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On Two-Step Methods for Stochastic Differential Equations

Rózsa Horváth Bokor *†

Abstract

The paper introduces a new two-step method. Its order of strong convergence is proved. In the approximation of solutions of some stochastic differential equations, this multistep method converges faster in mean $E|X - Y_N|$ than the One-step Milstein scheme with order 1.0 or Two-step Milstein scheme with order 1.0.

Keywords: Stochastic differential equations, strong solutions, numerical schemes

1 Introduction

The problem considered in this article is that of approximating strong solutions of the following type of the Itô stochastic differential equation:

$$dX_t = a(t, X_t)dt + b(t, X_t)dW_t, \text{ for } 0 \le t \le T, \ X_t \in \mathbf{R}^d, \tag{1}$$

where

$$a = (a_1 \dots a_d)^\tau, b = (b_1 \dots b_d)^\tau, X_0 = X (\in \mathbf{R}^d).$$

The above system is driven by the one-dimensional Brownian motion. Details about this stochastic object and corresponding calculus can be found in Karatzas and Shreve [2].

We suppose that throughout this paper $E \| X_0 \|^2 < +\infty$ and X_0 is independent of $\mathcal{F}_t = \sigma\{W_s : 0 \le s \le t\}$, the σ-algebra generated by the underlying process. Also, suppose that coefficients $a(t, x)$ and $b(t, x)$ satisfy conditions which guarantee the existence of the unique, strong solution of the stochastic differential equation.

The approximations considered here are evaluated at points of regular partition of the interval $[0, T]$; these have the form $(0, \Delta, 2\Delta, \dots, N\Delta)$, where N is a natural number and $\Delta = \frac{T}{N}$. We denote $n\Delta$ by τ_n, for $n = 0, 1, \dots, N$.

*Department of Mathematics University of Zagreb, Bijemicka 30, 10000 Zagreb, Croatia, email: bokor@math.hr

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Here we shall use the abbreviation Y_n to denote the value of the approximation at time $n\Delta$ and the following operators

$$L^0 = \frac{\partial}{\partial t} + \sum_{k=1}^{d} a_k \frac{\partial}{\partial x_k} + \frac{1}{2} \sum_{k,l=1}^{d} b^k b^l \frac{\partial^2}{\partial x_k \partial x_l}, \tag{2}$$

$$L^1 = \sum_{k=1}^{d} b^k \frac{\partial}{\partial x_k}. \tag{3}$$

To classify different methods with respect to the rate of strong convergence as in [3] we say that a discrete time approximation Y^Δ converges with strong order $\gamma > 0$ if there exist constants $\Delta_0 \in (0, +\infty)$ and $K < +\infty$, not depending on Δ, such that we have a mean global error

$$Eps(T) = E\left|X_T - Y_N^\Delta\right| \leq K\Delta^\gamma \text{ for all } \Delta \subset (0, \Delta_0).$$

The widely used method of order 1.0 is the Milstein method, which has the form

$$Y_{n+1}^M = Y_n^M + a(\tau_n, Y_n^M)\Delta + b(\tau_n, Y_n^M)\Delta W_n + \frac{1}{2} L^1 b(\tau_n; Y_n^M)((\Delta W_n)^2 - \Delta), \tag{4}$$

with $Y_0^M = X_0$. The two-step Milstein strong scheme, for which the k-th component in the general multidimensional case $d = 1, 2, \ldots$ is given by

$$\begin{aligned} Y_{n+1}^{k,T} &= (1-\gamma_k)Y_n^{k,T} + \gamma_k Y_{n-1}^{k,T} + a^k(\tau_n, Y_n^T)\Delta + V_n^k \\ &+ \gamma_k \left[\left((1-\alpha_k)a^k(\tau_n, Y_n^T) + \alpha_k a^k(\tau_{n-1}, Y_{n-1}^T) \right) \Delta + V_{n-1}^k \right], \end{aligned} \tag{5}$$

with

$$\begin{aligned} V_n^k &= b^k(\tau_n, Y_n^T)\Delta W_n + \frac{1}{2} L^1 b^k(\tau_n, Y_n^T)((\Delta W_n)^2 - \Delta), \\ Y_0^T &= X_0,\ Y_1^T = Y_1^M, \end{aligned}$$

where $\Delta W_n = W_{\tau_{n+1}} - W_{\tau_n}$, $n = 0, 1, \ldots, N-1, k = 1, \ldots, d$, and $\alpha_k, \gamma_k \in [0, 1]$.

