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Isomorphic representation of nondeterministic nilpotent automata

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Abstract

In this paper, we deal with nondeterministic nilpotent automata and give a characterization of their isomorphic embedding with respect to the direct product.

1 Introduction

Investigation on homomorphic or isomorphic embeddings into products of automata is the starting point in the study of homomophic or isomorphic completeness of certain classes of automata with respect to different kinds of products. The problem of decomposition has a general approach in [4], where an abstract notion of composition and a general decomposition theorem are presented. Regarding the subdirect representation, there are several works dealing with this topic, see [3], [5], [2], [9], [8].

In this paper, we generalize the notion of nilpotency for nondeterministic automata and study the isomorphic representation of nondeterministic nilpotent automata under the direct product. As it turns out, this case is much more complicated than the deterministic one presented in [3].

2 Preliminaries

By a *nondeterministic automaton* we mean a system $\mathbf{A} = (X, A, \delta)$, where X and A are nonempty finite sets, X is the set of *input signs*, A is the set of *states* and $\delta : A \times X \to \mathcal{P}(A)$ is the *transition function.* $\mathcal{P}(A)$ denotes here the power-set of A. For an input sign $x \in X$ and a state $a \in A$, $\delta(a, x)$ can be visualised as the set of all states in which the automaton goes when the current state is a and the input sign is x. For $\delta(a, x)$, the notation $ax^{\mathbf{A}}$ is frequently used. Let us suppose that $a \in A$ and $p \in X^+$. The transition function can be extended as follows:

$$ap^{\mathbf{A}} = \begin{cases} \bigcup\limits_{b \in aq^{\mathbf{A}}} bx^{\mathbf{A}} & \text{if } p = qx \text{ where } x \in X \text{ and } q \in X^+, \\ ax^{\mathbf{A}} & \text{if } p = x \text{ and } x \in X. \end{cases}$$

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If $M \subseteq A, x \in X$, then we denote by $Mx^{\mathbf{A}}$ the set $\bigcup_{a \in M} ax^{\mathbf{A}}$ and if $p \in X^+$, then $Mp^{\mathbf{A}} = \bigcup_{a \in M} ap^{\mathbf{A}}$.

Let now $\mathbf{A} = (X, A, \delta)$ and $\mathbf{B} = (X, B, \delta_B)$ be two nondeterministic automata. It is said that $\mathbf{B}$ is a *subautomaton* of $\mathbf{A}$ if $B \subseteq A$ and δ_B is the restriction of δ to $B \times X$, i.e. $ax^{\mathbf{B}} = ax^{\mathbf{A}} \cap B$ is valid, for all $a \in A$ and $x \in X$. For two nondeterministic automata $\mathbf{A} = (X, A, \delta)$ and $\mathbf{B} = (X, B, \delta_B)$, a mapping $\mu : A \to B$ is a *homomorphism* if $\mu(ax^{\mathbf{A}}) = \mu(a)x^{\mathbf{B}}$ holds, for all $a \in A$ and $x \in X$. If the homomorphism $\mu : A \to B$ is an onto mapping, then we say that $\mathbf{B}$ is the homomorphic image of $\mathbf{A}$; moreover μ is called an *isomorphism*, if μ is a one-to-one mapping of A onto B. In this case, we say that $\mathbf{A}$ and $\mathbf{B}$ are isomorphic.

Let k be an arbitrary positive integer and $\mathbf{A}_r = (X, A_r, \delta_r), r = 1, \dots, k$, be nondeterministic automata. By the *direct product* of these nondeterministic automata we mean the automaton $\mathbf{A} = (X, A, \delta)$ where $A = A_1 \times \cdots \times A_k$ and δ is defined as follows. For every $a = (a_1, \dots, a_k) \in A$ and $x \in X$, $\delta(a, x) = \delta_1(a_1, x) \times \cdots \times \delta_k(a_k, x)$ or using the other notation: $ax^{\mathbf{A}} = (a_1, \dots, a_k)x^{\mathbf{A}} = a_1 x^{\mathbf{A}_1} \times \cdots \times a_k x^{\mathbf{A}_k}$. For the direct product of $\mathbf{A}_1, \dots, \mathbf{A}_k$, we will use the notation $\mathbf{A}_1 \times \cdots \times \mathbf{A}_k$. It is said that a nondeterministic automaton $\mathbf{A}$ *can be embedded isomorphically* into the direct product of $\mathbf{A}_1, \dots, \mathbf{A}_k$ if $\mathbf{A}$ is isomorphic to a subautomaton $\mathbf{B}$ of $\mathbf{A}_1 \times \cdots \times \mathbf{A}_k$.

Let $\mathbf{A} = (X, A, \delta)$ be a nondeterministic automaton and θ an equivalence relation on A. For every $a \in A$, let $\theta(a)$ denote the class of the partition belonging to θ and containing a. We can define a *factor-automaton* based on θ as follows. Let $\mathbf{A}/\theta = (X, A/\theta, \delta')$ be the nondeterministic automaton where the transition is defined by $\theta(a)x^{\mathbf{A}/\theta} = \{\theta(a') | a' \in bx^{\mathbf{A}} \text{ and } b \in \theta(a)\}$, for all $a \in A$ and $x \in X$. It is important to remark that the mapping $a \to \theta(a), a \in A$ is not a homomorphism of $\mathbf{A}$ onto $\mathbf{A}/\theta$ in general.

Let us introduce some special equivalence relations on A. If $a, b \in A$, then let $\theta_{a,b}$ be the equivalence relation defined in the following way: $u \ \theta_{a,b} \ v$ if and only if $u = v$ or $u, v \in \{a, b\}$, for all $u, v \in A$. It is obvious that the factor-set $A/\theta_{a,b} = \{\{u\} | u \in A \setminus \{a, b\}\} \cup \{\{a, b\}\}$. For the sake of simplicity we will denote by u the classes of the form $\{u\}$ and by capital letters like U or V the classes of the form $\{a, b\}$ of the factor-set. We will use in the proofs of this paper the following important observation. If $\mathbf{A}$ has two states $c \neq d$ satisfying $cx^{\mathbf{A}} = dx^{\mathbf{A}}$, for all $x \in X$, then $\mathbf{A}/\theta_{c,d}$ is a homomorphic image of $\mathbf{A}$ under the homomorphism $a \to \theta_{c,d}(a), a \in A$.

Now, let $\theta_1, \dots, \theta_k$ be arbitrary equivalence relations on A. Then, the mapping $\mu : A \to A/\theta_1 \times \cdots \times A/\theta_k$ given by $\mu(a) = (\theta_1(a), \dots, \theta_k(a)), a \in A$, is called the *natural mapping*. It is easy to see that the natural mapping is a one-to-one mapping of A into $A/\theta_1 \times \cdots \times A/\theta_k$ if $\bigcap_{r=1}^{k} \theta_r = \Delta_A$, where Δ_A is the equality relation on A. If we consider the factor-automata $\mathbf{A}_1, \dots, \mathbf{A}_k$ based on arbitrary equivalence relations $\theta_1, \dots, \theta_k$, respectively, then one can see that the natural mapping is not a homomorphism of $\mathbf{A}$ into $\mathbf{A}_1 \times \cdots \times \mathbf{A}_k$, in general. In the constructive proofs given in this paper, we will have only natural mappings that are one-to-one mappings and are homomorphisms. By reducing the codomain of these mappings to $B = \mu(A)$ we obtain isomorphisms. Of course we will prove these features of the natural mapping in each case separately.

3 Nondeterministic nilpotent automata

Throughout the paper we restrict ourselves to the complete nondeterministic automata (see e.g. [1], [7]). Here, we recall its definition. A nondeterministic automaton $\mathbf{A} = (X, A, \delta)$ is called *complete* if $\delta(a, x) \neq \emptyset$ is valid, for all $a \in A$ and $x \in X$. In this paper, by nondeterministic automaton we mean a complete nondeterministic automaton. Moreover, we deal with a special class, the class of nondeterministic nilpotent automata. As a generalization of the traditional nilpotent automaton, it can be defined as follows.

