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On α_i-products of nondeterministic tree automata*

B. Imreh†

Abstract

In this paper, we characterize the isomorphically complete systems of nondeterministic tree automata with respect to the family of α_i-products. In particular, our characterization yields that any finite nondeterministic tree automata can be embedded isomorphically into a suitable serial product of two-state nondeterministic tree automata.

Keywords: nondeterministic tree automata, composition, completeness

1 Introduction

Isomorphic representation of automata by different compositions is one of the central problems in the theory of automata. One line of the researches is to characterize those systems of automata which are isomorphically complete, i.e., every automaton is an isomorphic image of a subautomaton of a product from them. Most of the studies regarding characterizations of isomorphically complete systems concern deterministic automata or deterministic tree automata. We quote only [1],[3],[4],[7],[9],[10],[11], and [15]. On the other hand, together with the spread of parallel computation, the importance of nondeterministic automata is increasing. This is the motivation to deal with the representations of nondeterministic automata. The first description of the isomorphically complete systems of nondeterministic automata with respect to the general product was given in [5]. In the work [6], it is proved that the cube-product is equivalent to the general product regarding isomorphically complete systems of nondeterministic automata. The isomorphic representation of a special class of nondeterministic automata is investigated in [12]. The notion of α_i-product (cf. [2],[3]) was extended to nondeterministic automata, and the isomorphically complete systems were characterized with respect to this hierarchy of products in [14]. From this characterization, it turns out that contrary to the deterministic case, in the nondeterministic case, there exist finite isomorphically complete systems with respect to the α_0-product, furthermore, the α_i-product is equivalent to the general product regarding isomorphically complete systems if $i \geq 1$. The isomorphically complete systems of nondeterministic tree automata

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†Dept. of Informatics, József Attila University, Árpád tér 2, H-6720 Szeged, Hungary

with respect to the general product and the cube-product are studied in [13] where it is proved that these compositions are equivalent regarding isomorphically complete systems. Here, using the characterization presented in [13] and extending the notion of α_i-product to nondeterministic tree automata, we generalize the result of [14] for nondeterministic tree automata. Namely, we prove that there exist finite isomorphically complete systems of nondeterministic tree automata with respect to the α_0-product, moreover, the α_i-product is equivalent to the general product regarding isomorphically complete systems of nondeterministic tree automata if $i \geq 1$.

The paper is organized as follows. In Section 2, the necessary notions and notations are introduced. The following part, Section 3, presents the characterization of the isomorphically complete systems of nondeterministic tree automata with respect to the α_0-product. Finally, Section 4 is devoted to the description of the isomorphically complete systems of nondeterministic tree automata regarding α_i-product with $i \geq 1$.

2 Preliminaries

To start the discussion, we introduce some notions and notations of relational systems (cf. [8]). By a set of *relational symbols*, we mean a nonempty union $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \ldots$ where Σ_m, $m = 1, 2, \ldots$, are pairwise disjoint sets of symbols. For any $m \geq 1$, the set Σ_m is called the set of *m-ary relational symbols*. It is said that the *rank* or *arity* of a symbol $\sigma \in \Sigma$ is m if $\sigma \in \Sigma_m$. Now, let a set Σ of relational symbols and a set R of positive integers be given. R is called the *rank-type* of Σ if, for any integer $m \geq 0$, $\Sigma_m \neq \emptyset$ if and only if $m \in R$. In the sequel, we shall work under a fixed rank-type R.

Now, let Σ be a set of relational symbols with rank-type R. By a *nondeterministic Σ-algebra* $\mathcal{A}$, we mean a pair consisting of a nonempty set A and a mapping that assigns to every relational symbol $\sigma \in \Sigma$ an m-ary relation $\sigma^{\mathcal{A}} \subseteq A^m$ where the arity of σ is m. The set A is called the *set of elements of* $\mathcal{A}$ and $\sigma^{\mathcal{A}}$ is the *realization of σ in* $\mathcal{A}$. The mapping $\sigma \to \sigma^{\mathcal{A}}$ will not be mentioned explicitly, we only write $\mathcal{A} = (A, \Sigma)$. For every $m \in R$, $\sigma \in \Sigma_m$, and $(a_1, \ldots, a_{m-1}) \in A^{m-1}$, we denote the set $\{a : a \in A \ \& \ \sigma^{\mathcal{A}}(a_1, \ldots, a_{m-1}, a)\}$ by $(a_1, \ldots, a_{m-1})\sigma^{\mathcal{A}}$. If $(a_1, \ldots, a_{m-1})\sigma^{\mathcal{A}}$ is a one-element set $\{a\}$, then we write $(a_1, \ldots, a_{m-1})\sigma^{\mathcal{A}} = a$.

It is said that a nondeterministic Σ-algebra $\mathcal{A}$ is *finite* if A is finite, and it is of *finite type* if Σ is finite. By a *nondeterministic tree automaton*, we mean a finite nondeterministic Σ-algebra of finite type. Finally, it is said that the *rank-type of a nondeterministic tree automaton* $\mathcal{A} = (A, \Sigma)$ is R if the rank-type of Σ is R.

Let $\mathcal{A} = (A, \Sigma)$ and $\mathcal{B} = (B, \Sigma)$ be nondeterministic tree automata with rank-type R. $\mathcal{B}$ is called a *subautomaton* of $\mathcal{A}$ if $B \subseteq A$ and, for all $m \in R$ and $\sigma \in \Sigma_m$, $\sigma^{\mathcal{B}}$ is the restriction of $\sigma^{\mathcal{A}}$ to B^m. A one-to-one mapping μ of A onto B is called an *isomorphism* of $\mathcal{A}$ onto $\mathcal{B}$ if $\sigma^{\mathcal{A}}(a_1, \ldots, a_m)$ if and only if $\sigma^{\mathcal{B}}(\mu(a_1), \ldots, \mu(a_m))$, for all $m \in R$, $(a_1, \ldots, a_m) \in A^m$, $\sigma \in \Sigma_m$. In this case, it is said that $\mathcal{A}$ and $\mathcal{B}$ are *isomorphic*. It is easy to see that μ is an isomorphism of $\mathcal{A}$ onto $\mathcal{B}$ if and only if $(a_1, \ldots, a_{m-1})\sigma^{\mathcal{A}}\mu = (\mu(a_1), \ldots, \mu(a_{m-1}))\sigma^{\mathcal{B}}$ holds, for all $m \in R$, $\sigma \in \Sigma_m$, $(a_1, \ldots, a_{m-1}) \in A^{m-1}$.

