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Complete Finite Automata Network Graphs with Minimal Number of Edges*

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Dedicated to Professor Ferenc Gécseg on his 60th birthday

Abstract

An automata network graph is said to be n-complete (under projection) if every automata network having underlying graph with n vertices can be simulated (under projection) on it. In this paper n-complete automata network graphs with minimal number of edges are completely characterized.

1 Basic Notions

Let $f : X_1 \times \ldots \times X_n \to X$ be a mapping having n variables for some positive integer n, moreover, let $t \in \{1, \ldots, n\}$. f is said to *really depend* on its t^{th} variable if there exist $x_1 \in X_1, \ldots, x_{t-1} \in X_{t-1}, x_t, x_t' \in X_t, x_{t+1} \in X_{t+1}, \ldots, x_n \in X_n$ having $f(x_1, \ldots, x_n) \neq f(x_1, \ldots, x_{t-1}, x_t', x_{t+1}, \ldots, x_n)$. If f does not have this property then we also say that f is *really independent* of its t^{th} variable. Moreover, if there is no danger of confusion then sometimes we omit the attribute "really".

For a given non-empty set X and positive integer n denote by X^n the n^{th} 0 power of X. Given a k-element subset H of $\{1, \ldots, n\}$, $H = \{i_1, \ldots, i_k\}$ ($i_1 < \ldots < i_k$), the *H-projection* of X^n is a mapping $pr_H : X^n \to X^k$ defined by $pr_H(x_1, \ldots, x_n) = (x_{i_1}, \ldots, x_{i_k})$, where $(x_1, \ldots, x_n) \in X^n$. The function $pr_H(F) : X^k \to X^k$ with $pr_H(F(x_1, \ldots, x_n)) = pr_H(F)(pr_H(x_1, \ldots, x_n)), (x_1, \ldots, x_n) \in X^n$ is called the *H-projection* of $F : X^n \to X^n$ (if it exists). If $H = \{h\}$ for

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some $h \in \{1, \ldots, n\}$, i.e., H is a singleton then sometimes we use the expression *h-projection* (of a vector or function) in the same sense as the concept "H-projection". (And in this case sometimes we use the notation pr_h instead of $pr_{\{h\}}$.) Moreover, for an arbitrary $i \in \{1, \ldots, n\}$, we define the i^{th} *component* of $F : X^n \to X^n$ as the function $cp_i(F) : X^n \to X$ with $cp_i(F)((x_1, \ldots, x_n)) = pr_i(F(x_1, \ldots, x_n))$ $(x_1, \ldots, x_n) \in X^n$.

For any pair $F_i : X^n \to X^n, i = 1, 2$, one denotes by $F_1 \circ F_2 : X^n \to X^n$ the function $F_1 \circ F_2(x_1, \ldots, x_n) = F_1(F_2(x_1, \ldots, x_n))$, $(x_1, \ldots, x_n) \in X^n$.

A (finite) *directed graph* (or, in short, a *digraph*) $\mathcal{D} = (V, E)$ (of order $n > 0$) is a pair of the sets of *vertices* $V = \{v_1, \ldots, v_n\}$ and *edges* $E \subseteq V \times V$. $v_i \in V$ is an *isolated vertex* if $(\{v_i\} \times V \cup V \times \{v_i\}) \cap E = \emptyset$. If $(v_i, v_j) \in E$ and $i = j$ then (v_i, v_j) is called *(self-) loop edge.* The digraph $\mathcal{D}' = (V', E')$ is a *subdigraph* of $\mathcal{D}$ if V' is a non-void subset of V, and $E' \subseteq E$. $\mathcal{D}$ is said to be *connected* for $v_i \in V$ if every vertex $v_j \in V$ has a (directed) path from v_i to v_j. $\mathcal{D}$ is called *strongly connected* if it is connected for all of its vertices. Moreover, $\mathcal{D}$ is *centralized* if there exists a $v_i \in V$ with $V \times \{v_i\} \subseteq E$ (including $(v_i, v_i) \in E$). In addition, a digraph $\mathcal{D} = (V, E)$ having a structure $V = \{v_1, \ldots, v_n\}$, $E = \{(v_i, v_{i+1(mod n)}) : i = 1, \ldots, n\}$ is called a *cycle (with n length).* We also say that a digraph $\mathcal{D}$ *has a cycle (with n length)* if there is a subdigraph of $\mathcal{D}$ which forms a cycle (with n length). A transformation $F : X^n \to X^n$ is said to be *compatible* with a digraph $\mathcal{D} = (V, E)$ (of order n) if F has the form $F(x_1, \ldots, x_n) = (f_1(x_1, \ldots, x_n), \ldots, f_n(x_1, \ldots, x_n))$ $((x_1, \ldots, x_n) \in X^n)$ and $f_i : X^n \to X$, $i = 1, \ldots, n$ may depend only on x_i and those x_j for which $(v_j, v_i) \in E$ (including the case $i = j$).

