to a \in P\}, \text{ for } 1 \leq i \leq n-1, \\ P_n &= \{A_n \to aB_1 \mid A \to aB \in P\} \cup \{A_n \to a \mid A \to a \in P\} \end{aligned}$$

By this construction any component performs exactly one step and the $i$-th, $1 \leq i \leq n-1$, and $n$-th component are followed by the $(i+1)$-st and first component, respectively. Thus, $L(G) = L_f(\Gamma, s-q)$, for all $q \geq 1$ and $f \in M_1$.

Now, it suffices to note that

$$L_f(\Gamma, s-q) = \begin{cases} \{a^{m_1}b^{m_2}a^{m_3}b^{m_4}\ldots b^{m_n} \mid m_i \geq 1, |m_i - m_j| \leq q, 1 \leq i,j \leq n\}, \\ \text{if } n \text{ is even} \\ \\ \{a^{m_1}b^{m_2}a^{m_3}b^{m_4}\ldots a^{m_n} \mid m_i \geq 1, |m_i - m_j| \leq q, 1 \leq i,j \leq n\}, \\ \text{if } n \text{ is odd} \end{cases}$$

for $f \in M_1$, are not regular languages.

In order to prove the last assertion let us remark that for any $k \geq 2$, every language in $CDL_n(REG, =k, s-q) \cup CDL_n(REG, \geq k, s-q)$ contains words of length greater than $k$, only. Therefore, $REG \setminus (CDL_n(REG, =k, s-q) \cup CDL_n(REG, \geq k, s-q)) \neq \emptyset$, for all $n \geq 2, q \geq 1$. The grammar system considered in the proof of the first statement provides languages for the converse part. Consequently, the proof is complete. □

**Theorem 2** *(i)* $CDL_2(REG, f, w-0) \subset REG$, *for* $f \in \{t, \geq 1\}$.

*(ii) The families* $CDL_n(REG, f, w-0)$ *and* $REG$ *are incomparable, for all* $n \geq 2$ *and* $f \in \bigcup_{k \geq 1}\{\leq k, =k\}$.

*(iii) For* $f \in \{t, \geq 1\}$ *and* $n \geq 4$, $REG$ *and* $CDL_n(REG, f, w-0)$ *are incomparable.*

*Proof.* (i) For $\Gamma = (N, T, S, P_1, P_2)$ consider the right linear grammars

$$G_1 = (\{S\} \cup \{A', A" \mid A \in N\}, T, S, P)$$

where

$$\begin{aligned} P &= \{A' \to aB' \mid A \to aB \in P_1\} \cup \{A' \to aB' \mid A \to aB \in P_2, A \neq S\} \cup \\ &\cup \{A' \to a \mid A \to a \in P_2\} \cup \{A" \to aB" \mid A \to aB \in P_2\} \cup \{S \to S', S \to S"\} \\ &\cup \{A" \to aB" \mid A \to aB \in P_1, A \neq S\} \cup \{A" \to a \mid A \to a \in P_1\} \end{aligned}$$

and

$$G_2 = (\{S\} \cup \{A_i, A'_i \mid A \in N, i = 1,2\}, T, S, P \cup \{S \to S_1, S \to S'_2\})$$

where

$$\begin{array}{rcl} P & = & \{A_1 \to aB_1 | A \to aB \in P_1, B \in dom(P_1)\} \cup \\ & \cup & \{A_1 \to aB_2 | A \to aB \in P_1, B \notin dom(P_1)\} \cup \\ & \cup & \{A_2 \to aB_2 | A \to aB \in P_2, B \in dom(P_2)\} \cup \\ & \cup & \{A_2 \to aB_1 | A \to aB \in P_2, B \notin dom(P_2)\} \cup \\ & \cup & \{A_2 \to a | A \to a \in P_2\} \cup \\ & \cup & \{A'_2 \to aB'_2 | A \to aB \in P_2, B \in dom(P_2)\} \cup \\ & \cup & \{A'_2 \to aB'_1 | A \to aB \in P_2, B \notin dom(P_2)\} \cup \\ & \cup & \{A'_1 \to aB'_1 | A \to aB \in P_1, B \in dom(P_1)\} \cup \\ & \cup & \{A'_1 \to aB'_2 | A \to aB \in P_1, B \notin dom(P_1)\} \cup \\ & \cup & \{A'_1 \to a | A \to a \in P_1\} \end{array}$$

The equalities

$$L_{\geq 1}(\Gamma, w-0) = L(G_1) \quad \text{and} \quad L_t(\Gamma, w-0) = L(G_2)$$

follow immediately. The inclusions are proper since any language in $CDL_2(REG, f, w-0)$, $f \in \{t, \geq 1\}$, contains no word of length 1.