In the case of classical automata, a composition of automata can be visualized as a network in which each vertex denotes an automaton and the actual input sign of a component automaton may depend on the input sign of the whole composition and only on those automata which have a direct connection to the component automaton under consideration. From practical point of view, one of the most self-evident networks is the well-known serial or cascade connection. In this case, the composition can be considered as a chain in which each machine has a direct connection with all the previous ones. Generalizing this concept, F. Gécseg [2] introduced a family of compositions, the α_i-products, where i is a nonnegative integer which denotes the maximal admissible length of feedbacks. Now, we extend the notion of α_i-product to nondeterministic tree automata.

Let us denote the class of all nondeterministic tree automata with rank-type R by $\mathbf{U}_R$. In general, a composition of nondeterministic tree automata from $\mathbf{U}_R$ can be visualized as a network in which each vertex denotes a nondeterministic tree automaton in $\mathbf{U}_R$ and the actual relation of a component automaton may depend on the relational symbol of the whole composition and only on those nondeterministic tree automata which have a direct connection to the component under consideration. In particular, the formal definition of the α_i-product of nondeterministic tree automata can be given as follows.

Let i be an arbitrary nonnegative integer. Let us consider the nondeterministic tree automata $\mathcal{A} = (A, \Sigma) \in \mathbf{U}_R$ and $\mathcal{A}_j = (A_j, \Sigma^{(j)}) \in \mathbf{U}_R$, $j = 1, \ldots, n$. Furthermore, let us take a family Ψ of mappings

$$\Psi_{mj} : (A_1 \times \ldots \times A_{j+i-1})^{m-1} \times \Sigma_m \to \Sigma_m^{(j)}, \quad m \in R, \ 1 \le j \le n .$$

It is said that $\mathcal{A}$ is the α_i-product of $\mathcal{A}_j$, $j = 1, \ldots, n$, with respect to Ψ if the following conditions are satisfied:

(i) $\quad A = \prod_{j=1}^{n} A_j ,$

(ii) for any $m \in R$, $\sigma \in \Sigma_m$ and $((a_{1,1}, \ldots, a_{1,n}), \ldots, (a_{m-1,1}, \ldots, a_{m-1,n})) \in A^{m-1}$,

$((a_{1,1}, \ldots, a_{1,n}), \ldots, (a_{m-1,1}, \ldots, a_{m-1,n}))\sigma^{\mathcal{A}} =$

$(a_{1,1}, \ldots, a_{m-1,1})\sigma_1^{\mathcal{A}_1} \times \ldots \times (a_{1,n}, \ldots, a_{m-1,n})\sigma_n^{\mathcal{A}_n}$,
where

$$\sigma_j = \Psi_{mj}((a_{1,1},\ldots,a_{1,j+i-1}),\ldots,(a_{m-1,1},\ldots,a_{m-1,i+j-1}),\sigma), \qquad j=1,\ldots,n\,.$$

We shall use the notation

$$\prod_{j=1}^{n} \mathcal{A}_j(\Sigma,\Psi)$$

for the product introduced above. In particular, if $\mathcal{A}_j$, $j=1,\ldots,n$, are identical copies of some nondeterministic tree automaton $\mathcal{B}$, then we speak of an *α_i-power* and we write $\mathcal{B}^n(\Sigma,\Psi)$ for $\prod_{j=1}^{n} \mathcal{A}_j(\Sigma,\Psi)$.

Let $\mathbf{B}$ be a system of nondeterministic tree automata from $\mathbf{U}_R$. It is said that $\mathbf{B}$ is *isomorphically complete for* $\mathbf{U}_R$ *with respect to the α_i-product* if any nondeterministic tree automaton from $\mathbf{U}_R$ is isomorphic to a subautomaton of an α_i-product of nondeterministic tree automata in $\mathbf{B}$.

3 α_0-product

In this section, we deal with the first member of this family of products, the α_0-product, which correspondes to the serial composition. In this case, the feedback functions can be given as follows:

$$\Psi_{1j} : \Sigma_1 \to \Sigma_1^{(j)},\ j=1,\ldots,n,\ \text{ if }\ 1 \in R,$$

$$\Psi_{m1} : \Sigma_m \to \Sigma_m^{(1)},\ \ 1 \neq m \in R,$$

$$\Psi_{mj} : (A_1 \times \ldots \times A_{j-1})^{m-1} \times \Sigma_m \to \Sigma_m^{(j)}, \quad 1 \neq m \in R,\ \ 2 \leq j \leq n\,.$$

In what follows, we need a special two-state nondeterministic tree automaton which is defined in the following way. For all $m \in R$, let us assign a symbol to each m-ary relation on $\{0,1\}$. Let $\bar{\Sigma}_m$ denote the set of these relational symbols and let $\bar{\Sigma} = \cup_{m\in R}\bar{\Sigma}_m$. Let us define the nondeterministic tree automaton $\mathcal{G} = (\{0,1\},\bar{\Sigma})$ such that, for every $m \in R$ and $\sigma \in \bar{\Sigma}_m$, $\sigma^{\mathcal{G}}$ is the corresponding m-ary relation on $\{0,1\}$.

The following theorem provides necessary and sufficient conditions for a system of nondeterministic tree automata from $\mathbf{U}_R$ to be isomorphically complete for $\mathbf{U}_R$ with respect to the α_0-product.

Theorem 1. *A system* $\mathbf{B}$ *of nondeterministic tree automata from* $\mathbf{U}_R$ *is isomorphically complete for* $\mathbf{U}_R$ *with respect to the α_0-product if and only if*

(a) *there exists a nondeterministic tree automaton $\mathcal{A}^* = (A^*, \Sigma^*) \in \mathbf{B}$ such that $\mathcal{A}^*$ has two different elements a_0^*, a_1^*, and for every $1 \neq m \in R$, there is a $\sigma_m \in \Sigma_m^*$ for which $(a_{s_1}^*, \ldots, a_{s_{m-1}}^*)\sigma_m^{\mathcal{A}^*} \supseteq \{a_0^*, a_1^*\}$ is valid, for all $(s_1, \ldots s_{m-1}) \in \{0,1\}^{m-1}$, furthermore, there is a $\sigma_1 \in \Sigma_1^*$ with $\{a_0^*, a_1^*\} \subseteq \sigma_1^{\mathcal{A}^*}$ if $1 \in R$,*

(b) *for all $m \in R$ and $\mathbf{i} = (i_1, \ldots, i_m) \in \{0,1\}^m$, $\mathbf{B}$ contains a nondeterministic tree automaton $\mathcal{A}^{(\mathbf{i})} = (A^{(\mathbf{i})}, \Sigma^{(\mathbf{i})})$ satisfying the following conditions:*

(b1) *$A^{(\mathbf{i})}$ has two different elements $a_0^{(\mathbf{i})}$ and $a_1^{(\mathbf{i})}$,*