In the general multidimensional case with $d = 1, 2, \ldots$ the k-th component of the new multistep scheme takes the form

$$\begin{aligned} Y_{n+1}^k &= (1-\gamma_k)Y_n^k + \gamma_k Y_{n-1}^k + a^k(\tau_n, Y_n)\Delta + b^k(\tau_n, Y_n)\Delta W_n \\ &+ \frac{1}{2} L^1 b^k(\tau_n, Y_n)((\Delta W_n)^2 - \Delta) \\ &+ \gamma_k \left[\left((1-\alpha_k)a^k(\tau_n, Y_n) + \alpha_k a^k(\tau_{n-1}, Y_{n-1}) \right) \Delta \right. \\ &+ \left. \frac{1}{2} \left(b^k(\tau_n, Y_n) + b^k(\tau_{n-1}, Y_{n-1}) \right) \Delta W_{n-1}, \right. \end{aligned}$$

$$
\begin{aligned}
&- \quad \frac{1}{2}L^1 b^k(\tau_{n-1}, Y_{n-1})\Delta \Bigg], \qquad (6)\\
Y_0 \quad &= \quad X_0,\ Y_1 = Y_1^M,
\end{aligned}
$$

where $\Delta W_n = W_{\tau_{n+1}} - W_{\tau_n}$, $\Delta = \tau_{n+1} - \tau_n$, $n = 0, 1, \ldots, N-1, k = 1, \ldots, d$ and $\alpha_k, \gamma_k \in [0, 1]$.

During the last years several authors have proposed multistep methods for stochastic differential equations with respect to strong convergence criterious.

I refer here to the books of Kloeden and Platen [3], Boulean and Lépingle [1] and the paper of Lépingle and Ribémont [4].

2 The Main Results

Now we are able to state the corresponding convergence theorem for the multistep method (6):

Theorem 2.1 *Consider the Itô equation (1). Let*

$$
\frac{\partial a}{\partial t}, \frac{\partial a}{\partial x_i}, \frac{\partial^2 a}{\partial x_i \partial x_j}, \frac{\partial b}{\partial t}, \frac{\partial b}{\partial x_i}, \frac{\partial^2 b}{\partial t^2}, \frac{\partial^2 b}{\partial t \partial x_i},
$$
$$
\frac{\partial^2 b}{\partial x_i \partial t}, \frac{\partial^2 b}{\partial x_i \partial x_j}, \frac{\partial^3 b}{\partial x_i \partial x_j \partial x_k} \in C_b([0,T] \times \mathbf{R}^d, \mathbf{R}^d),
$$

be given for all $1 \leq i, j, k \leq d$, where $C_b([0,T] \times \mathbf{R}^d, \mathbf{R}^d)$ denotes the set of continuous and bounded functions from $[0,T] \times \mathbf{R}^d$ to $\mathbf{R}^d$, and functions $L^0 a, L^0 b, L^1 a, L^0 L^1 b, L^1 L^1 b$ fulfill the linear growth condition

$$
\| f(t,x) \| \leq K_1(1 + \| x \|),
$$

for every $t \in [0,T], x \in \mathbf{R}^d$, where K_1 is a positive constant. Under the assumptions the multistep method converges with strong order $\gamma = 1.0$, that is for all $n = 0, 1, \ldots, N$ and step size $\Delta = \frac{T}{N}$, $N = 2, 3 \ldots$

$$
E(\| X_{\tau_n} - Y_n \|) \leq K_2(1 + E \| X_0 \|)\Delta^{1.0},
$$

where K_2 does not depend on Δ.

Remarks 2.2 **(1)** *In computation, the boundedness assumption is no restriction since any number generated by the computer is bounded by the capacity of the computer.*

(2) $\| \cdot \|$ *is a norm in $\mathbf{R}^d$.*

(3) *We would prove the statement of the theorem for the scheme (6), where $\alpha_k = 0.0$. For $\alpha_k \in (0,1]$ we prove the statement of the theorem on the same way. For $\alpha_k = 0.0$ the scheme (5) equals (4) if $Y_0^T = Y_0^M$ and $Y_1^T = Y_1^M$.*

To prove Theorem (2.1), we recall the following lemmas:

Lemma 2.3 *For all natural number $N = 1, 2, \ldots$ and for all $k = 0, 1, \ldots, N$ are valid the next inequalities*

$$E(\| Y_k^M \|^2) \leq K_3(1 + E\|X_0\|^2),$$

$$E(\| Y_k^T \|^2) \leq K_3(1 + E\|X_0\|^2).$$

Lemma 2.4 *Under the assumptions of Theorem 2.1 the Milstein approximation Y_n^M converges with strong order 1.0 that is*

$$E \| X_T - Y_N^M \|^2 \leq K_5 \Delta^{2.0}(1 + E \| X_0 \|^2) + K_6 E \| X_0 - Y_0^M \|^2$$

where the constants K_5, K_6 do not depend on Δ.

Proof

Since the first-order partial derivatives of a and b are bounded, there exists a $K_7 < +\infty$ such that for all $x, y \in \mathbf{R}^d$, (see details in Newton [5])

$$\begin{aligned} \| a(t,x) - a(t,y) \| &\leq K_7 \| x - y \|, \\ \| b(t,x) - b(t,y) \| &\leq K_7 \| x - y \|, \\ \| L^1 b(t,x) - L^1 b(t,y) \| &\leq K_7 \| x - y \|, \\ \| a(t,x) \| + \| b(t,x) \| + \| L^1 b(t,x) \| &\leq K_7(1 + \| x \|). \end{aligned}$$

We introduce the Milstein approximation (4) in the form

$$\begin{aligned} Y_{n+1}^{k,M} &= (1-\gamma_k) Y_n^{k,M} + a^k(\tau_n, Y_n^M)\Delta + b^k(\tau_n, Y_n^M)\Delta W_n \\ &+ \frac{1}{2} L^1 b^k(\tau_n, Y_n)((\Delta W_n)^2 - \Delta) + \gamma_k Y_n^{k,M} \\ &= (1-\gamma_k) Y_n^{k,M} + a^k(\tau_n, Y_n^M)\Delta + b^k(\tau_n, Y_n^M)\Delta W_n \\ &+ \frac{1}{2} L^1 b^k(\tau_n, Y_n^M)((\Delta W_n)^2 - \Delta) + \gamma_k \Bigg(Y_{n-1}^{k,M} + a^k(\tau_{n-1}, Y_{n-1}^M)\Delta \\ &+ b^k(\tau_{n-1}, Y_{n-1}^M)\Delta W_{n-1} + \frac{1}{2} L^1 b^k(\tau_{n-1}, Y_{n-1}^M)((\Delta W_{n-1})^2 - \Delta) \Bigg). \end{aligned}$$

Taylor's expansion is used to give the term $b^k(\tau_{n-1}, Y_{n-1}^M)$ around (τ_n, Y_n^M) and

$$\begin{aligned} b^k(\tau_{n-1}, Y_{n-1}^M) &= b^k(\tau_n, Y_n^M) + \frac{\partial}{\partial t} b^k(\tau_n, Y_n^M)(\tau_{n-1} - \tau_n) \\ &+ \sum_{i=1}^{d} \frac{\partial b^k}{\partial x_i}(\tau_n, Y_n^M)(Y_{n-1}^{i,M} - Y_n^{i,M}) \end{aligned}$$

$$
\begin{aligned}
&+ \frac{1}{2}\frac{\partial^2 b^k}{\partial t^2}(\tau_n^*, Y_n^{*,M})(\tau_{n-1} - \tau_n)^2 \\
&+ \sum_{i=1}^{d} \frac{\partial^2 b^k}{\partial t \partial x_i}(\tau_n^*, Y_n^{*,M})(\tau_{n-1} - \tau_n)(Y_{n-1}^{i,M} - Y_n^{i,M}) \\
&+ \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial^2 b^k}{\partial x_i \partial x_j}(\tau_n^*, Y_n^{*,M})(Y_{n-1}^{i,M} - Y_n^{i,M})(Y_{n-1}^{j,M} - Y_n^{j,M}),
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial b^k}{\partial x_i}(\tau_n, Y_n^M) &= \frac{\partial b^k}{\partial x_i}(\tau_{n-1}, Y_{n-1}^M) + \frac{\partial^2 b^k}{\partial t \partial x_i}(\tau_{n-1}^{*,*}, Y_{n-1}^{*,*,M})(\tau_n - \tau_{n-1}) \\
&+ \sum_{j=1}^{d} \frac{\partial^2 b^k}{\partial x_j \partial x_i}(\tau_{n-1}^{*,*}, Y_{n-1}^{*,*,M})(Y_{n-1}^{j,M} - Y_n^{j,M}).
\end{aligned}
$$