A nondeterministic automaton $\mathbf{A} = (X, A, \delta)$ is called *nilpotent* if there exist a positive integer n and a state a_0, such that $Ap^{\mathbf{A}} = \{a_0\}$ is valid, for all $p \in X^+$ with $|p| \geq n$ ($|p|$ denotes the length of the word p). The distinguished state a_0 is called the *absorbent state* of $\mathbf{A}$.

Now, let us define the following relation on A: $a \leq b$ if and only if $a = b$ or there is a $p \in X^+$ such that $b \in ap^{\mathbf{A}}$. It is easy to see that the introduced relation is a partial ordering on A since $\mathbf{A}$ is nilpotent. If one of the relations $a \leq b$ or $b \leq a$ is valid, then a and b are called *comparable*. In the opposite case, we say that they are *incomparable* and it is denoted by $a \bowtie b$. Furthermore, the absorbent state a_0 is the greatest element in $(A, \leq)$ and if A has at least two elements, then there exists a $b_0 \neq a_0 \in A$ such that b_0 is a maximal element in $(A \setminus \{a_0\}, \leq)$. From the maximality of a_0 and b_0, it follows that $a_0 x^{\mathbf{A}} = b_0 x^{\mathbf{A}}$ is valid, for all $x \in X$. Note that if there exists only one pair of states $a \neq b$ which satisfies $ax^{\mathbf{A}} = bx^{\mathbf{A}}$ for all $x \in X$, then these two states must be a_0 and b_0. Throughout the paper we will express this in a shorter way, using the sentence "$\mathbf{A}$ has exactly one pair of different states a_0 and b_0 for which $a_0 x^{\mathbf{A}} = b_0 x^{\mathbf{A}}$ holds, for all $x \in X$". In this case, b_0 is the greatest element in $(A \setminus \{a_0\}, \leq)$, because it is the only maximal element in this set.

The following statement is a consequence of Lemma 2 in [6] and it follows from the observation taken at the end of the previous section.

Lemma 1 *Let c and d be two different states of a nondeterministic nilpotent automaton $\mathbf{A} = (X, A, \delta)$. If $cx^{\mathbf{A}} = dx^{\mathbf{A}}$ is valid, for all $x \in X$, then the factor-automaton $\mathbf{A}/\theta_{c,d}$ is also a nondeterministic nilpotent automaton.*

In particular, if $\{c, d\} = \{a_0, b_0\}$, then it is easy to see that the corresponding factor-automaton is a nondeterministic nilpotent automaton with the absorbent state $\theta_{a_0,b_0}(a_0) = \{a_0, b_0\}$.

Using the notion of incomparability, an other similar factor-automaton can be defined as follows. Let $\mathbf{A} = (X, A, \delta)$ be a nondeterministic nilpotent automaton with $|A| \geq 3$ and let $c \bowtie d \in A$ be incomparable states of A. Then, one can prove that $\mathbf{A}/\theta_{c,d}$ is a homomorphic image of $\mathbf{A}$ under the homomorphism $a \to \theta_{c,d}(a), a \in A$. This observation leads to the following statement:

Lemma 2 *If a nondeterministic nilpotent automaton* $\mathbf{A} = (X, A, \delta)$ *with* $|A| \geq 3$ *has two incomparable states c and d, then the factor-automaton* $\mathbf{A}/\theta_{c,d}$ *is also a nondeterministic nilpotent automaton.*

The following property is very important with respect to the inner structure of the nondeterministic nilpotent automata. It shows that no "loops" or "circuits" may appear on the states of a nondeterministic nilpotent automaton except for the absorbent state.

Lemma 3 *If* $\mathbf{A} = (X, A, \delta)$ *is a nondeterministic nilpotent automaton with the absorbent state* a_0*, then* $a \notin ap^{\mathbf{A}}$ *holds, for all* $a \in A \setminus \{a_0\}$ *and* $p \in X^+$.

We will also refer to the next Lemma, whose proof needs the following observation which is a direct consequece of the definitions. If $\mathbf{A}_1, \ldots, \mathbf{A}_k$ are nondeterministic nilpotent automata with the absorbent states $a_1^0, \ldots, a_k^0$, respectively, then their direct product is a nondeterministic nilpotent automaton with the absorbent state $(a_1^0, \ldots, a_k^0)$. Furthermore, every complete subautomaton of the direct product is nilpotent with the absorbent state $(a_1^0, \ldots, a_k^0)$. This also means that if a nondeterministic nilpotent automaton $\mathbf{A} = (X, A, \delta)$ with the absorbent state a_0 can be embedded into $\mathbf{A}_1 \times \cdots \times \mathbf{A}_k$ under the isomorphism μ, then $\mu(a_0) = (a_1^0, \ldots, a_k^0)$.

Lemma 4 *Assume that* $\mathbf{A} = (X, A, \delta)$ *is a nondeterministic nilpotent automaton with the absorbent state* a_0 *and* $\mathbf{A}$ *has exactly one pair of different states* a_0, b_0 *for which* $a_0x^{\mathbf{A}} = b_0x^{\mathbf{A}}$ *holds, for all* $x \in X$. *Let* $\mathbf{A}_r = (X, A_r, \delta_r), r = 1, \ldots, k$, $(k \geq 2)$ *be nondeterministic nilpotent automata with the absorbent states* $a_1^0, \ldots, a_k^0$, *respectively, and let* $\mu : A \to B \subseteq A_1 \times \cdots \times A_k$ *be an isomorphism of* $\mathbf{A}$ *into* $\mathbf{A}_1 \times \cdots \times \mathbf{A}_k$. *If we denote the image of* b_0 *under* μ *by* $(b_1^0, \ldots, b_k^0)$ *and the set of indices* $\{i \in \{1, \ldots, k\} | a_i^0 \neq b_i^0\}$ *by* I, *then the components* a_i^0, b_i^0 *with* $i \in I$ *may appear in no other elements of* $\mathbf{B}$ *but* $\mu(a_0)$ *and* $\mu(b_0)$.

4 Isomorphic representation of nondeterministic nilpotent automata

Theorem 1 *Let* $\mathbf{A} = (X, A, \delta)$ *be a nondeterministic nilpotent automaton with* $|A| \geq 3$*, the absorbent state* a_0*, and let* b_0 *be a maximal element in* $(A \setminus \{a_0\}, \leq)$. *If there exist* $a_1, b_1 \in A$, $a_1 \neq b_1$, $\{a_1, b_1\} \neq \{a_0, b_0\}$ *such that beside* $a_0x^{\mathbf{A}} = b_0x^{\mathbf{A}}$, $a_1x^{\mathbf{A}} = b_1x^{\mathbf{A}}$ *also holds for all* $x \in X$, *then* $\mathbf{A}$ *can be embedded isomorphically into a direct product of nondeterministic nilpotent automata having fewer states than* $|A|$.