(b2) *there exists a $\sigma_{\mathbf{i}} \in \Sigma_m^{(\mathbf{i})}$ with $(a_{i_1}^{(\mathbf{i})}, \ldots, a_{i_{m-1}}^{(\mathbf{i})})\sigma_{\mathbf{i}}^{\mathcal{A}^{(\mathbf{i})}} \cap \{a_0^{(\mathbf{i})}, a_1^{(\mathbf{i})}\} = \{a_{i_m}^{(\mathbf{i})}\}$,*

(b3) *for all $1 \neq u \in R$ and $\mathbf{s} = (s_1, \ldots, s_{u-1}) \in \{0,1\}^{u-1}$, there is a $\sigma_{\mathbf{i},\mathbf{s}} \in \Sigma_u^{(\mathbf{i})}$ for which $\{a_0^{(\mathbf{i})}, a_1^{(\mathbf{i})}\} \subseteq (a_{s_1}^{(\mathbf{i})}, \ldots, a_{s_{u-1}}^{(\mathbf{i})})\sigma_{\mathbf{i},\mathbf{s}}^{\mathcal{A}^{(\mathbf{i})}}$, furthermore, there is a $\bar{\sigma}_{\mathbf{i}} \in \Sigma_1^{(\mathbf{i})}$ with $\{a_0^{(\mathbf{i})}, a_1^{(\mathbf{i})}\} \subseteq \bar{\sigma}_{\mathbf{i}}^{\mathcal{A}^{(\mathbf{i})}}$ if $1 \in R$.*

Proof. To prove the necessity, let us suppose that $\mathbf{B}$ is an isomorphically complete system of nondeterministic tree automata for $\mathbf{U}_R$ with respect to the α_0-product. Then there are $\mathcal{A}_j = (A_j, \Sigma^{(j)}) \in \mathbf{B}$, $j = 1, \ldots, n$, such that $\mathcal{G}$ is isomorphic to a subautomaton $\mathcal{A} = (A, \bar{\Sigma})$ of an α_0-product $\prod_{j=1}^n \mathcal{A}_j(\bar{\Sigma}, \Psi)$. Let μ denote a suitable isomorphism and let

$$\mu(0) = (a_{0,1}, \ldots, a_{0,n}) \text{ and } \mu(1) = (a_{1,1}, \ldots, a_{1,n}).$$

Let us denote by k the smallest index with $a_{0,k} \neq a_{1,k}$. Then we prove that $\mathcal{A}_k$ satisfies condition (a). For this purpose, we distinguish two cases depending on m.

Let us suppose that $m \neq 1$. By the definition of $\mathcal{G}$, each m-ary relation on $\{0,1\}$ has a relational symbol in $\bar{\Sigma}_m$. Thus, there exists a $\bar{\sigma}_m \in \bar{\Sigma}_m$ such that $\bar{\sigma}_m^{\mathcal{G}}$ is the complete m-ary relation on $\{0,1\}$. This means that $\bar{\sigma}_m^{\mathcal{G}}(s_1, \ldots, s_m)$ is valid, for all $(s_1, \ldots, s_m) \in \{0,1\}^m$. Therefore, $(s_1, \ldots, s_{m-1})\bar{\sigma}_m^{\mathcal{G}} = \{0,1\}$, and thus, $(s_1, \ldots, s_{m-1})\bar{\sigma}_m^{\mathcal{G}}\mu = \{0,1\}\mu = \{\mu(0), \mu(1)\}$ is valid, for all $(s_1, \ldots, s_{m-1}) \in \{0,1\}^{m-1}$. Since μ is an isomorphism, we have $(s_1, \ldots, s_{m-1})\bar{\sigma}_m^{\mathcal{G}}\mu = (\mu(s_1), \ldots, \mu(s_{m-1}))\bar{\sigma}_m^{\mathcal{A}}$. Consequently,

$$(\mu(s_1), \ldots, \mu(s_{m-1}))\bar{\sigma}_m^{\mathcal{A}} = \{\mu(0), \mu(1)\}$$

is valid, for all $(s_1, \ldots, s_{m-1}) \in \{0,1\}^{m-1}$. By the definition of the α_0-product, the above equality implies

$$\{a_{0,k}, a_{1,k}\} \subseteq (a_{s_1,k}, \ldots, a_{s_{m-1},k})\sigma_{\mathbf{s},k}^{\mathcal{A}_k}$$

where $\mathbf{s} = (s_1, \ldots, s_{m-1})$ and

$$\sigma_{\mathbf{s},k} = \Psi_{mk}((a_{s_1,1}, \ldots, a_{s_1,k-1}), \ldots, (a_{s_{m-1},1}, \ldots, a_{s_{m-1},k-1}), \bar{\sigma}_m).$$

If $k = 1$, then $\sigma_{\mathbf{s},k} = \Psi_{m1}(\bar{\sigma}_m)$. If $k > 1$, then let us observe that, by the definition of k, $a_{s_t,j} = a_{0,j}$, $t = 1, \ldots, m-1$, is valid, for all j, $j = 1, \ldots, k-1$. Therefore,

$$\sigma_{\mathbf{s},k} = \Psi_{mk}((a_{0,1},\ldots,a_{0,k-1}),\ldots,(a_{0,1},\ldots,a_{0,k-1}),\bar{\sigma}_m).$$

In both cases, we obtain that $\sigma_{\mathbf{s},k}$ does not depend on s, and thus, there exists a $\sigma_m \in \Sigma_m^{(k)}$ such that

$$\{a_{0,k},a_{1,k}\} \subseteq (a_{s_1,k},\ldots,a_{s_{m-1},k})\sigma_m^{\mathcal{A}_k}$$

holds, for all $(s_1,\ldots,s_{m-1}) \in \{0,1\}^{m-1}$ which yields the validity of (a) if $m \neq 1$.

Now, let us suppose that $1 \in R$ and $m = 1$. By the definition of $\mathcal{G}$, there exists a $\bar{\sigma} \in \bar{\Sigma}_1$ such that $\bar{\sigma}^{\mathcal{G}}(0)$ and $\bar{\sigma}^{\mathcal{G}}(1)$ are valid. Since μ is an isomorphism, we obtain that $\bar{\sigma}^{\mathcal{A}}(\mu(0))$ and $\bar{\sigma}^{\mathcal{A}}(\mu(1))$ are also valid. Therefore, $\bar{\sigma}^{\mathcal{A}} = \{\mu(0),\mu(1)\}$. This equality implies $\{a_{0,k},a_{1,k}\} \subseteq \bar{\sigma}_1^{\mathcal{A}_k}$ where $\bar{\sigma}_1 = \Psi_{1k}(\bar{\sigma})$, and thus, $\mathcal{A}_k$ satisfies (a) in this case, too.

Regarding validity of (b), it follows from the proof of Theorem 1 in [13]. For the sake of completeness, we present its proof here as well. For this purpose, let us denote the set $\{k : 1 \leq k \leq n \;\&\; a_{0,k} \neq a_{1,k}\}$ by K. Obviously, $K \neq \emptyset$. Now, let $m \in R$ and $\mathbf{i} = (i_1,\ldots,i_m) \in \{0,1\}^m$ be arbitrary elements. We distinguish the following two cases depending on m.