(ii) The second statement is completely proved if we provide, for all $k \geq 1, n \geq 2$, a non-regular language in $CDL_n(REG, \leq k, w-0)$. To this end, let us consider two cases.

• $n = 2$. The grammar system $\Gamma$ identified by the following regular components

$$P_1 = \{S \to aS, S \to aB\}$$

$$P_2 = \{B \to bB, B \to b\}$$

generates in the $\leq k$-mode the non-regular language

$$L_{\leq k}(\Gamma, w-0) = \{a^t b^m | 1 \leq t \leq m \leq kt \text{ or } 1 \leq m \leq t \leq km\}$$

• $n > 2$. The grammar system $\Gamma$ identified by the following regular components

$$\begin{array}{rcl} P_1 & = & \{S \to aS_2\} \\ P_i & = & \{S_i \to aS_{i+1}\}, 2 \leq i \leq n-2, \\ P_{n-1} & = & \{S_{n-1} \to aS, S_{n-1} \to aB\} \\ P_n & = & \{B \to bB, B \to b\} \end{array}$$

Observe that

$$L_{\leq k}(\Gamma, w-0) = \{a^{t(n-1)} b^m | 1 \leq t \leq m \leq kt \text{ or } 1 \leq m \leq t \leq km\}$$

which concludes the proof of the second item.

(iii) Any language in $CDL_n(REG, f, w-0)$, $f \in \{t, \geq 1\}$, contains only words of length at least $n$ since any component has to be applied at least once. Thus the regular language $\{a, a^2, \ldots, a^n\}$ does not belong to the class $CDL_n(REG, f, w-0)$.

If $n = 4$, then the statement follows from Example 1.

If $n > 4$, then we subsitute the component $P_2$ of the grammar system $\Gamma_1$ in Example 1 by the components

$$\begin{aligned} P_{2,1} &= \{A' \to aA_2\}, \\ P_{2,i} &= \{A_i \to aA_{i+1}\} \quad \text{for} \quad 2 \leq i \leq n-4, \\ P_{2,n-3} &= \{A_{n-3} \to aA\} \end{aligned}$$

and obtain the grammar system $\Gamma'_1$ which generates the non-regular 0-fair language

$$L_f(\Gamma'_1, f, w-0) = \{a^{(n-2)m}b^{2m} \mid m \geq 1\}.$$

□

By using similar ideas as those involved in the previous proofs one can get:

**Theorem 3** *For all $q \geq 1$ we have:*

$$\begin{aligned} (i)\ REG &= CDL_2(REG, f, w-q), f \in \{t, \geq 1\}. \\ (ii)\ REG &\subset CDL_n(REG, \leq k, w-q), n \geq 2, k \geq 1. \\ (iii)\ REG &\subset CDL_m(REG, f, w-q), m \geq 4, f \in \{t, \geq 1\}. \end{aligned}$$

At the end of this section we would like to mention that when considering grammar systems with right-linear components (i. e. containing rules of the forms $A \to xB, A \to x$, $x \in T^*$, $A, B \in N$) the results are similar to those given in the following section for the context-free case.

# 4 The context-free case

We start with a theorem which states a situation for context-free grammar systems which differs from that in the regular case.

**Theorem 4** *For all $n \geq 2$, $q \geq 0$, $u \in \{w, s\}$ and $f \in M$,*

$$CDL_n(CF, f, u-q) \subseteq CDL_{n+1}(CF, f, u-q).$$

*Proof.* • $u = w$. For a CD grammar system $\Gamma = (N, T, S, P_1, P_2, \ldots, P_n)$, we construct the system

$$\Gamma' = (N', T, S, P'_1, P_2, P_3, \ldots, P_n, P'_{n+1}),$$

with

$$\begin{aligned} N' &= N \cup \{X\} \quad \text{with } X \notin N, \\ P_1' &= P_1 \cup \{A \to wX \mid A \to w \in P_1\}, \\ P_{n+1}' &= \{X \to \lambda\} \end{aligned}$$

Obviously, $L_f(\Gamma) = L_f(\Gamma')$ and, because $P_1'$ can be used as often as $P_{n+1}'$, we obtain $L_f(\Gamma, w-q) = L_f(\Gamma', w-q)$.