Proof. Case 1. $\{a_0, b_0\} \cap \{a_1, b_1\} = \emptyset$. Let $\mathbf{A}_1 = \mathbf{A}/\theta_{a_0,b_0}$ and $\mathbf{A}_2 = \mathbf{A}/\theta_{a_1,b_1}$. Then, $\mathbf{A}$ can be embedded isomorphically into $\mathbf{A}_1 \times \mathbf{A}_2$ under the natural mapping denoted by μ. Let $B = \mu(A)$. Since $\theta_{a_0,b_0} \cap \theta_{a_1,b_1} = \Delta_A$, μ is a one-to-one mapping. To simplify the notation we will use U_0 for $\theta_{a_0,b_0}(a_0) = \{a_0, b_0\}$ and U_1 for $\theta_{a_1,b_1}(a_1) = \{a_1, b_1\}$. To prove that μ is an isomorphism, we have to show that $\mu(ax^{\mathbf{A}}) = \mu(a)x^{\mathbf{A}_1 \times \mathbf{A}_2} \cap B$, for all $a \in A$ and $x \in X$. We can do this by evaluating $\mu(ax^{\mathbf{A}})$ and $\mu(a)x^{\mathbf{A}_1 \times \mathbf{A}_2} \cap B$ in the following cases: $a \in A \setminus (U_0 \cup U_1)$ and $ax^{\mathbf{A}} \cap (U_0 \cup U_1) = \emptyset$; $a \in A \setminus (U_0 \cup U_1)$ and $ax^{\mathbf{A}} \cap U_0 = \emptyset$ and $ax^{\mathbf{A}} \cap U_1 \neq \emptyset$; $a \in A \setminus (U_0 \cup U_1)$ and $ax^{\mathbf{A}} \cap U_0 \neq \emptyset$ and $ax^{\mathbf{A}} \cap U_1 = \emptyset$; $a \in A \setminus (U_0 \cup U_1)$ and $ax^{\mathbf{A}} \cap U_0 \neq \emptyset$ and $ax^{\mathbf{A}} \cap U_1 \neq \emptyset$; $a \in U_1$ and $ax^{\mathbf{A}} \cap U_0 = \emptyset$ and $ax^{\mathbf{A}} \cap U_1 = \emptyset$; $a \in U_1$ and $ax^{\mathbf{A}} \cap U_0 = \emptyset$ and $ax^{\mathbf{A}} \cap U_1 \neq \emptyset$; $a \in U_1$ and $ax^{\mathbf{A}} \cap U_0 \neq \emptyset$ and $ax^{\mathbf{A}} \cap U_1 = \emptyset$; $a \in U_1$ and $ax^{\mathbf{A}} \cap U_0 \neq \emptyset$ and $ax^{\mathbf{A}} \cap U_1 \neq \emptyset$; $a \in U_0$.

Case 2. $\{a_0, b_0\} \cap \{a_1, b_1\} \neq \emptyset$. $\{a_0, b_0\} \neq \{a_1, b_1\}$, hence $\{a_0, b_0\} \cap \{a_1, b_1\}$ contains exactly one element. Let $\{b_1\} = \{a_0, b_0\} \cap \{a_1, b_1\}$. There are two possibilities: $b_1 = a_0$ or $b_1 = b_0$. In both cases, $a_1 x^{\mathbf{A}} = b_0 x^{\mathbf{A}} = a_0 x^{\mathbf{A}} = \{a_0\}$, for all $x \in X$. Consequently, the factor-automata $\mathbf{A}_1 = \mathbf{A}/\theta_{a_0,b_0}$, $\mathbf{A}_2 = \mathbf{A}/\theta_{a_0,a_1}$ and $\mathbf{A}_3 = \mathbf{A}/\theta_{b_0,a_1}$ are nondeterministic nilpotent automata. We show that $\mathbf{A}$ can be embedded isomorphically into $\mathbf{A}_1 \times \mathbf{A}_2 \times \mathbf{A}_3$ under the natural mapping denoted by μ. Let $B = \mu(A) \subseteq A_1 \times A_2 \times A_3$. The mapping μ is one-to-one since $\theta_{a_0,b_0} \cap \theta_{a_0,a_1} \cap \theta_{b_0,a_1} = \Delta_A$. It also satisfies $\mu(ax^{\mathbf{A}}) = \mu(a)x^{\mathbf{A}_1 \times \mathbf{A}_2 \times \mathbf{A}_3} \cap B$, for all $a \in A$ and $x \in X$. To prove this, we must evaluate once again $\mu(ax^{\mathbf{A}})$ and $\mu(a)x^{\mathbf{A}_1 \times \mathbf{A}_2 \times \mathbf{A}_3} \cap B$ in the following cases: $a \in A \setminus \{a_0, b_0, a_1\}$ and $ax^{\mathbf{A}} \cap \{a_0, b_0, a_1\} = \{a_0\}$ (or $\{b_0\}$, or $\{a_1\}$); $a \in A \setminus \{a_0, b_0, a_1\}$ and $ax^{\mathbf{A}} \cap \{a_0, b_0, a_1\} = \{a_0, b_0\}$ (or $\{a_0, a_1\}$, or $\{b_0, a_1\}$); $a \in A \setminus \{a_0, b_0, a_1\}$ and $ax^{\mathbf{A}} \cap \{a_0, b_0, a_1\} = \{a_0, b_0, a_1\}$; $a \in \{a_0, b_0, a_1\}$. □

Corollary 1 *If a nondeterministic nilpotent automaton* $\mathbf{A} = (X, A, \delta)$ *cannot be embedded into a direct product of nondeterministic nilpotent automata having less states than* $|A|$, *then* $\mathbf{A}$ *has exactly one pair of different states* a_0, b_0 *for which* $a_0 x^{\mathbf{A}} = b_0 x^{\mathbf{A}}$ *holds, for all* $x \in X$. *These states are* a_0, *the absorbent state and* b_0, *the greatest element in* $(A \setminus \{a_0\}, \leq)$.

It is worth noting that, unlike for deterministic automata (see [3]), the converse statement is not true.

Lemma 5 *Let* $\mathbf{A} = (X, A, \delta)$ *be a nondeterministic nilpotent automaton* ($|A| \geq 3$) *that has exactly one pair of different states* a_0, b_0 *for which* $a_0 x^{\mathbf{A}} = b_0 x^{\mathbf{A}}$ *holds, for all* $x \in X$. *If* $\mathbf{A}$ *can be embedded isomorphically into a direct product of nondeterministic nilpotent automata having fewer states than* $|A|$, *then there must exist a pair of incomparable states* $c \bowtie d \in A$.

Proof. Let $\mu : A \to B \subseteq A_1 \times \cdots \times A_k$ be an embedding isomorphism, where $k \geq 2$ is fixed and $|A_r| < |A|$, for all $r = 1, \ldots, k$. Since $|A_r| < |A|$, there exists a state $\bar{a}_r \in A_r$, such that $\bar{a}_r$ occurs in at least two different elements of B on the r-th position, for all $r = 1, \ldots, k$. If we use the notation $(a_1^0, \ldots, a_k^0)$ for $\mu(a_0)$ and $(b_1^0, \ldots, b_k^0)$ for $\mu(b_0)$, then, because $a_0 \neq b_0$ and μ is an isomorphism, there has to be an index $i \in \{1, \ldots, k\}$ such that $a_i^0 \neq b_i^0$. Let $\mathbf{b}'$ and $\mathbf{b}''$ be two different elements of B in which $\bar{a}_i$ occurs on the i-th position, and let $c, d \in A$ be the states for which $\mu(c) = \mathbf{b}'$ and $\mu(d) = \mathbf{b}''$. μ is an isomorphism, hence $c \neq d$. We will show now, that c and d are incomparable in $(A, \leq)$. Assume to the contrary that

$c \leq d$ (or $d \leq c$). Since $c \neq d$, there exists $p \in X^+$, such that $d \in cp^{\mathbf{A}}$. In the meantime, μ is an isomorphism, so we can conclude that $\mu(d) \in \mu(c)p^{\mathbf{B}}$, which implies $\overline{a}_i \in \overline{a}_i p^{\mathbf{A}_i}$. Thus, due to Lemma 3, the only possibility for $\overline{a}_i$ is $\overline{a}_i = a_i^0$. Since b_0 is the greatest element in $(A \setminus \{a_0\}, \leq)$, it is obvious that $d \leq b_0$, i.e. $d = b_0$ or there exists a $q \in X^+$ such that $b_0 \in dq^{\mathbf{A}}$. If $d = b_0$, then $b_i^0 = a_i^0$ follows immediately and contradicts the assumption $b_i^0 \neq a_i^0$. If $b_0 \in dq^{\mathbf{A}}$, then under the isomorphism μ we have $\mu(b^0) \in \mu(d)q^{\mathbf{B}}$, which for the i-th component means $b_i^0 \in a_i^0 q^{\mathbf{A}_i}$. This contradicts the nilpotency of $\mathbf{A}_i$. □

Corollary 2 *If the partially ordered set $(A, \leq)$ of a nondeterministic nilpotent automaton $\mathbf{A} = (X, A, \delta)$ with $|A| \geq 3$ is a chain, then $\mathbf{A}$ cannot be embedded into a direct product of nondeterministic nilpotent automata having less states than $|A|$.*