Case 1: $m > 1$. By the definition of $\mathcal{G}$, there is a $\bar{\sigma}_m \in \bar{\Sigma}_m$ with $(i_1,\ldots,i_{m-1})\bar{\sigma}_m^{\mathcal{G}} = i_m$. Since μ is an isomorphism, this yields

$$(\mu(i_1),\ldots,\mu(i_{m-1}))\bar{\sigma}_m^{\mathcal{A}} = \mu(i_m).$$

Therefore, $a_{i_m,k} \in (a_{i_1,k},\ldots,a_{i_{m-1},k})\sigma_k^{\mathcal{A}_k}$ holds, for all $k \in K$, where

$$\sigma_k = \Psi_{mk}((a_{i_1,1},\ldots,a_{i_1,k-1}),\ldots,(a_{i_{m-1},1},\ldots,a_{i_{m-1},k-1}),\bar{\sigma}_m).$$

But then there exists at least one index $l \in K$ such that

$$(a_{i_1,l},\ldots,a_{i_{m-1},l})\sigma_l^{\mathcal{A}_l} \cap \{a_{0,l},a_{1,l}\} = \{a_{i_m,l}\}.$$

Consequently, $\mathcal{A}^{(l)}$ satisfies (b1) and (b2). To prove (b3), let $1 \neq u \in R$ and $\mathbf{s} = (s_1,\ldots,s_{u-1}) \in \{0,1\}^{u-1}$ be arbitrary elements. By the definition of $\mathcal{G}$, there exists a $\sigma_{\mathbf{s}} \in \bar{\Sigma}_u$ with $(s_1,\ldots,s_{u-1})\sigma_{\mathbf{s}}^{\mathcal{G}} = \{0,1\}$. Since μ is an isomorphism, this implies

$$(\mu(s_1),\ldots,\mu(s_{u-1}))\sigma_{\mathbf{s}}^{\mathcal{A}} = \{\mu(0),\mu(1)\}.$$

Then $\{a_{0,k},a_{1,k}\} \subseteq (a_{s_1,k},\ldots,a_{s_{u-1},k})\sigma_{\mathbf{s},k}^{\mathcal{A}_k}$ holds, for all $k \in K$, where

$$\sigma_{\mathbf{s},k} = \Psi_{uk}((a_{s_1,1},\ldots,a_{s_1,k-1}),\ldots,(a_{s_{u-1},1},\ldots,a_{s_{u-1},k-1}),\sigma_{\mathbf{s}}).$$

Therefore, $\{a_{0,l},a_{1,l}\} \subseteq (a_{s_1,l},\ldots,a_{s_{u-1},l})\sigma_{\mathbf{s},l}^{\mathcal{A}_l}$. If $1 \in R$ and $u = 1$, then, by the definition of $\mathcal{G}$, there is a $\sigma_1 \in \bar{\Sigma}_1$ with $\sigma_1^{\mathcal{G}} = \{0,1\}$. But then $\sigma_1^{\mathcal{A}} = \{\mu(0),\mu(1)\}$, and consequently, $\{a_{0,k},a_{1,k}\} \subseteq \bar{\sigma}_k^{\mathcal{A}_k}$, for all $k \in K$, where $\bar{\sigma}_k = \Psi_{1k}(\sigma_1)$. Thus

$\{a_{0,l}, a_{1,l}\} \subseteq \bar{\sigma}_l^{\mathcal{A}_l}$, i.e., $\mathcal{A}^{(l)}$ satisfies (b3) as well. This completes the proof of the necessity when $m \neq 1$.

Case 2: $1 \in R$ and $m = 1$. By the definition of $\mathcal{G}$, there is a $\bar{\sigma}_1 \in \bar{\Sigma}_1$ with $\bar{\sigma}_1^{\mathcal{G}} = i_1$. But then $\bar{\sigma}_1^{\mathcal{A}} = \mu(i_1)$. Therefore, $a_{i_1,k} \in \sigma_k^{\mathcal{A}_k}$ is valid, for all $k \in K$, where $\sigma_k = \Psi_{1k}(\bar{\sigma}_1)$. From this it follows that there exists at least one $l \in K$ such that

$$\sigma_l^{\mathcal{A}_l} \cap \{a_{0,l}, a_{1,l}\} = \{a_{i_1,l}\}.$$

Now, let $u \in R$ and $\mathbf{s} = (s_1, \ldots, s_{u-1}) \in \{0,1\}^{u-1}$ be fixed arbitrarily. In a similar way as above, it is easy to see that there is a $\sigma_{\mathbf{s},l} \in \Sigma_u^{(l)}$ such that $\{a_{0l}, a_{1l}\} \subseteq (a_{s_1 l}, \ldots, a_{s_{u-1} l})\sigma_{\mathbf{s},l}^{\mathcal{A}_l}$ if $u \neq 1$, and there is a $\sigma_l^* \in \Sigma_1^{(l)}$ with $\{a_{0,l}, a_{1,l}\}\sigma_l^{*\mathcal{A}_l}$ if $u = 1$. This completes the proof of the necessity.

For proving the sufficiency, let us assume that $\mathbf{B}$ satisfies the conditions of Theorem 1. Let us define the sets W and W' by

$$W = \{\{0,1\}^m : m \in R\} \quad \text{and} \quad W' = \{(i_1, \ldots, i_m) : (i_1, \ldots, i_m) \in W \ \&\ i_m = 0\}.$$

Let $|W'| = n$, and let γ denote a one-to-one mapping of $\{1, \ldots, n\}$ onto W'. By our assumption on $\mathbf{B}$, for any $p \in \{1, \ldots, n\}$, there exists a nondeterministic tree automaton $\mathcal{A}^{(\gamma(p))} = (A^{(\gamma(p))}, \Sigma^{(\gamma(p))}) \in \mathbf{B}$ satisfying conditions (b1), (b2), and (b3) with $\mathbf{i} = (i_1, \ldots, i_m) = \gamma(p)$ where $i_m = 0$. For the sake of simplicity, let us denote the elements $a_0^{(\gamma(p))}$ and $a_1^{(\gamma(p))}$ by 0 and 1, respectively. Furthermore, let us denote by $\mathcal{A}^* = (A^*, \Sigma^*)$ an automaton of $\mathbf{B}$ satisfying (a), moreover, let 0 and 1 denote the elements a_0^* and a_1^*, respectively.

Now, let $\mathcal{C} = (C, \Sigma) \in \mathbf{U}_R$ be an arbitrary nondeterministic tree automaton with $C = \{c_1, \ldots, c_r\}$. We prove that $\mathcal{C}$ can be embedded isomorphically into an α_0-product of nondeterministic tree automata from $\{\mathcal{A}^*\} \cup \{\mathcal{A}^{(\gamma(p))} : 1 \le p \le n\}$.