• $u = s$. For a CD grammar system $\Gamma = (N, T, S, P_1, P_2, \ldots, P_n)$, we construct the system

$$\Gamma'' = (N'', T, S', P_1'', P_2'', \ldots, P_n'', P_{n+1}''),$$

with

$$\begin{aligned} N'' &= N \cup \{X, S'\} \quad \text{with } X, S' \notin N, \\ P_i'' &= P_i \cup \{S' \to wX \mid S \to w \in P_i\}, \quad 1 \le i \le n, \\ P_{n+1}'' &= \{X \to X, X \to \lambda\} \end{aligned}$$

Obviously, $L_f(\Gamma) = L_f(\Gamma')$ and, because the new introduced component $P_{n+1}''$ can work as long as we want, we infer $L_f(\Gamma, s-q) = L_f(\Gamma'', s-q)$. □

**Theorem 5** *i) For $n \ge 2$, $q \ge 0$, $u \in \{w, s\}$ and $f \in M_1$,*

$$CDL_n(CF, f) \subseteq CDL_n(CF, f, u-q).$$

*ii) The aforementioned inclusion is proper in the following cases:*

| | | | | |
|---|---|---|---|---|
| *a)* | $u = w,$ | $n \ge 4,$ | $f \ne t,$ | $q \ge 0,$ |
| *b)* | $u = s,$ | $n \ge 2,$ | $f \ne t,$ | $q \ge 0,$ |
| *c)* | $u = w,$ | $n \ge 6,$ | $f = t,$ | $q = 0,$ |
| *d)* | $u = s,$ | $n = 2$ *or* $n \ge 7,$ | $f = t,$ | $q = 0.$ |

*Proof.* First we recall that, for $n \ge 2$, $m \ge 3$ and $f \in M_1 \setminus \{t\}$,

$$CDL_2(CF, t) = CDL_n(CF, f) = CF \quad \text{and} \quad CDL_m(CF, t) = ETOL. \tag{1}$$

i) First we consider the case $u = w$ and $f \ne t$.

For a context-free grammar $G = (N, T, S, P)$, we construct the CD grammar system $\Gamma$ with the following two components

$$\begin{aligned} P_1 &= \{A \to wX \mid A \to w \in P\}, \\ P_2 &= \{X \to \lambda\} \end{aligned}$$

where $X$ is an additional nonterminal. Obviously, $G$ and $\Gamma$ generate the same language and, moreover, any word can be derived in $\Gamma$ by using each component exactly once. This proves $L = L(G) = L_f(\Gamma, w-q)$.

Therefore, from (1) and Theorem 4, it follows that

$$CDL_n(CF,f) = CF \subseteq CDL_2(CF,f,w-q) \subseteq CDL_n(CF,f,w-q).$$

This proof can be carried over the cases $u = w$, $f = t$ and $n = 2$.

We now consider the case $u = w$, $f = t$ and $n \geq 3$. Let $L \in CDL_n(CF,t)$. By (1), $L$ is generated by some ETOL system

$$G = (V,T,S,T_1,T_2,\ldots,T_m)$$

with the alphabet $V$, containing the set $T$ of terminals, the start word $S$ which can be assumed without loss of generality as an element of $V \setminus T$ and the tables $T_1, T_2, \ldots T_m$.

For $a \in V$, $0 \leq i \leq m$ and $1 \leq k \leq 5$, we introduce the new letters $a_i^k$ and extend this inductively to words by

$$w_i^k = \begin{cases} \lambda & \text{for } w = \lambda \\ a_i^k & \text{for } w = a \\ v_i^k a_i^k & \text{for } w = va,\ v \in V^*,\ a \in V \end{cases} .$$