Lemma 6 *Let $\mathbf{A} = (X, A, \delta)$ be a nondeterministic nilpotent automaton ($|A| \geq 4$) that has exactly one pair of different states a_0, b_0 for which $a_0 x^{\mathbf{A}} = b_0 x^{\mathbf{A}}$ holds, for all $x \in X$. If there exist $c \bowtie d \in A$ such that for all $x \in X$, $cx^{\mathbf{A}} \cap \{a_0, b_0\} \neq \emptyset$ and $dx^{\mathbf{A}} \cap \{a_0, b_0\} \neq \emptyset$ jointly imply $cx^{\mathbf{A}} \cap \{a_0, b_0\} = dx^{\mathbf{A}} \cap \{a_0, b_0\}$, then $\mathbf{A}$ can be embedded isomorphically into a direct product of nondeterministic nilpotent automata having fewer states than $|A|$.*

Proof. Since c and d are incomparable, $\{c, d\} \cap \{a_0, b_0\} = \emptyset$. Let $\mathbf{A}_1 = \mathbf{A}/\theta_{a_0,b_0}$ and $\mathbf{A}_2 = \mathbf{A}/\theta_{c,d}$. Since $c \bowtie d$ and $a_0 x^{\mathbf{A}} = b_0 x^{\mathbf{A}}$, for all $x \in X$, both $\mathbf{A}/\theta_{a_0,b_0}$ and $\mathbf{A}/\theta_{c,d}$ are nondeterministic nilpotent automata. Now, we will prove that $\mathbf{A}$ can be embedded isomorphically into $\mathbf{A}_1 \times \mathbf{A}_2$ under the natural mapping denoted by μ. Since $\theta_{a_0,b_0} \cap \theta_{c,d} = \Delta_A$, μ is a one-to-one mapping of A into $A/\theta_{a_0,b_0} \times A/\theta_{c,d}$. To prove that μ is an isomorphism, let $B = \mu(A)$. For the sake of simplity, let us denote by U $\theta_{a_0,b_0}(a_0) = \{a_0, b_0\}$ and by V $\theta_{c,d}(c) = \{c, d\}$. Then, we have to evaluate $\mu(ax^{\mathbf{A}})$ and $\mu(a)x^{\mathbf{A}_1 \times \mathbf{A}_2} \cap B$ in the following cases: $a \in A \setminus (U \cup V)$ and $ax^{\mathbf{A}} \cap (U \cup V) = \emptyset$; $a \in A \setminus (U \cup V)$ and $ax^{\mathbf{A}} \cap U = \emptyset$ and $ax^{\mathbf{A}} \cap V \neq \emptyset$; $a \in A \setminus (U \cup V)$ and $ax^{\mathbf{A}} \cap U \neq \emptyset$ and $ax^{\mathbf{A}} \cap V = \emptyset$; $a \in A \setminus (U \cup V)$ and $ax^{\mathbf{A}} \cap U \neq \emptyset$ and $ax^{\mathbf{A}} \cap V \neq \emptyset$; $a \in U$; $a = c$ (or d) and $cx^{\mathbf{A}} \cap U = \emptyset$ and $dx^{\mathbf{A}} \cap U = \emptyset$; $a = c$ (or d) and $cx^{\mathbf{A}} \cap U = \emptyset$ and $dx^{\mathbf{A}} \cap U \neq \emptyset$; $a = c$ (or d) and $cx^{\mathbf{A}} \cap U \neq \emptyset$ and $dx^{\mathbf{A}} \cap U = \emptyset$; $a = c$ (or d) and $cx^{\mathbf{A}} \cap U \neq \emptyset$ and $dx^{\mathbf{A}} \cap U \neq \emptyset$. In this latter case, we have to use the assumption of the lemma, namely, if $cx^{\mathbf{A}} \cap U \neq \emptyset$ and $dx^{\mathbf{A}} \cap U \neq \emptyset$, then $cx^{\mathbf{A}} \cap U = dx^{\mathbf{A}} \cap U$. □

Corollary 3 *Let $\mathbf{A} = (X, A, \delta)$ be a nondeterministic nilpotent automaton with $|A| \geq 4$ that has exactly one pair of different states a_0, b_0 for which $a_0 x^{\mathbf{A}} = b_0 x^{\mathbf{A}}$ holds, for all $x \in X$. If $\mathbf{A}$ has exactly one pair of incomparable states $c \bowtie d$ and $\mathbf{A}$ cannot be embedded into a direct product of nondeterministic nilpotent automata having fewer states than $|A|$, then there has to be $\overline{x} \in X$ such that $c\overline{x}^{\mathbf{A}} \cap \{a_0, b_0\} \neq \emptyset$, $d\overline{x}^{\mathbf{A}} \cap \{a_0, b_0\} \neq \emptyset$ and $c\overline{x}^{\mathbf{A}} \cap \{a_0, b_0\} \neq d\overline{x}^{\mathbf{A}} \cap \{a_0, b_0\}$ hold.*

Lemma 7 *Assume that $\mathbf{A} = (X, A, \delta)$ is a nondeterministic nilpotent automaton ($|A| \geq 4$) that has exactly one pair of different states a_0, b_0 for which $a_0 x^{\mathbf{A}} = b_0 x^{\mathbf{A}}$ holds, for all $x \in X$, and that $\mathbf{A}$ can be embedded isomorphically into a direct* *product of nondeterministic nilpotent automata having fewer states than $|A|$. If for $c \bowtie d \in A$, there exists $\overline{x} \in X$ such that $c\overline{x}^{\mathbf{A}} \cap \{a_0, b_0\} \neq \emptyset$, $d\overline{x}^{\mathbf{A}} \cap \{a_0, b_0\} \neq \emptyset$ and $c\overline{x}^{\mathbf{A}} \cap \{a_0, b_0\} \neq d\overline{x}^{\mathbf{A}} \cap \{a_0, b_0\}$ simultaneously hold, then there exists $e \bowtie f \in A$ such that $\{e, f\} \neq \{c, d\}$.*

Proof. We use the same notations as in the proof of Lemma 5. Let $\mu(c) = \mathbf{b}' = (b'_1, \ldots, \overline{a}_i, \ldots, b'_k)$ and $\mu(d) = \mathbf{b}'' = (b'_1, \ldots, \overline{a}_i, \ldots, b''_k)$. We must analyse the cases in which the given conditions $c\overline{x}^{\mathbf{A}} \cap \{a_0, b_0\} \neq \emptyset$, $d\overline{x}^{\mathbf{A}} \cap \{a_0, b_0\} \neq \emptyset$ and $c\overline{x}^{\mathbf{A}} \cap \{a_0, b_0\} \neq d\overline{x}^{\mathbf{A}} \cap \{a_0, b_0\}$ hold. Let us assume that $c\overline{x}^{\mathbf{A}} = \{a_0\}$ and $d\overline{x}^{\mathbf{A}} = \{b_0\}$ (all other cases can be proved similarly to this one). This yields that $\{a_i^0, b_i^0\} \subseteq \overline{a}_i\overline{x}^{\mathbf{A}_i}$ and that there exists an index $j \neq i$, such that $b'_j \neq b''_j$ and $a_j^0 \neq b_j^0$. For this index j, the relations $a_j^0 \in b'_j\overline{x}^{\mathbf{A}_j}$, $b_j^0 \notin b'_j\overline{x}^{\mathbf{A}_j}$, $a_j^0 \notin b''_j\overline{x}^{\mathbf{A}_j}$ and $b_j^0 \in b''_j\overline{x}^{\mathbf{A}_j}$ also hold. Consider now $\overline{a}_j$ and two different elements of B in which $\overline{a}_j$ occurs on the j-th position. Let e and f be those states of A whose images under the isomorphism μ are these two elements. Like in the proof of Lemma 5, one can see that $e \bowtie f$. We still have to prove that $\{e, f\} \neq \{c, d\}$. For $\overline{a}_j$, there are three possibilities: $\overline{a}_j \notin \{b'_j, b''_j\}$, $\overline{a}_j = b'_j$ or $\overline{a}_j = b''_j$. In the first case, we have $\{e, f\} \cap \{c, d\} = \emptyset$, in the second case, $\{e, f\} \cap \{c, d\} = \{c\}$ and in the third case, $\{e, f\} \cap \{c, d\} = \{d\}$. Consequently, $\{e, f\} \neq \{c, d\}$. □