For this purpose, let us take all the r-dimensional column vectors over $\{0,1\}$ and order them in lexicographically increasing order. Let $\mathbf{Q}^{(r)}$ denote the matrix formed by these column vectors. Then $\mathbf{Q}^{(r)}$ is a matrix of type $r \times 2^r$ over $\{0,1\}$, the row vectors of $\mathbf{Q}^{(r)}$ are pairwise different, moreover, for any subset V of $\{1, \ldots, r\}$, there exists exactly one index $k \in \{1, \ldots, 2^r\}$ such that, for all $t \in \{1, \ldots, r\}$, $t \in V$ if and only if $q_{tk}^{(r)} = 0$. Let

$$\mathbf{Q} = (\mathbf{Q}^{(r)} \ldots \mathbf{Q}^{(r)})$$

where the number of the occurences of $\mathbf{Q}^{(r)}$ is $n+1$ in the partitioned form of $\mathbf{Q}$. Finally, let us define the one-to-one mapping μ of $\{c_1, \ldots, c_r\}$ onto the set of the row vectors of $\mathbf{Q}$ by $\mu(c_i) = (q_{i,1}, \ldots, q_{i,(n+1)2^r})$, $i = 1, \ldots, r$, and let $M = \{\mu(c_i) : i = 1, \ldots, r\}$.

Now, let us construct the α_0-product $\mathcal{A} = (A, \Sigma) =$

$$\underbrace{A^* \times \cdots \times A^*}_{2^r times} \times \underbrace{A^{(\gamma(1))} \times \cdots \times A^{(\gamma(1))}}_{2^r times} \times \cdots \times \underbrace{A^{(\gamma(n))} \times \cdots \times A^{(\gamma(n))}}_{2^r times}(\Sigma, \Psi)$$

in the following way. First of all, let us observe that $M \subseteq A$. To define the feedback functions, let us consider the following two cases.

Case 1: $1 \in R$ and $m = 1$. Let $\sigma \in \Sigma_1 (\subseteq \Sigma)$ be an arbitrary relational symbol, furthermore, let $\sigma^C = \{c_{k_1}, \ldots, c_{k_l}\}$ where $0 \le l \le r$. Since $1 \in R$, the vector $\mathbf{i} = (0)$ is contained in W', and thus, there exists a $p_0 \in \{1, \ldots, n\}$ such that $\gamma(p_0) = (0)$. On the other hand, by the definition of $\mathbf{Q}^{(r)}$, there exists exactly one index $d \in \{1, \ldots, 2^r\}$ such that, for each $s \in \{0, \ldots, n\}$, the following assertion is valid:

$$\text{for all } t \in \{1, \ldots, r\},\ q_{t,s2^r+d} = 0 \text{ if and only if } t \in \{k_1, \ldots, k_l\}.$$

Now, the feedback functions Ψ_{1j}, $j = 1, \ldots, (n+1)2^r$, are defined as follows:

$$\Psi_{1j}(\sigma) = \begin{cases} \sigma_1 & \text{if } 1 \le j \le 2^r, \\ \sigma_{(0)} & \text{if } j = p_0 2^r + d, \\ \bar{\sigma}_{(0)} & \text{if } p_0 2^r < j \le (p_0+1)2^r \ \ \&\ \ j \ne p_0 2^r + d, \\ \bar{\sigma}_{\gamma(p)} & \text{if } p_0 \ne p \in \{1, \ldots, n\} \ \ \&\ \ p2^r < j \le (p+1)2^r, \end{cases}$$

where $\sigma_1 \in \Sigma_1^*$ satisfying (a), $\sigma_{(0)} \in \Sigma_1^{((0))}$ satisfying (b2), $\bar{\sigma}_{(0)} \in \Sigma_1^{((0))}$ satisfying (b3), finally, $\bar{\sigma}_{\gamma(p)} \in \Sigma_1^{(\gamma(p))}$ satisfying (b3).

Case 2: $1 \ne m \in R$. Let $\sigma \in \Sigma_m (\subseteq \Sigma)$ be an arbitrary m-ary relational symbol and let us consider $m-1$ elments from M denoted by $(q_{i_t,1}, \ldots, q_{i_t,(n+1)2^r})$, $t = 1, \ldots, m-1$. Then, $\mu(c_{i_t}) = (q_{i_t,1}, \ldots, q_{i_t,(n+1)2^r})$, $t = 1, \ldots, m-1$. Let us suppose that $(c_{i_1}, \ldots, c_{i_{m-1}})\sigma^C = \{c_{k_1}, \ldots, c_{k_l}\}$ where $0 \le l \le r$. Then there is one and only one integer $d \in \{1, \ldots, 2^r\}$ such that, for every $s \in \{0, \ldots, n\}$, we have the following assertion:

$$\text{for all } t \in \{1, \ldots, r\},\ q_{t,s2^r+d} = 0 \text{ if and only if } t \in \{k_1, \ldots, k_l\}.$$

On the other hand, let us observe that, for any $v \in \{1, \ldots, 2^r\}$, the column vectors of $\mathbf{Q}$ with indices $s2^r+v$, $s = 0, \ldots, n$, are identical copies of some r-dimensional vector over $\{0,1\}$. Consequently, the vectors $(q_{i_1,s2^r+v}, \ldots, q_{i_{m-1},s2^r+v})$, $s = 0, \ldots, n$, are the copies of an $(m-1)$-dimensional vector over $\{0,1\}$. Let us denote the vector $(q_{i_1,v}, \ldots, q_{i_{m-1},v})$ by $\mathbf{s}_v$ if $1 \le v \le 2^r$, $v \ne d$, and the vector $(q_{i_1,d}, \ldots, q_{i_{m-1},d})$ by $(i'_1, \ldots, i'_{m-1})$. Let $\mathbf{i} = (i'_1, \ldots, i'_{m-1}, 0)$. Then $\mathbf{i} \in W'$, and thus, there is a $p_0 \in \{1, \ldots, n\}$ with $\gamma(p_0) = \mathbf{i}$. Now, we define the feedback functions as follows. For any $j \in \{1, \ldots, (n+1)2^r\}$, let

$$\Psi_{mj}((q_{i_1,1}, \ldots, q_{i_1,j-1}), \ldots, (q_{i_{m-1},1}, \ldots, q_{i_{m-1},j-1}), \sigma) =$$

$$= \begin{cases} \sigma_m & \text{if } 1 \le j \le 2^r, \\ \sigma_{\mathbf{i}} & \text{if } j = p_0 2^r + d, \\ \sigma_{\gamma(p),\mathbf{s}_v} & \text{if } j \ne p_0 2^r + d \;\&\; v \equiv j (\text{mod } 2^r) \;\&\; p2^r < j \le (p+1)2^r \\ & \quad \& \; p \in \{1, \ldots, n\} \end{cases}$$

where $\sigma_m \in \Sigma_m^*$ satisfying (a), $\sigma_{\mathbf{i}} \in \Sigma_m^{(\mathbf{i})}$ satisfying (b2), and $\sigma_{\gamma(p),\mathbf{s}_v} \in \Sigma_m^{(\gamma(p))}$ satisfying (b3).
In all the remaining cases, let us define the feedback functions Ψ_{mj} arbitrarily in accordance with the definition of the α_0-product.