Also; used the fact that

$$
\begin{aligned}
Y_{n-1}^{j,M} - Y_n^{j,M} &= -a^j(\tau_{n-1}, Y_{n-1}^M)\Delta - b^j(\tau_{n-1}, Y_{n-1}^M)\Delta W_{n-1} \\
&- \frac{1}{2}L^1 b^j(\tau_{n-1}, Y_{n-1}^M)((\Delta W_{n-1})^2 - \Delta).
\end{aligned}
$$

When these are substituted into the expression $Y_{n+1}^{k,M}$ and assumptions of the theorem are used we get

$$
\begin{aligned}
Y_{n+1}^k &- Y_{n+1}^{k,M} = (1-\gamma_k)(Y_n^k - Y_n^{k,M}) + (a^k(\tau_n, Y_n) - a^k(\tau_n, Y_n^M))\Delta \\
&+ (b^k(\tau_n, Y_n) - b^k(\tau_n, Y_n^M))\Delta W_n \\
&+ \frac{1}{2}\left(L^1 b^k(\tau_n, Y_n) - L^1 b^k(\tau_n, Y_n^M)\right)((\Delta W_n)^2 - \Delta) \\
&+ \gamma_k\Bigg(Y_{n-1}^k - Y_{n-1}^{k,M} + (a^k(\tau_{n-1}, Y_{n-1}) - a^k(\tau_{n-1}, Y_{n-1}^M))\Delta \\
&+ \frac{1}{2}\left[b^k(\tau_n, Y_n) - b^k(\tau_n, Y_n^M) + b^k(\tau_{n-1}, Y_{n-1}) - b^k(\tau_{n-1}, Y_{n-1}^M)\right]\Delta W_{n-1} \\
&- \frac{1}{2}\left[L^1 b^k(\tau_{n-1}, Y_{n-1}) - L^1 b^k(\tau_{n-1}, Y_{n-1}^M)\right]\Delta\Bigg) \\
&+ f_1(\tau_{n-1}, \tau_n, Y_{n-1}^M, Y_n^M)(\Delta \cdot \Delta W_{n-1}) \\
&+ f_2(\tau_{n-1}, \tau_n, Y_{n-1}^M, Y_n^M)((\Delta^2 \cdot \Delta W_{n-1}) \\
&+ f_3(\tau_{n-1}, \tau_n, Y_{n-1}^M, Y_n^M)(\Delta \cdot (\Delta W_{n-1})^2) \\
&+ f_4(\tau_{n-1}, \tau_n, Y_{n-1}^M, Y_n^M)(\Delta W_{n-1})^3 \\
&+ f_5(\tau_{n-1}, \tau_n, Y_{n-1}^M, Y_n^M)(\Delta \cdot (\Delta W_{n-1)}^3) \\
&+ f_6(\tau_{n-1}, \tau_n, Y_{n-1}^M, Y_n^M)(\Delta W_{n-1})^5,
\end{aligned}
$$

where $\| f_i(\tau_{n-1}, \tau_n, Y_{n-1}^M, Y_n^M) \|^2 \leq C_i(1 + \| Y_{n-1}^M \|^2), i = 1, 2, 3, 4, 5, 6.$

Squaring both sides of the equation, taking expectation and from Lemma (2.3) we get

$$\begin{aligned} E(\| Y_{n+1}^k - Y_{n+1}^{k,M} \|^2) &\leq E(\| Y_n^k - Y_n^{k,M} \|^2)(K_8 + K_9\Delta + K_{10}\Delta^2) \\ &+ E(\| Y_{n-1}^k - Y_{n-1}^{k,M} \|^2)(K_{11} + K_{12}\Delta + K_{13}\Delta^2) + K_{14}\Delta^3, \end{aligned}$$

where $K_8, K_9, K_{10}, K_{11}, K_{12}, K_{13}$ and K_{14} do not depend on Δ.

Using for the starting routine Milstein approximation i.e. $Y_0^k = Y_0^{k,M}$ and $Y_1^k = Y_1^{k,M}$ we get that for all $n = 0, 1, \ldots, N$

$$E(\| Y_n^k - Y_n^{k,M} \|^2) \leq K_{15}\Delta^2,$$

where K_{15} does not depend on Δ.