Corollary 4 *Let $\mathbf{A} = (X, A, \delta)$ be a nondeterministic nilpotent automaton with $|A| \geq 4$ that has exactly one pair of different states a_0, b_0 for which $a_0x^{\mathbf{A}} = b_0x^{\mathbf{A}}$ holds, for all $x \in X$. If $\mathbf{A}$ has exactly one pair of incomparable states $c \bowtie d$ and there exists $\overline{x} \in X$ such that $c\overline{x}^{\mathbf{A}} \cap \{a_0, b_0\} \neq \emptyset$, $d\overline{x}^{\mathbf{A}} \cap \{a_0, b_0\} \neq \emptyset$ and $c\overline{x}^{\mathbf{A}} \cap \{a_0, b_0\} \neq d\overline{x}^{\mathbf{A}} \cap \{a_0, b_0\}$ are true, then $\mathbf{A}$ cannot be embedded into a direct product of nondeterministic nilpotent automata having fewer states than $|A|$.*

Lemma 8 *Let $\mathbf{A} = (X, A, \delta)$ be a nondeterministic nilpotent automaton ($|A| \geq 4$) that has exactly one pair of different states a_0, b_0 for which $a_0x^{\mathbf{A}} = b_0x^{\mathbf{A}}$ holds, for all $x \in X$. If there are $c \bowtie d \in A$ and $e \bowtie f \in A$ such that $\{c, d\} \cap \{e, f\} = \emptyset$, then $\mathbf{A}$ can be embedded isomorphically into a direct product of nondeterministic nilpotent automata having fewer states than $|A|$.*

Proof. Let $\mathbf{A}_1 = \mathbf{A}/\theta_{a_0,b_0}$, $\mathbf{A}_2 = \mathbf{A}/\theta_{c,d}$ and $\mathbf{A}_3 = \mathbf{A}/\theta_{e,f}$. Since $a_0x^{\mathbf{A}} = b_0x^{\mathbf{A}}$, for all $x \in X$, and $c \bowtie d$, $e \bowtie f$, the factor-automata $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3$ are nondeterministic nilpotent automata. On the other hand, $\theta_{a_0,b_0} \cap \theta_{c,d} \cap \theta_{e,f} = \Delta_A$, and thus, the natural mapping denoted by μ is a one-to-one mapping. To prove that μ is an isomorphism, as in the constructive proofs given above, we must calculate $\mu(ax^{\mathbf{A}})$ and $\mu(a)x^{\mathbf{A}_1 \times \mathbf{A}_2 \times \mathbf{A}_3} \cap B$ in the corresponding cases, where $B = \mu(A)$. □

Lemma 9 *Let $\mathbf{A} = (X, A, \delta)$ be a nondeterministic nilpotent automaton ($|A| \geq 5$) that has exactly one pair of different states a_0, b_0 for which $a_0x^{\mathbf{A}} = b_0x^{\mathbf{A}}$ holds, for all $x \in X$. If there are three pairwise incomparable states $c \bowtie d \in A$, $d \bowtie e \in A$ and $e \bowtie c \in A$, then $\mathbf{A}$ can be embedded isomorphically into the direct product of nondeterministic nilpotent automata having fewer states than $|A|$.*

Proof. We construct the following three factor-automata: $\mathbf{A}_1 = \mathbf{A}/\theta_{c,d}$, $\mathbf{A}_2 = \mathbf{A}/\theta_{d,e}$ and $\mathbf{A}_3 = \mathbf{A}/\theta_{e,c}$. Since $c \bowtie d$, $d \bowtie e$ and $e \bowtie c$, each of the automata $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3$ is a nondeterministic nilpotent automaton. It can be proved that the natural mapping is an isomorphism in this case. The proof goes similarly to the cases mentioned above. □

Note that the condition that $\mathbf{A}$ has exactly one pair of different states a_0, b_0 for which $a_0x^{\mathbf{A}} = b_0x^{\mathbf{A}}$ holds, for all $x \in X$, was not used in this proof, hence, it is not necessary.

Corollary 5 *Let $\mathbf{A} = (X, A, \delta)$ be a nondeterministic nilpotent automaton with $|A| \geq 5$ that has exactly one pair of different states a_0, b_0 for which $a_0x^{\mathbf{A}} = b_0x^{\mathbf{A}}$ holds, for all $x \in X$. If $\mathbf{A}$ has at least two incomparable states and $\mathbf{A}$ cannot be embedded into a direct product of nondeterministic nilpotent automata having fewer states than $|A|$, then there has to be $c \in A \setminus \{a_0, b_0\}$ such that for every $e \bowtie f \in A$, $c \in \{e, f\}$ must hold.*

Lemma 10 *Let $\mathbf{A} = (X, A, \delta)$ be a nondeterministic nilpotent automaton with $|A| \geq 5$ that has exactly one pair of different states a_0, b_0 for which $a_0x^{\mathbf{A}} = b_0x^{\mathbf{A}}$ holds, for all $x \in X$. If there are a natural number $k \geq 2$ and $c, e_1, \ldots, e_k \in A$ with $c \bowtie e_r, r = 1, \ldots, k$, such that*

$$\forall x \in X \left(\exists i \in \{1, \ldots, k\} : e_ix^{\mathbf{A}} \cap \{a_0, b_0\} \subseteq cx^{\mathbf{A}} \cap \{a_0, b_0\}\right) \text{ holds,}$$

then $\mathbf{A}$ can be embedded isomorphically into a direct product of nondeterministic nilpotent automata having fewer states than $|A|$.

Proof. We construct the following $k+1$ automata: let $\mathbf{A}_0 = \mathbf{A}/\theta_{a_0,b_0}$ and for every $r = 1, \ldots, k$, let $\mathbf{A}_r = \mathbf{A}/\theta_{c,e_r}$.

Since $a_0x^{\mathbf{A}} = b_0x^{\mathbf{A}}$ for all $x \in X$ and $c \bowtie e_r, r = 1, \ldots, r$, each of the automata $\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_k$ is a nondeterministic nilpotent automaton. We prove now that the natural mapping is an isomorphism of $\mathbf{A}$ into $\mathbf{A}_0 \times \mathbf{A}_1 \times \cdots \times \mathbf{A}_k$. Since $\theta_{a_0,b_0} \cap \theta_{c,e_1} \cap \cdots \cap \theta_{c,e_k} = \Delta_A$, μ is a one-to-one mapping. To prove that μ is an isomorphism, we have to investigate more cases. For the sake of simplity, we shall use the notations: $U = \theta_{a_0,b_0}(a_0) = \{a_0, b_0\}$ and for $r = 1, \ldots, k$, $V_r = \theta_{c,e_r}(c) = \{c, e_r\}$. We will give the proof in detail for the following two cases.