Regarding above definition, we have to verify that it is really an α_0-product. If $1 \in R$ and $m = 1$, then our definition is obviously correct. Now, let $1 \ne m \in R$. Then Ψ_{mj} depends only on m if $1 \le j \le 2^r$. Let us consider the case when $2^r < j \le (n+1)2^r$. Since the row vectors of $\mathbf{Q}^{(r)}$ are pairwise different, each element of M is uniquely determined by its first 2^r components. Therefore, the indices $i_1, \ldots, i_{m-1}$ are uniquely determined. Then $k_1, \ldots, k_l$ are determined by σ. Furthermore, d, $\mathbf{i}$ and p_0 are determined uniquely by $k_1, \ldots k_l$, the definition of $\mathbf{Q}^{(r)}$, and the first 2^r components of the elements in M under consideration. Now, if $j = p_0 2^r + d$, then the definition of Ψ_{mj} is in accordance with the definition of the α_0-product. If $j \ne p_0 2^r + d$, then j determines v and p uniquely, furthermore, $\mathbf{s}_v$ is determined by v and the first 2^r components of the considered elements of M. Consequently, the definition of Ψ_{mj} correspondes to the definition of the α_0-product in this case as well.

By the above observations, we have that $\mathcal{A}$ is an α_0-product of nondeterministic tree automata from $\{\mathcal{A}^*\} \cup \{\mathcal{A}^{(\gamma(p))} : 1 \le p \le n\}$. Let us consider the subautomaton of $\mathcal{A}$ determined by M and denote this subautomaton by $\mathcal{M} = (M, \Sigma)$. We prove that $\mathcal{C}$ and $\mathcal{M}$ are isomorphic, moreover, the mapping μ is a suitable isomorphism.

First, let us suppose that $1 \in R$ and $m = 1$. Let $\sigma \in \Sigma_1$ be an arbitrary relational symbol. We have to prove that $\sigma^{\mathcal{C}}(c_k)$ if and only if $\sigma^{\mathcal{M}}(\mu(c_k))$, for all $c_k \in C$, or equivalently, $\sigma^{\mathcal{C}}\mu = \sigma^{\mathcal{M}}$. We distinguish the following two cases.

Let us suppose that $\sigma^{\mathcal{C}} = \emptyset$. Then $d = 2^r$, furthermore, $\Psi_{1,(p_0+1)2^r}(\sigma) = \sigma_{(0)}$, and thus, the $(p_0+1)2^r$-th component of each element of $\sigma^{\mathcal{A}}$ is not equal to 1. On the other hand, the $(p_0+1)2^r$-th component of each element of M is equal to 1. Therefore, $\emptyset = \sigma^{\mathcal{A}} \cap M = \sigma^{\mathcal{M}}$. Conversely, let us assume that $\sigma^{\mathcal{M}} = \emptyset$. If $\sigma^{\mathcal{C}} \ne \emptyset$, then $\sigma^{\mathcal{C}} = \{c_{k_1}, \ldots c_{k_l}\}$ for some $1 \le l \le r$. Then, by the definition of Ψ_{1j}, $j = 1, \ldots, (n+1)2^r$, we obtain that

$$\sigma^{\mathcal{A}} \supseteq \{0,1\}^{p_0 2^r + d - 1} \times \{0\} \times \{0,1\}^{(n+1)2^r - p_0 2^r - d},$$

and the right-side set of the above inclusion contains $\mu(c_{k_t})$, for all t, $t = 1, \ldots, l$. Therefore, $\sigma^{\mathcal{A}} \cap M = \sigma^{\mathcal{M}} \ne \emptyset$ which is a contradiction. Consequently, $\sigma^{\mathcal{C}} = \emptyset$.

Now, let us suppose that $\sigma^{\mathcal{C}} = \{c_{k_1}, \ldots, c_{k_l}\}$ for some $1 \le l \le r$. Then

$$\sigma^{\mathcal{A}} \supseteq \{0,1\}^{p_0 2^r + d - 1} \times \{0\} \times \{0,1\}^{(n+1)2^r - p_0 2^r - d},$$

and the right-side set contains $\mu(c_{k_t})$, for all t, $t = 1, \ldots, l$. On the other hand, by the definition of d, for all $t \in \{1, \ldots, r\}$, $q_{t, p_0 2^r + d} = 0$ if and only if $t \in \{k_1, \ldots, k_l\}$. This yields that $\sigma^{\mathcal{A}} \cap M = \{\mu(c_{k_1}), \ldots, \mu(c_{k_l})\}$, i.e., $\sigma^{\mathcal{M}} = \{\mu(c_{k_1}), \ldots, \mu(c_{k_l})\}$. Consequently, $\sigma^{\mathcal{C}}\mu = \sigma^{\mathcal{M}}$.

Now, let $1 \neq m \in R$, $\sigma \in \Sigma_m$, $c_{i_t} \in C$, $t = 1, \ldots, m-1$, be arbitrary elements. We have to show that

$$(c_{i_1}, \ldots, c_{i_{m-1}})\sigma^{\mathcal{C}}\mu = (\mu(c_{i_1}), \ldots, \mu(c_{i_{m-1}}))\sigma^{\mathcal{M}}$$

is valid. Let $(c_{i_1}, \ldots, c_{i_{m-1}})\sigma^{\mathcal{C}} = \{c_{k_1}, \ldots, c_{k_l}\}$ for some integer $0 \leq l \leq r$. Then, by the definition of Ψ_{mj}, $j = 1, \ldots, (n+1)2^r$,

$$(\mu(c_{i_1}), \ldots, \mu(c_{i_{m-1}}))\sigma^{\mathcal{A}} \supseteq \{0,1\}^{p_0 2^r + d - 1} \times \{0\} \times \{0,1\}^{(n+1)2^r - p_0 2^r - d},$$

furthermore, $\{\mu(c_{k_1}), \ldots, \mu(c_{k_l})\} = \{(q_{k_t,1}, \ldots, q_{k_t,(n+1)2^r} : 1 \leq t \leq l\}$ is a subset of the right-side set. By the definition of d, for all $t \in \{1, \ldots, r\}$, $q_{t, p_0 2^r + d} = 0$ if and only if $t \in \{k_1, \ldots, k_l\}$. This yields that

$$(\mu(c_{i_1}), \ldots, \mu(c_{i_{m-1}}))\sigma^{\mathcal{A}} \cap M = \{(q_{k_t,1}, \ldots, q_{k_t,(n+1)2^r}) : 1 \leq t \leq l\} =$$

$$= \{\mu(c_{k_1}), \ldots, \mu(c_{k_l})\}.$$

Consequently, $(c_{i_1}, \ldots, c_{i_{m-1}})\sigma^{\mathcal{C}}\mu = (\mu(c_{i_1}), \ldots, \mu(c_{i_{m-1}}))\sigma^{\mathcal{M}}$, and thus, μ is an isomorphism of $\mathcal{C}$ onto $\mathcal{M}$.