We now construct the CD grammar system

$$\Gamma = (N,T,S_0^1,P_1,P_2,P_3\}$$

with

$$\begin{aligned}
N &= \{F\} \cup \{a_i^k \mid a \in V, 0 \leq i \leq m, 1 \leq k \leq 5\},\\
P_1 &= \{a_i^1 \to a_i^2 \mid 0 \leq i \leq m, a \in V\} \cup \{a_i^4 \to a_i^5 \mid 0 \leq i \leq m, a \in V\},\\
P_2 &= \{a_i^2 \to a_i^3 \mid 0 \leq i \leq m, a \in V\} \cup \{a_i^1 \to a_i^4 \mid 0 \leq i \leq m, a \in V\},\\
P_3 &= \bigcup_{i=1}^{m} \{a_i^3 \to w_0^1 \mid a \to w \in T_i\} \cup \{a_0^3 \to a \mid a \in T\}\\
&\qquad \cup \{a_0^3 \to F \mid a \in V \setminus T\} \cup \{a_i^5 \to a_{i+1}^1 \mid 1 \leq i \leq m, a \in V\}.
\end{aligned}$$

Let us consider a word of the form $x_i^1$. Note that the axiom is of this form.

If we apply the component $P_1$, we obtain $x_i^2$ and we have to apply $P_2$ and $P_3$ in succession. If $i \geq 1$, then this yields $y_0^1$ where $x \Longrightarrow_{T_i} y$ is a derivation step in the ETOL system $G$, i.e. we have simulated the application of table $T_i$ to $x$. If $i = 0$ then we obtain $x$ or a word containing $F$ according whether $x \in T^+$ or not and the derivation is finished.

If we apply the component $P_2$, we have to apply $P_1$ and $P_3$ and obtain $x_{i+1}^1$.

From this explanation it is easy to see that we can simulate the application of an arbitrary sequence of tables and finish the derivation by using any of the three components exactly once for the simulation of one step. Hence

$$L_t(\Gamma, w-q) = L_t(\Gamma, w-0) = L(G) = L$$

holds for any $q$ which proves the statement.

Now let $u = s$ and $f \in M_1$. Let $\Gamma = (N, T, S, P_1, \ldots, P_n)$, be a CD grammar system. The grammar system

$$\Gamma' = (N', T, S, P_1', \ldots, P_n'),$$

with

$$\begin{aligned} N' &= N \cup \{X\} \quad \text{with } X \notin N, \\ P_i' &= P_i \cup \{A \to wX \mid A \to w \in P_i\} \cup \{X \to X, X \to \lambda\} \quad \text{for } 1 \leq i \leq n \end{aligned}$$

generates the same language as $\Gamma$. Therefore

$$L_f(\Gamma', f, s - q) \subseteq L_f(\Gamma)$$

for all $q \geq 0$.

By using the rules $X \to X$ and $X \to \lambda$, for any $w \in L_f(\Gamma')$, we can find a derivation $D$ such that $ds(D) = 0$. Hence

$$L_f(\Gamma) = L_f(\Gamma') \subseteq L_f(\Gamma', s - q).$$

Consequently, for any $q \geq 0$,

$$L_f(\Gamma) = L_f(\Gamma', s - q)$$

which concludes the inclusions $CDL_n(CF, f) \subseteq CDL_n(CF, f)$ for all $n \geq 2$.

ii) a) The CD grammar system $\Gamma$ with the components

$$\begin{aligned} P_1 &= \{S \to aA_1c, A \to aA_1c\}, \\ P_i &= \{A_i \to A_{i+1}\}, \quad 2 \leq i \leq n-3, \\ P_{n-2} &= \{A_{n-3} \to A\}, \\ P_{n-1} &= \{A \to bB', B \to bB'\}, \\ P_n &= \{B' \to B, B' \to \lambda\} \end{aligned}$$

generates in any mode $f \in M_1 \setminus \{t\}$ the weakly $q$-fair language

$$L_f(\Gamma, w - q) = \{a^n b^m c^n \mid n \geq 1, m \geq 1, |n - m| \leq q\}$$

which is not context-free. Now the statement follows from (1).

b) The statement follows from Example 2, Theorem 4 and (1).

c) We consider the CD grammar system

$$\Gamma = (\{A, A', A'', B, B', D, D', F\}, \{a, b, d\}, A, P_1, P_2, P_3, P_4, P_5, P_6)$$

with

$$\begin{aligned}
P_1 &= \{A \to BA', A \to BA''\},\\
P_2 &= \{A' \to A, A'' \to D\},\\
P_3 &= \{A \to F, A' \to F, A'' \to F, B \to B'b, D \to dD'\},\\
P_4 &= \{B' \to B, D' \to D, D' \to d\},\\
P_5 &= \{B' \to a, D' \to D\},\\
P_6 &= \{B \to F, B' \to F, D \to dD'\}.
\end{aligned}$$