Assume that $a = c$ and $cx^{\mathbf{A}} \cap \{a_0, b_0\} \neq \emptyset$. Then,

$$\mu(cx^{\mathbf{A}}) = \mu(cx^{\mathbf{A}} \setminus \{a_0, b_0\}) \cup \mu(cx^{\mathbf{A}} \cap \{a_0, b_0\}) =$$

$$= \{(b, \ldots, b) | b \in cx^{\mathbf{A}} \setminus U\} \cup \{(U, b, \ldots, b) | b \in cx^{\mathbf{A}} \cap U\}.$$

On the other hand,

$$\mu(c)x^{\mathbf{A}_0 \times \mathbf{A}_1 \times \cdots \times \mathbf{A}_k} = (c, V_1, V_2, \ldots, V_k)x^{\mathbf{A}_0 \times \mathbf{A}_1 \times \cdots \times \mathbf{A}_k} =$$

$$= cx^{\mathbf{A}_0} \times V_1x^{\mathbf{A}_1} \times \cdots \times V_kx^{\mathbf{A}_k} =$$

$$= \left((cx^{\mathbf{A}} \setminus U) \cup \{U\}\right) \times (cx^{\mathbf{A}} \cup e_1x^{\mathbf{A}}) \times \cdots \times (cx^{\mathbf{A}} \cup e_kx^{\mathbf{A}}) =$$

$$= \big((cx^{\mathbf{A}} \setminus U) \times (cx^{\mathbf{A}} \cup e_1 x^{\mathbf{A}}) \times \cdots \times (cx^{\mathbf{A}} \cup e_k x^{\mathbf{A}})\big) \cup$$
$$\cup \big(\{U\} \times (cx^{\mathbf{A}} \cup e_1 x^{\mathbf{A}}) \times \cdots \times (cx^{\mathbf{A}} \cup e_k x^{\mathbf{A}})\big),$$

which implies

$$\mu(c)x^{\mathbf{A}_0 \times \mathbf{A}_1 \times \cdots \times \mathbf{A}_k} \cap B =$$
$$= \{(b,\ldots,b) | b \in cx^{\mathbf{A}} \setminus U\} \cup \{(U,b,\ldots,b) | b \in \big(\bigcap_{r=1}^{k} (cx^{\mathbf{A}} \cup e_r x^{\mathbf{A}})\big) \cap U\} =$$
$$= \{(b,\ldots,b) | b \in cx^{\mathbf{A}} \setminus U\} \cup \{(U,b,\ldots,b) | b \in (cx^{\mathbf{A}} \cap U) \cup \big(\bigcap_{r=1}^{k} (e_r x^{\mathbf{A}} \cap U)\big)\}.$$

Since there exists an index i such that $e_i x^{\mathbf{A}} \cap \{a_0, b_0\} \subseteq cx^{\mathbf{A}} \cap \{a_0, b_0\}$, the inclusion $\bigcap_{r=1}^{k} (e_r x^{\mathbf{A}} \cap U) \subseteq cx^{\mathbf{A}} \cap U$ also holds and $\mu(cx^{\mathbf{A}}) = \mu(c)x^{\mathbf{A}_0 \times \mathbf{A}_1 \times \cdots \times \mathbf{A}_k} \cap B$.

Now, assume that $a = e_j$ for some $j \in \{1,\ldots,k\}$ and $e_j x^{\mathbf{A}} \cap \{a_0, b_0\} \neq \emptyset$. Let $I = \{l | e_l \in e_j x^{\mathbf{A}}\}$ and $J = \{l | e_l \notin e_j x^{\mathbf{A}}\}$. Then,

$$\mu(e_j x^{\mathbf{A}}) = \mu\big(e_j x^{\mathbf{A}} \setminus (\{a_0,b_0\} \cup \{e_l | l \in I\})\big) \cup \mu\big(e_j x^{\mathbf{A}} \cap \{a_0,b_0\}\big) \cup \mu\big(\{e_l | l \in I\}\big) =$$
$$= \{(b,b,\ldots,b) | b \in e_j x^{\mathbf{A}} \setminus (\{a_0,b_0\} \cup \{e_l | l \in I\})\} \cup$$
$$\cup \{x^{\mathbf{A}}(U,b,\ldots,b) | b \in e_j x^{\mathbf{A}} \cap \{a_0,b_0\}\} \cup \bigcup_{l \in I} \{(e_l, e_l, \ldots, e_l, V_l, e_l, \ldots, e_l)\},$$

where V_l occurs on the $(l+1)$-th position of the element $(e_l, e_l, \ldots, e_l, V_l, e_l, \ldots, e_l)$.

$$\mu(e_j)x^{\mathbf{A}_0 \times \mathbf{A}_1 \times \cdots \times \mathbf{A}_k} = (e_j, e_j, \ldots, e_j, V_j, e_j, \ldots, e_j)x^{\mathbf{A}_0 \times \mathbf{A}_1 \times \ldots \times \mathbf{A}_k} =$$
$$= e_j x^{\mathbf{A}_0} \times e_j x^{\mathbf{A}_1} \times \cdots \times e_j x^{\mathbf{A}_{j-1}} \times V_j x^{\mathbf{A}_j} \times e_j x^{\mathbf{A}_{j+1}} \times \cdots \times e_j x^{\mathbf{A}_k} =$$
$$= \big((e_j x^{\mathbf{A}} \setminus U) \cup \{U\}\big) \times M_1 \times \cdots \times M_{j-1} \times (cx^{\mathbf{A}} \cup e_j x^{\mathbf{A}}) \times M_{j+1} \times \cdots \times M_k,$$

where

$$M_l = \begin{cases} e_j x^{\mathbf{A}} & \text{if } l \in J, \\ (e_j x^{\mathbf{A}} \setminus V_l) \cup \{V_l\} & \text{if } l \in I, \end{cases}$$

for all $l \in \{1,\ldots,j-1,j+1,\ldots,k\}$. Therefore,

$$\mu(e_j)x^{\mathbf{A}_0 \times \mathbf{A}_1 \times \cdots \times \mathbf{A}_k} \cap B =$$
$$= \{(b,\ldots,b) | b \in e_j x^{\mathbf{A}} \setminus (U \cup \bigcup_{l \in I} V_l)\} \cup \{(U,b,\ldots,b) | b \in \bigcap_{l \in I} (e_j x^{\mathbf{A}} \setminus V_l) \cap \{a_0,b_0\}\} \cup$$
$$\cup \bigcup_{l \in I} \{(e_l, e_l, \ldots, e_l, V_l, e_l, \ldots, e_l)\}$$

and the proof of $\mu(cx^{\mathbf{A}}) = \mu(c)x^{\mathbf{A}_0 \times \mathbf{A}_1 \times \cdots \times \mathbf{A}_k} \cap B$ is complete in this case, too.

The proof of $\mu(ax^{\mathbf{A}}) = \mu(a)x^{\mathbf{A}_0 \times \mathbf{A}_1 \times \cdots \times \mathbf{A}_k} \cap B$ is similar in all other cases. □

By studying in detail all the cases of the proof of Lemma 10, we can state the following result.

Corollary 6 *Let* $\mathbf{A} = (X, A, \delta)$ *be a nondeterministic nilpotent automaton with* $|A| \geq 5$ *that has exactly one pair of different states* a_0, b_0 *for which* $a_0 x^{\mathbf{A}} = b_0 x^{\mathbf{A}}$ *holds, for all* $x \in X$. *If there exists a natural number* $k \geq 2$ *and there are* $c, e_1, \ldots, e_k \in A$ *with* $c \bowtie e_r, r = 1, \ldots, k$, *such that for all* $x \in X$,

$$cx^{\mathbf{A}} \cap \{a_0, b_0\} = \emptyset \text{ or } \exists i \in \{1, \ldots, k\} \text{ such that } e_i x^{\mathbf{A}} \cap \{a_0, b_0\} \subseteq cx^{\mathbf{A}} \cap \{a_0, b_0\}$$

holds, then $\mathbf{A}$ *can be embedded isomorphically into a direct product of nondeterministic nilpotent automata having fewer states than* $|A|$.

Theorem 2 *Let* $\mathbf{A} = (X, A, \delta)$ *be a nondeterministic nilpotent automaton with* $|A| \geq 4$ *that has exactly one pair of different states* a_0, b_0 *for which* $a_0 x^{\mathbf{A}} = b_0 x^{\mathbf{A}}$ *holds, for all* $x \in X$. *If there exists* $c \in A$ *such that:*

(a) *for all* $e \bowtie f \in A$, $c \in \{e, f\}$, *and*

(b) *there exists* $\overline{x} \in X$ *such that* $c\overline{x}^{\mathbf{A}} \cap \{a_0, b_0\} \neq \emptyset$ *and for all* $e \in A$ *with* $c \bowtie e$, $e\overline{x}^{\mathbf{A}} \cap \{a_0, b_0\} \not\subseteq c\overline{x}^{\mathbf{A}} \cap \{a_0, b_0\}$ *holds,*

then $\mathbf{A}$ *cannot be embedded into a direct product of nondeterministic nilpotent automata having fewer states than* $|A|$.