This completes the proof of Theorem 1.

Remark. In particular, if $R = \{2\}$, then $\mathbf{U}_R$ is the class of the nondeterministic automata. Then as a special case of Theorem 1, we obtain the characterization of the isomorphically complete systems of nondeterministic automata with respect to the α_0-product which was presented in [14].

It is easy to observe that the nondeterministic tree automaton $\mathcal{G}$ satisfies the conditions of Theorem 1. Therefore, every nondeterministic tree automaton from $\mathbf{U}_R$ can be embedded into an α_0-power of $\mathcal{G}$. This implies the following corollary.

Corollary 1. *Every nondeterministic tree automaton from* $\mathbf{U}_R$ *can be embedded isomorphically into an* α_0*-product of suitable two-state nondeterministic tree automata.*

4 α_i-product with $i \geq 1$

In this section, we study the α_i-product with $i \geq 1$. For this reason, let $i > 0$ be an arbitrarily fixed integer. Then the isomorphically complete systems of nondeterministic tree automata with respect to the α_i-product can be characterized as follows.

Theorem 2. *A system* $\mathbf{B}$ *of nondeterministic tree automata from* $\mathbf{U}_R$ *is isomorphically complete for* $\mathbf{U}_R$ *with respect to the* α_i*-product if and only if, for all* $m \in R$ *and* $\mathbf{i} = (i_1, \ldots, i_m) \in \{0,1\}^m$, $\mathbf{B}$ *contains a nondeterministic tree automaton* $\mathcal{A}^{(\mathbf{i})} = (A^{(\mathbf{i})}, \Sigma^{(\mathbf{i})})$ *satisfying the following conditions:*

(I) $A^{(\mathbf{i})}$ *has two different elements* $a_0^{(\mathbf{i})}$ *and* $a_1^{(\mathbf{i})}$,

(II) *there exists a* $\sigma_{\mathbf{i}} \in \Sigma_m^{(\mathbf{i})}$ *with* $(a_{i_1}^{(\mathbf{i})}, \ldots, a_{i_{m-1}}^{(\mathbf{i})})\sigma_{\mathbf{i}}^{\mathcal{A}^{(\mathbf{i})}} \cap \{a_0^{(\mathbf{i})}, a_1^{(\mathbf{i})}\} = \{a_{i_m}^{(\mathbf{i})}\}$,

(III) *for all* $1 \neq u \in R$ *and* $\mathbf{s} = (s_1, \ldots, s_{u-1}) \in \{0,1\}^{u-1}$, *there is a* $\sigma_{\mathbf{i},\mathbf{s}} \in \Sigma_u^{(\mathbf{i})}$ *for which* $\{a_0^{(\mathbf{i})}, a_1^{(\mathbf{i})}\} \subseteq (a_{s_1}^{(\mathbf{i})}, \ldots, a_{s_{u-1}}^{(\mathbf{i})})\sigma_{\mathbf{i},\mathbf{s}}^{\mathcal{A}^{(\mathbf{i})}}$, *furthermore, there is a* $\bar{\sigma}_{\mathbf{i}} \in \Sigma_1^{(\mathbf{i})}$ *with* $\{a_0^{(\mathbf{i})}, a_1^{(\mathbf{i})}\} \subseteq \bar{\sigma}_{\mathbf{i}}^{\mathcal{A}^{(\mathbf{i})}}$ *if* $1 \in R$.

Proof. The necessity of the conditions follows from Theorem 1 in [13]; the proof has the same idea as the proof of the necessity of (b) in Theorem 1 of Section 3. In order to prove the sufficiency, let us suppose that $\mathbf{B}$ satisfies the conditions of Theorem 2. Let us define the sets W and W' as above, i.e., let

$$W = \{\{0,1\}^m : m \in R\} \quad \text{and} \quad W' = \{(i_1, \ldots, i_m) : (i_1, \ldots, i_m) \in W \ \& \ i_m = 0\}.$$

Let $|W'| = n$, and let γ denote a one-to-one mapping of $\{1, \ldots, n\}$ onto W'. By our assumption on $\mathbf{B}$, for any $p \in \{1, \ldots, n\}$, there exists a nondeterministic tree automaton $\mathcal{A}^{(\gamma(p))} = (A^{(\gamma(p))}, \Sigma^{(\gamma(p))}) \in \mathbf{B}$ satisfying conditions (I), (II), and (III) with $\mathbf{i} = (i_1, \ldots, i_m) = \gamma(p)$ where $i_m = 0$. Again, let us denote the elements $a_0^{(\gamma(p))}$ and $a_1^{(\gamma(p))}$ by 0 and 1, respectively.

Now, let $\mathcal{C} = (C, \Sigma) \in \mathbf{U}_R$ be an arbitrary nondeterministic tree automaton with $C = \{c_1, \ldots, c_r\}$. We prove that $\mathcal{C}$ can be embedded isomorphically into an α_i-product of nondeterministic tree automata from $\{\mathcal{A}^{(\gamma(p))} : 1 \leq p \leq n\}$.

For this purpose, let

$$\mathbf{Q}' = (\mathbf{Q}^{(r)} \ldots \mathbf{Q}^{(r)})$$

where the number of the occurences of $\mathbf{Q}^{(r)}$ is $n+1$ in the partitioned form of $\mathbf{Q}'$. Furthermore, let us define the one-to-one mapping μ of $\{c_1, \ldots, c_r\}$ onto the set of the row vectors of $\mathbf{Q}'$ by $\mu(c_i) = (q_{i,1}, \ldots, q_{i,(n+1)2^r})$, $i = 1, \ldots, r$, and let $M' = \{\mu(c_i) : i = 1, \ldots, r\}$.

Let us construct the α_i-product $\mathcal{A} = (A, \Sigma) =$

$$\underbrace{\mathcal{A}^{(\gamma(1))} \times \cdots \times \mathcal{A}^{(\gamma(1))}}_{2^r times} \times \underbrace{\mathcal{A}^{(\gamma(1))} \times \cdots \times \mathcal{A}^{(\gamma(1))}}_{2^r times} \times \cdots \times \underbrace{\mathcal{A}^{(\gamma(n))} \times \cdots \times \mathcal{A}^{(\gamma(n))}}_{2^r times}(\Sigma, \Psi)$$

in the following way. First of all, let us observe that $M' \subseteq A$. To define the feedback functions, let us consider the following two cases.