From Lemma 2.4

$$E(\| X_{\tau_n} - Y_n^M \|^2) \leq K_{16}(1 + E \| X_0 \|^2)\Delta^2,$$

where K_{16} does not depend on Δ (see in [3]), we apply these results to prove finally the strong order $\gamma = 1.0$ of the multistep method, as is claimed in Theorem 1.

3 Some Experiments

Let us consider the Milstein approximation (4), two-step order 1.0 strong scheme (5) and the approximation set out above (6). The three approximations set out above were each tested on the following examples.

Example 3.1

$$\begin{aligned} dX_t &= 1.5X_t dt + X_t dW_t \\ X_0 &= 1.0, \end{aligned} \tag{7}$$

where (W_t) is a Wiener process.
The solution of (7) is $X_t = X_0 \exp(t + W_t)$

Example 3.2

$$\begin{aligned} dX_t &= \left(\frac{\alpha X_t}{1+t} + X_0(1+t)^\alpha \right) dt + X_0(1+t)^\alpha dW_t \\ X_0 &= 1.0 \text{ and } \alpha = 2.0 \end{aligned} \tag{8}$$

where (W_t) is a Wiener process.
The solution of (8) is $X_t = (1+t)^2\,(W_t + t + 1.0)$

In each case the mean-square error $E\| X_1 - Y_1 \|^2$ at the final time $(T = 1)$ is estimated in the following way. A set of 20 blocks, each consisting of 100 outcomes $(1 \leq i \leq 20,\ 1 \leq j \leq 100)$, were simulated and for each block the estimator

$$\varepsilon_i = \frac{1}{100} \sum_{j=1}^{100} \| X_1(\omega_{i,j}) - Y_N(\omega_{i,j}) \|^2$$

was found. Next the means and variances of these estimators were themselves estimated in the usual way:

$$\varepsilon = \frac{1}{20} \sum_{i=1}^{20} \varepsilon_i$$

and

$$\sigma^2 = \frac{1}{19} \sum_{i=1}^{20} (\varepsilon - \varepsilon_i)^2 .$$

According to the central limit theorem, the ε_i should be nearly Gaussian and so approximate 90 percent confidence limits for $E\| X_1 - Y_N \|^2$ can be found from the Gaussian distribution; these were calculated according to the formula $\varepsilon \pm 1.73\sqrt{\frac{\sigma^2}{20}}$.

The results of the simulations for Examples 3.1 and 3.2 are shown in Table 1 and 2. These results are gotten for $\alpha = 0,\ \gamma = 1.0$ in Example 3.1 and for $\alpha = 0,\ \gamma = 1.0$ and $\alpha = 0.5,\ \gamma = 1.0$ in Example 3.2. There is no sense to take γ near zero, because then the new term can be neglected, so the new scheme behaves as Milstein 1.0. The meaning of the headers in the tables is:

Δ - time step size of the strong approximation;

ε - absolute errors for different time step sizes;

L - half of the confidence interval lengths.

For example, we can see from the tables that in Example 3.2 for $\Delta = 2^{-8}$ and $\alpha = 0.0$ and $\gamma = 1.0$ the absolute error by Milstein method (4) is $3.42858 \cdot 10^{-2}$, by Two-step Milstein method (5) is $9.45832 \cdot 10^{-3}$, while by the new scheme (6) is $6.81161 \cdot 10^{-3}$. Also, the length of the confidence interval by the new scheme is smaller than by Milstein 1.0 and Two-step Milstein methods. This statement is also true for the Example 3.1.

Table 1: Example 3.1
Milstein method (4).

Δ	ε	L
1.00000E+00	2.27665E+00	1.47186E-01
5.00000E-01	1.97078E+00	2.40568E-01
2.50000E-01	1.20429E+00	8.45154E-02
1.25000E-01	7.37239E-01	5.64921E-02
6.25000E-02	3.82413E-01	3.99189E-02
3.12500E-02	2.39074E-01	6.31194E-02
1.56250E-02	1.10807E-01	1.27486E-02
7.81250E-03	5.60566E-02	8.09157E-03
3.90625E-03	2.53057E-02	3.36756E-03

Multistep method (6) for $\alpha = 0$ and $\gamma = 1.0$.