Proof. Assume to the contrary that $\mathbf{A}$ can be embedded into the direct product of $k \geq 2$ nondeterministic nilpotent automata having fewer states than $|A|$. Let $A_1, \ldots, A_k$ be these nondeterministic nilpotent automata with $|A_r| < |A|$, $r = 1, \ldots, k$, and let $\mu : A \to B \subseteq A_1 \times \cdots \times A_k$ be the embedding isomorphism. Let $\mathbf{B}$ denote the subautomaton of $\mathbf{A}_1 \times \cdots \times \mathbf{A}_k$ with state set B. Let $\mu(a_0) = (a_1^0, \ldots, a_k^0)$ and $\mu(b_0) = (b_1^0, \ldots, b_k^0)$. We consider the following set of indices: $I = \{r \in \{1, \ldots, k\} | \ a_r^0 \neq b_r^0\}$. Since $a_0 \neq b_0$ and μ is an isomorphism, $I \neq \emptyset$. Without loss of generality, we may assume that $I = \{1, \ldots, m\}$ where $m \leq k$ and m is a fixed value.

For all $i \in I$, let us examine the state $\overline{a}_i$, i.e. the state which occurs in at least two different elements of B on the i-th position. The existence of $\overline{a}_i$ follows from the fact that $|A_i| < |A|$. Due to Lemma 4 and the definition of I, $\overline{a}_i$ will appear neither in the element $(a_1^0, \ldots, a_k^0)$, nor in $(b_1^0, \ldots, b_k^0)$. According to this and to Lemma 3 that guarantees that in a nondeterministic nilpotent automaton no state may have "loops"or "circuits" except for the absorbent state, the ancestors with respect to μ of these elements are incomparable in $\mathbf{A}$. Consequently, for every $\overline{a}_i, i = 1, \ldots, m$, we have in $\mathbf{A}$ a pair of incomparable states, that don't have to be different for different $\overline{a}_i$-s. Due to (a) these states are of the form $c \bowtie e_1, \ldots, c \bowtie e_\nu$, where $\nu \leq m$ and ν is a fixed value. According to this, $\mu(c) = (\overline{a}_1, \ldots, \overline{a}_m, c_{m+1}, \ldots, c_k)$.

In the same time, $c\overline{x}^{\mathbf{A}} \cap \{a_0, b_0\} \neq \{a_0, b_0\}$ and also $e_i\overline{x}^{\mathbf{A}} \cap \{a_0, b_0\} \neq \emptyset$ holds for all $i \in I$, because otherwise there would exist $e \in A$ such that $e\overline{x}^{\mathbf{A}} \cap \{a_0, b_0\} \subseteq c\overline{x}^{\mathbf{A}} \cap \{a_0, b_0\}$ would be true in a trivial way and condition (b) would not be satisfied. $c\overline{x}^{\mathbf{A}} \cap \{a_0, b_0\} \neq \emptyset$ stated by (b) and $c\overline{x}^{\mathbf{A}} \cap \{a_0, b_0\} \neq \{a_0, b_0\}$ implies $c\overline{x}^{\mathbf{A}} \cap \{a_0, b_0\} = \{a_0\}$ or $c\overline{x}^{\mathbf{A}} \cap \{a_0, b_0\} = \{b_0\}$.

Let us analyse the case $c\overline{x}^{\mathbf{A}} \cap \{a_0, b_0\} = \{a_0\}$. Because μ is an isomorphism and $\mathbf{B}$ is a subautomaton of $\mathbf{A}_1 \times \cdots \times \mathbf{A}_k$, $a_i^0 \in \overline{a}_i\overline{x}^{\mathbf{A}_i}$, for all $i = 1, \ldots, m$, and $a_l^0 \in c_l\overline{x}^{\mathbf{A}_l}$, for all $l = m+1, \ldots, k$, must be true. Remember that $a_l^0 = b_l^0$, for all $l = m+1, \ldots, k$. This implies the existence of a $j \in I$ such that $b_j^0 \notin \overline{a}_j\overline{x}^{\mathbf{A}_j}$ because, otherwise $b_i^0 \in \overline{a}_i\overline{x}^{\mathbf{A}_i}$, for all $i = 1, \ldots, m$, and $b_l^0 \in c_l\overline{x}^{\mathbf{A}_l}$, for all $l = m+1, \ldots, k$ would hold, which would imply $b_0 \in c\overline{x}^{\mathbf{A}}$. Let us denote by J the following set of indices $J = \{i \in I |\ b_j^0 \notin \overline{a}_j\overline{x}^{\mathbf{A}_j}\}$. Then, by the statements above, it is obvious that $J \neq \emptyset$. On the other hand, all e_j with $j \in J$ must satisfy (b). This means that $e_j\overline{x}^{\mathbf{A}} \cap \{a_0, b_0\} \not\subseteq \{a_0\}$, for all $j \in J$. We already observed that $e_i\overline{x}^{\mathbf{A}} \cap \{a_0, b_0\} \neq \emptyset$ for all $i \in I$, thus the single possibility for e_j is $b_j^0 \in \overline{a}_j\overline{x}^{\mathbf{A}_j}$, for all $j \in J$, which contradicts the definition and the nonemptyness of the set J. The analysis of the case $c\overline{x}^{\mathbf{A}} \cap \{a_0, b_0\} = \{b_0\}$ is similar. □

Theorem 3 *Assume that $\mathbf{A} = (X, A, \delta)$ is a nondeterministic nilpotent automaton ($|A| \geq 5$) that has exactly one pair of different states a_0, b_0 for which $a_0x^{\mathbf{A}} = b_0x^{\mathbf{A}}$ holds, for all $x \in X$. If $\mathbf{A}$ cannot be embedded into a direct product of nondeterministic nilpotent automata having fewer states than $|A|$ and A has at least two pairs of incomparable states, then there exists $c \in A \setminus \{a_0, b_0\}$ such that the following statements are simultaneously true:*

(a) *for all $e \bowtie f \in A$, $c \in \{e, f\}$, and*

(b) *there exists $\overline{x} \in X$ such that $c\overline{x}^{\mathbf{A}} \cap \{a_0, b_0\} \neq \emptyset$ and for all $e \in A$ with $c \bowtie e$, $e\overline{x}^{\mathbf{A}} \cap \{a_0, b_0\} \not\subseteq c\overline{x}^{\mathbf{A}} \cap \{a_0, b_0\}$.*

Proof. Let our starting assumption be that $\mathbf{A}$ has exactly one pair of different states a_0, b_0 for which $a_0x^{\mathbf{A}} = b_0x^{\mathbf{A}}$ holds, for all $x \in X$, that $|A| \geq 5$, that A has at least two pairs of incomparable states and that $\mathbf{A}$ cannot be embedded into a direct product of nondeterministic nilpotent automata. The proof of (a) is given by Corollary 5. To prove (b) assume to the contrary that for all $x \in X$, $cx^{\mathbf{A}} \cap \{a_0, b_0\} = \emptyset$ or there exists $e_x \in A$ with $c \bowtie e_x$ such that $e_xx^{\mathbf{A}} \cap \{a_0, b_0\} \subseteq cx^{\mathbf{A}} \cap \{a_0, b_0\}$.

Let E be the set of all states of A that are incomparable with c, i.e. $E = \{e_1, \ldots, e_k\}$. By (a) we know that there are no other pairs of incomparable states but $c \bowtie e_1, \ldots, c \bowtie e_k$, where $k \geq 2$. Thus, we can reformulate the converse of (b) as follows: there exist $k \geq 2$ and $e_1, \ldots, e_k \in A$ such that $c \bowtie e_r, r = 1, \ldots, k$, and for all $x \in X$, $cx^{\mathbf{A}} \cap \{a_0, b_0\} = \emptyset$ or there exists $i \in \{1, \ldots, k\}$ such that $e_ix^{\mathbf{A}} \cap \{a_0, b_0\} \subseteq cx^{\mathbf{A}} \cap \{a_0, b_0\}$ holds. Due to Corollary 6 $\mathbf{A}$ can be embedded isomorphically into a direct product of nondeterministic nilpotent automata having fewer states than $|A|$ which contradicts the starting assumption. □

Now, we can prove our main result.