It is easy to see that any derivation where any component works exactly $n$ times is given by the sequence

$$(P_1P_2)^n(P_3P_4)^m P_3P_5(P_3P_4)^{n-m-1}(P_6P_5)^{n-1}P_6P_4$$

of components where $m < n$. Hence $\Gamma$ generates in the $t$-mode the weakly 0-fair language

$$L' = \{(ab^{m+1})^n d^{2n+1} \mid n \geq m+1 \geq 1\}.$$

Using the closure properties of the family of ETOL languages and Corollary 2.2 of Part V in [7] we obtain that $L' \notin ETOL$. By i), (1) and Theorem 4, for $n \geq 6$,

$$CD_n(CF,t) = ETOL \subset CDL_6(CF,t,0-q) \subseteq CDL_n(CF,t,0-q)$$

follows.

d) The strict inclusion for $n = 2$ follows from Example 2.

We shall prove the strict inclusion for $n = 7$, hence all inclusions for $n \geq 7$ are consequences of the previous theorem.

Let us consider the CD grammar system

$$\begin{aligned}
P_1 &= \{S \to CAZXY, Y \to Y\},\\
P_2 &= \{C \to BC', X \to C'C'AZX', Y \to Y\},\\
P_3 &= \{C' \to C, X' \to X, Y \to Y\},\\
P_4 &= \{B \to \lambda, C \to \lambda, Z \to \lambda\},\\
P_5 &= \{A \to aA', A' \to bD', D \to bD', Y \to Y\},\\
P_6 &= \{D' \to D, Y \to Y, X \to \lambda\},\\
P_7 &= \{D \to \lambda, Y \to Y, Y \to \lambda\}.
\end{aligned}$$

Here are some explanations about the working mode of the above system. The sets $P_1, P_2, P_3$ are used in order to obtain strings $\alpha$ with

$$|\alpha|_{\{B,C\}} = m^2 \quad \text{and} \quad |\alpha|_A = |\alpha|_Z = m$$

for all $m \geq 1$. Every terminal derivation has to use the component $P_4$ only once but for $m(m+1)$ steps. On the other hand, each component $P_i$, $i \in \{1,2,3\}$ can be used either once or several times for a total amount of $m(m+1)$ steps.

The component $P_5$ is used first time for at least $2m$ steps and all the other times for at least $m$ steps.

The component $P_6$ is used each time for at least $m$ steps. Because, $P_5$ and $P_6$ are used together for introducing $b$'s, the total number of $b$'s in the terminal words of $L_t(\Gamma, s-0)$ is $m^2$. In conclusion,

$$L_t(\Gamma, s-0) = \{(ab^n)^m \mid m \geq n \geq 1\}$$

which is not an $ET0L$ language (see [7]). Now the result follows due to (1). □

As one can see there remain plenty of open problems in this area. We do not list them since the reader can easily identify them or can invent others.

Finally let us mention that there also are some other concepts of fairness which can be introduced.

- For a given CD grammar system $\Gamma = (N, T, S, P_1, P_2, \ldots, P_n)$, we fix some integers $r_1, r_2, \ldots, r_n$ and require that, for $1 \leq i \leq n$, the component $P_i$ is applied at most or exactly $r_i$ times. However, since the application of a component in one of the modes is equivalent to a finite or context-free substitution we obtain only finite or context-free languages and, obviously, we obtain all languages of this type.

- To each component $P_i$, $1 \leq i \leq n$, and to any moment $l$ of time, $l \geq 0$ (this corresponds to the number of applications of components), we associate an integer $t_i(l)$ in the following way: Let $t_i(0) = 0$. If we apply the component $P_j$ in the moment $l$, then we set $t_j(l+1) = 0$ and increase the number of all the other components, i.e. $t_i(l+1) = t_i(l) + 1$ for $i \neq j$. The fairness now consists in the requirement that in each moment $l$ we can only apply a component $P_i$ such that $t_i(l) = \max_{1 \leq p \leq n} t_p(l)$. The number $t_i(l)$ can be interpreted as the period during that the component $P_i$ was not active, i.e. it is the waiting time of the component, and we can apply a component only if it has been waiting a maximal amount of time.

  Clearly, after the first activation, any component has to work after waiting $n-1$ steps. Thus this concept is nearly related to the weak fairness. We only mention that – by using the same proofs – one can show that similar statements as for weak fairness hold.

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