Δ	ε	L
1.00000E+00	2.51146E+00	1.98164E-01
5.00000E-01	1.41485E+00	9.57135E-02
2.50000E-01	6.39612E-01	5.46793E-02
1.25000E-01	3.21211E-01	2.94124E-02
6.25000E-02	1.50961E-01	8.22891E-03
3.12500E-02	7.51688E-02	5.73330E-03
1.56250E-02	3.92063E-02	2.09849E-03
7.81250E-03	2.00488E-02	1.25050E-03
3.90625E-03	9.94833E-03	6.94911E-04

Two-step Milstein (5) for $\alpha = 0$ and $\gamma = 1.0$.

Δ	ε	L
1.00000E+00	2.37813E+00	1.87704E-01
5.00000E-01	1.45746E+00	1.12863E-01
2.50000E-01	8.02364E-01	9.38468E-02
1.25000E-01	4.91936E-01	6.26155E-02
6.25000E-02	2.36993E-01	2.86351E-02
3.12500E-02	1.22735E-01	6.91430E-03
1.56250E-02	6.22639E-02	5.80727E-03
7.81250E-03	3.31988E-02	2.88916E-03
3.90625E-03	1.65349E-02	1.28400E-03

Table 2: Example 3.2

Milstein method (4).

Δ	ε	L
1.00000E+00	4.21558E+00	9.211294E-02
5.00000E-01	2.90298E+00	7.181054E-02
2.50000E-01	1.77082E+00	4.158990E-02
1.25000E-01	9.78134E-01	2.936154E-02
6.25000E-02	5.27383E-01	1.338104E-02
3.12500E-02	2.75086E-01	7.950747E-03
1.56250E-02	1.36424E-01	3.334465E-03
7.81250E-03	6.97031E-02	1.644745E-03
3.90625E-03	3.42858E-02	7.971471E-04

Multistep method (6) for $\alpha = 0$ and $\gamma = 1.0$.

Δ	ε	L
1.00000E+00	4.27766E+00	1.03425E-01
5.00000E-01	1.70013E+00	3.85968E-02
2.50000E-01	6.21525E-01	1.72348E-02
1.25000E-01	2.60004E-01	8.05579E-03
6.25000E-02	1.16169E-01	3.57810E-03
3.12500E-02	5.50517E-02	1.51257E-03
1.56250E-02	2.71983E-02	9.70674E-04
7.81250E-03	1.33966E-02	4.02296E-04
3.90625E-03	6.81160E-03	2.22766E-04

Multistep method (6) for $\alpha = 0.5$ and $\gamma = 1.0$.

Δ	ε	L
1.00000E+00	4.17855E+00	1.05099E-01
5.00000E-01	2.22505E+00	4.56814E-02
2.50000E-01	1.15922E+00	3.09267E-02
1.25000E-01	5.91574E-01	1.29769E-02
6.25000E-02	2.90397E-01	5.80337E-03
3.12500E-02	1.43653E-01	3.21847E-03
1.56250E-02	7.27217E-02	2.00281E-03
7.81250E-03	3.50626E-02	8.46181E-04
3.90625E-03	1.77133E-02	3.59233E-04

Two-step Milstein (5) for $\alpha = 0$ and $\gamma = 1.0$.

Δ	ε	L
1.00000E+00	4.24832E+00	9.85367E-02
5.00000E-01	1.77406E+00	5.03204E-02
2.50000E-01	7.62093E-01	1.71932E-02
1.25000E-01	3.37591E-01	1.07679E-02
6.25000E-02	1.60081E-01	5.27565E-03
3.12500E-02	7.77709E-02	2.43576E-03
1.56250E-02	3.73556E-02	1.02419E-03
7.81250E-03	1.96293E-02	5.93383E-04
3.90625E-03	9.45832E-03	2.41391E-04

Two-step Milstein (5) for $\alpha = 0.5$ and $\gamma = 1.0$.

Δ	ε	L
1.00000E+00	4.23623E+00	8.01164E-02
5.00000E-01	2.28984E+00	4.36308E-02
2.50000E-01	1.21665E+00	3.63160E-02
1.25000E-01	6.34940E-01	1.90075E-02
6.25000E-02	3.13706E-01	8.61972E-03
3.12500E-02	1.60810E-01	4.54545E-03
1.56250E-02	8.02790E-02	2.21567E-03
7.81250E-03	4.09332E-02	1.03573E-03
3.90625E-03	2.04743E-02	4.97450E-04

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