Theorem 4 *A nondeterministic nilpotent automaton $\mathbf{A} = (X, A, \delta)$ with $|A| \geq 2$ cannot be embedded into a direct product of nondeterministic nilpotent automata having fewer states than $|A|$, if and only if $\mathbf{A}$ has exactly one pair of different states a_0, b_0 for which $a_0x^{\mathbf{A}} = b_0x^{\mathbf{A}}$ holds, for all $x \in X$, and one of the following statements is true:*

(1) *The partially ordered set* $(A, \leq)$ *is a chain,*

(2) $\mathbf{A}$ *has exactly one pair of incomparable states* $c \bowtie d$ *and there exists* $\overline{x} \in X$ *such that* $c\overline{x}^{\mathbf{A}} \cap \{a_0, b_0\} \neq \emptyset$, $d\overline{x}^{\mathbf{A}} \cap \{a_0, b_0\} \neq \emptyset$ *and* $c\overline{x}^{\mathbf{A}} \cap \{a_0, b_0\} \neq d\overline{x}^{\mathbf{A}} \cap \{a_0, b_0\}$ *hold,*

(3) $\mathbf{A}$ *has at least two pairs of incomparable states and there exists* $c \in A$ *such that the following two statements are valid:*

   (a) *for all* $e \bowtie f \in A$, $c \in \{e, f\}$, *and*

   (b) *there exists* $\overline{x} \in X$ *such that* $c\overline{x}^{\mathbf{A}} \cap \{a_0, b_0\} \neq \emptyset$ *and for all* $e \in A$ *with* $c \bowtie e$, $e\overline{x}^{\mathbf{A}} \cap \{a_0, b_0\} \not\subseteq c\overline{x}^{\mathbf{A}} \cap \{a_0, b_0\}$ *holds.*

Proof. First, let us prove the necessity. Assume that $\mathbf{A}$ is a nondeterministic nilpotent automaton with $|A| \geq 2$ and that $\mathbf{A}$ cannot be embedded into a direct product of nondeterministic nilpotent automata having fewer states than $|A|$. The fact that there is exactly one pair of different states a_0, b_0 for which $a_0x^{\mathbf{A}} = b_0x^{\mathbf{A}}$ holds, for all $x \in X$, is guaranteed by Corollary 1. Now, we will show that one of the statements (1),(2) or (3) holds. Examining $(A, \leq)$ we can distinguish the following three cases: there are no incomparable states at all, there is exactly one pair of incomparable states and there are at least two pairs of incomparable states.

If there are no incomparable states at all, then $(A, \leq)$ is a chain and (1) holds. If there is exactly one pair of incomparable states, then, by Corollary 3, (2) holds. If there are at least two pairs of incomparable states, then, by Theorem 3, (3) holds and the proof of the necessity is complete.

Now, let us prove the sufficiency. Assume that $\mathbf{A}$ is a nondeterministic nilpotent automaton with $|A| \geq 2$ that has exactly one pair of different states a_0, b_0 for which $a_0x^{\mathbf{A}} = b_0x^{\mathbf{A}}$ holds, for all $x \in X$, and that also one of (1),(2) or (3) is true.

If (1) holds, then due to Corollary 2 $\mathbf{A}$ cannot be embedded into a direct product of nondeterministic nilpotent automata having fewer states than $|A|$. If (2) holds, then $\mathbf{A}$ cannot be embedded into a direct product of nondeterministic nilpotent automata having fewer states than $|A|$ due to Corollary 4. If (3) is true, then the fact that $\mathbf{A}$ cannot be embedded into a direct product of nondeterministic nilpotent automata having fewer states than $|A|$ follows from Theorem 2; and the proof of Theorem 4 is complete. □

The following statement presents the characterization of the isomorphic decomposability.

Theorem 5 *A nondeterministic nilpotent automaton* $\mathbf{A} = (X, A, \delta)$ *with* $|A| > 2$ *can be embedded isomorphically into a direct product of nondeterministic nilpotent automata having fewer states than* $|A|$ *if and only if it fulfills one of the following conditions:*

(i) $|A| \geq 3$, a_0 *is the absorbent state,* b_0 *is a maximal element in* $(A \setminus \{a_0\})$ *and there exist* $a_1, b_1 \in A$, $a_1 \neq b_1$, $\{a_1, b_1\} \neq \{a_0, b_0\}$ *such that beside* $a_0x^{\mathbf{A}} = b_0x^{\mathbf{A}}$, $a_1x^{\mathbf{A}} = b_1x^{\mathbf{A}}$ *also holds, for all* $x \in X$.

(ii) **A** *(with* $|A| \geq 4$*) has exactly one pair of different states* a_0, b_0 *for which* $a_0 x^{\mathbf{A}} = b_0 x^{\mathbf{A}}$ *holds, for all* $x \in X$, *and there exist* $c \bowtie d \in A$ *such that for all* $x \in X$, *the implication* $(cx^{\mathbf{A}} \cap \{a_0, b_0\} \neq \emptyset \wedge dx^{\mathbf{A}} \cap \{a_0, b_0\} \neq \emptyset) \Rightarrow cx^{\mathbf{A}} \cap \{a_0, b_0\} = dx^{\mathbf{A}} \cap \{a_0, b_0\}$ *holds.*

(iii) **A** *(with* $|A| \geq 4$*) has exactly one pair of different states* a_0, b_0 *for which* $a_0 x^{\mathbf{A}} = b_0 x^{\mathbf{A}}$ *holds, for all* $x \in X$, *and there exist* $c \bowtie d \in A$ *and* $e \bowtie f \in A$ *such that* $\{c, d\} \cap \{e, f\} = \emptyset$.

(iv) **A** *(with* $|A| \geq 5$*) has exactly one pair of different states* a_0, b_0 *for which* $a_0 x^{\mathbf{A}} = b_0 x^{\mathbf{A}}$ *holds, for all* $x \in X$, *and there are three pairwise incomparable states* $c \bowtie d \in A$, $d \bowtie e \in A$ *and* $e \bowtie c \in A$.

(v) **A** *(with* $|A| \geq 5$*) has exactly one pair of different states* a_0, b_0 *for which* $a_0 x^{\mathbf{A}} = b_0 x^{\mathbf{A}}$ *holds, for all* $x \in X$, *and there exist a natural number* $k \geq 2$ *and* $c, e_1, \ldots, e_k \in A$ *with* $c \bowtie e_r, r = 1, \ldots, k$, *such that, for all* $x \in X$,

$$cx^{\mathbf{A}} \cap \{a_0, b_0\} = \emptyset \text{ or } \exists i \in \{1, \ldots, k\} : e_i x^{\mathbf{A}} \cap \{a_0, b_0\} \subseteq cx^{\mathbf{A}} \cap \{a_0, b_0\}$$

holds.

Proof. We will prove the necessity by contradiction. One can see that in the class of nondeterministic nilpotent automata the following eqivalence holds:

$$\neg\big((\text{i}) \vee (\text{ii}) \vee (\text{iii}) \vee (\text{iv}) \vee (\text{v})\big) \Leftrightarrow \big(S \wedge ((1) \vee (2) \vee (3))\big),$$

where S is the statement that there is exactly one pair of different states a_0, b_0 in **A** for which $a_0 x^{\mathbf{A}} = b_0 x^{\mathbf{A}}$ holds, for all $x \in X$. It yields hence, that by Theorem 4, **A** cannot be embedded into a direct product of nondeterministic nilpotent automata having fewer states than $|A|$, which is a contradiction.

The sufficiency immediately follows from Theorem 1, Lemma 6, Lemma 8, Lemma 9 and Corollary 6, which imply (i), (ii), (iii), (iv) and (v), respectively. □

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