Case 1: $1 \in R$ and $m = 1$. Let $\sigma \in \Sigma_1 (\subseteq \Sigma)$ be an arbitrary relational symbol, furthermore, let $\sigma^C = \{c_{k_1}, \ldots, c_{k_l}\}$ where $0 \leq l \leq r$. Since $1 \in R$, the vector $\mathbf{i} = (0)$ is contained in W', and thus, there exists a $p_0 \in \{1, \ldots, n\}$ such that $\gamma(p_0) = (0)$. On the other hand, by the definition of $\mathbf{Q}^{(r)}$, there exists exactly one index $d \in \{1, \ldots, 2^r\}$ such that, for each $s \in \{0, \ldots, n\}$, the following assertion is valid:

$$\text{for all } t \in \{1, \ldots, r\},\ q_{t,s2^r+d} = 0 \text{ if and only if } t \in \{k_1, \ldots, k_l\}.$$

Let $j_0 = p_0 2^r + d$. Now, the feedback functions Ψ_{1j}, $j = 1, \ldots, (n+1)2^r$, are defined as follows:

$$\Psi_{1j}(\sigma) = \begin{cases} \bar{\sigma}_{\gamma(1)} & \text{if } 1 \leq j \leq 2^r, \\ \sigma_{(0)} & \text{if } j = j_0, \\ \bar{\sigma}_{\gamma(p)} & \text{if } j \neq j_0 \ \&\ p2^r < j \leq (p+1)2^r \text{ for some } p \in \{1, \ldots, n\}. \end{cases}$$

where $\bar{\sigma}_{\gamma(1)} \in \Sigma_1^{(\gamma(1))}$ satisfying (III), $\sigma_{(0)} \in \Sigma_1^{((0))}$ satisfying (II), and $\bar{\sigma}_{\gamma(p)} \in \Sigma_1^{(\gamma(p))}$ satisfying (III).

Case 2: $1 \neq m \in R$. Let $\sigma \in \Sigma_m (\subseteq \Sigma)$ be an arbitrary m-ary relational symbol and let us consider $m-1$ elments from M' denoted by $(q_{i_t,1}, \ldots, q_{i_t,(n+1)2^r})$, $t = 1, \ldots, m-1$. Then, $\mu(c_{i_t}) = (q_{i_t,1}, \ldots, q_{i_t,(n+1)2^r})$, $t = 1, \ldots, m-1$. Let us suppose that $(c_{i_1}, \ldots, c_{i_{m-1}})\sigma^C = \{c_{k_1}, \ldots, c_{k_l}\}$ where $0 \leq l \leq r$. Then there is one and only one integer $d \in \{1, \ldots, 2^r\}$ such that, for every $s \in \{0, \ldots, n\}$, we have the following assertion:

$$\text{for all } t \in \{1, \ldots, r\},\ q_{t,s2^r+d} = 0 \text{ if and only if } t \in \{k_1, \ldots, k_l\}.$$

On the other hand, let us observe that, for any $v \in \{1, \ldots, 2^r\}$, the column vectors of $\mathbf{Q}'$ with indices $s2^r + v$, $s = 0, \ldots, n$, are identical copies of some r-dimensional vector over $\{0, 1\}$. Consequently, the vectors $(q_{i_1,s2^r+v}, \ldots, q_{i_{m-1},s2^r+v})$, $s = 0, \ldots, n$, are the copies of an $(m-1)$-dimensional vector over $\{0, 1\}$. Let us denote the vector $(q_{i_1,v}, \ldots, q_{i_{m-1},v})$ by $\mathbf{s}_v$ if $1 \leq v \leq 2^r$, $v \neq d$, and the vector $(q_{i_1,d}, \ldots, q_{i_{m-1},d})$ by $(i'_1, \ldots, i'_{m-1})$. Let $\mathbf{i} = (i'_1, \ldots, i'_{m-1}, 0)$. Then $\mathbf{i} \in W'$, and thus, there is a $p_0 \in \{1, \ldots, n\}$ with $\gamma(p_0) = \mathbf{i}$. Let $j_0 = p_0 2^r + d$ again. We define the feedback functions in the following way. For any $j \in \{1, \ldots, (n+1)2^r\}$, let

$$\Psi_{mj}((q_{i_1,1}, \ldots, q_{i_1,j+i-1}), \ldots, (q_{i_{m-1},1}, \ldots, q_{i_{m-1},j+i-1}), \sigma) =$$

$$= \begin{cases} \sigma_{\gamma(1),\mathbf{s}_j} & \text{if } 1 \le j \le 2^r, \\ \sigma_{\gamma(1),(i'_1,\ldots,i'_{m-1})} & \text{if } j = d, \\ \sigma_{\mathbf{i}} & \text{if } j = j_0, \\ \sigma_{\gamma(p),\mathbf{s}_v} & \text{if } j \ne j_0 \ \& \ v \equiv j \ (\mathrm{mod}\ 2^r) \ \& \ p2^r < j \le (p+1)2^r \\ & \text{for some}; p \in \{1,\ldots,n\}, \end{cases}$$

where $\sigma_{\gamma(1),\mathbf{s}_j}, \sigma_{\gamma(1),(i'_1,\ldots,i'_{m-1})} \in \Sigma_m^{(\gamma(1))}$ satisfying (III), $\sigma_{\mathbf{i}} \in \Sigma_m^{(\mathbf{i})}$ satisfying (II), and $\sigma_{\gamma(p),\mathbf{s}_v} \in \Sigma_m^{(\gamma(p))}$ satisfying (III). In all the remaining cases, let us define the feedback functions Ψ_{mj} in accordance with the definition of the α_i-product.

Regarding above definition, it is easy to verify that it is really an α_i-product, and thus, $\mathcal{A}$ is an α_i-product of nondeterministic tree automata from $\{\mathcal{A}^{(\gamma(p))} : 1 \le p \le n\}$. Let us consider the subautomaton of $\mathcal{A}$ determined by M'. Let $\mathcal{M}' = (M', \Sigma)$ denote this subautomaton. Then it is easy to prove that μ is an isomorphism of $\mathcal{C}$ onto $\mathcal{M}'$.

This completes the proof of Theorem 2.

Since the characterization of the isomorphically complete systems of nondeterministic tree automata with respect to the general product (see Theorem 1 in [13]) contains the same conditions as Theorem 2, we immediately obtain the following corollary.

Corollary 2. *The α_i-product is equivalent to the general product regarding isomorphically complete systems of nondeterministic tree automata provided that $i \ge 1$.*

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