A *word* (over X) is a finite sequence of elements of some finite non-empty set X. We call the set X an *alphabet,* the elements of X *letters.* If u and v are words over an alphabet X, then their *catenation* uv is also a word over X. Especially, for every word u over X, $u\lambda = \lambda u = u$, where λ denotes the empty word having no letters. The *length* $|w|$ of a word w is the number of letters in w, where each letter is counted as many times as it occurs. Thus $|\lambda| = 0$. By the *free monoid* X^* *generated by* X we mean the set of all words (including the *empty word* λ) having catenation as multiplication. We set $X^+ = X^* \setminus \{\lambda\}$, where the subsemigroup X^+ of X^* is said to be *free semigroup generated by* X.

By an *automaton* $\mathcal{A} = (A, X, \delta)$ we mean a finite automaton without outputs. Here A is the (finite non-empty) *state set,* X is the *input alphabet* and $\delta : A \times X \to A$ is the *transition function.* We also use δ in an extended sense, i.e., as a mapping $\delta : A \times X^* \to A$, where $\delta(a, \lambda) = a$ $(a \in A)$ and $\delta(a, px) = \delta(\delta(a, p), x)$ $(a \in A, p \in X^*, x \in X)$. For a given word $p \in X^*$, the *transition induced by* p is the function $\delta_p : A \to A$ that takes any state $a \in A$ to $\delta(a, p)$.

If $A = Z^n$ for some $|Z| \geq 1$ and $n \geq 1$ (where $|Z|$ denotes the cardinality, i.e., the number of elements in Z) then we say that $\mathcal{A}$ is a *finite state-0 automata network* (*of size n with respect to the basic local state set* Z). Then the *underlying graph* $\mathcal{D}_{\mathcal{A}} = (V_{\mathcal{A}}, E_{\mathcal{A}})$ of $\mathcal{A}$ is defined by $V_{\mathcal{A}} = \{1, \ldots, n\}, E_{\mathcal{A}} = \{(i, j) \mid \exists x \in X : cp_j(\delta_x)$ really depends on its i^{th} variable$\}$. $\mathcal{A}$ is a *$\mathcal{D}$-network* if $\mathcal{D} = (V, E)$ is a digraph with $V = V_{\mathcal{A}}$ and $E \supseteq E_{\mathcal{A}}$. In other words, $\mathcal{A}$ is a $\mathcal{D}$-network if every mapping $\delta_x : A \to A$ $(x \in X)$ is *compatible* with $\mathcal{D}$. Note that a size n automata network may be regarded as comprising n component automata $\mathcal{A}_i = (Z, Z^n \times X, \delta_i)$, $i \in \{1, \ldots, n\}$, where the δ_i are defined by

$$\delta(z, x) = (\delta_1(z_1, (z, x)), \ldots, \delta_n(z_n, (z, x))),$$

for $z = (z_1, \ldots, z_n) \in Z^n$, $x \in X$. One may of course suppress the components of Z^n in the inputs to $\mathcal{A}_i$ upon which δ_i does not really depend.

If $n = 1$ or $|Z| = 1$ then we say that $\mathcal{A} = (Z^n, X, \delta)$ is a *trivial* automata network. The purpose of this paper is to investigate the state-homogeneous automata networks having state sets of the form Z^n, for a positive integer $n > 1$ and fixed finite set Z of cardinality at least two. Therefore, by an automata network we shall mean a non-trivial finite state-homogeneous network.

Let $\mathcal{A} = (Z^n, X, \delta), \mathcal{B} = (Z^m, Y, \delta')$ be networks (having the same basic set Z). We say that $\mathcal{B}$ *simulates* $\mathcal{A}$ *by projection* if there exists an $H \subseteq \{1, \ldots, m\}$ such that every $\delta_x : Z^n \to Z^n$ $(x \in X)$ is an H-projection of a mapping $\delta'_p : Z^m \to Z^m$ $(p \in Y^+)$. If there exists a $\mathcal{D}$-network $\mathcal{B}$ which simulates a given network $\mathcal{A}$ by projection then it is said that $\mathcal{A}$ *can be simulated* on $\mathcal{D}$ *by projection.* A digraph $\mathcal{D}$ is called n*-complete (with respect to simulation by projection)* if every network of size n can be simulated on $\mathcal{D}$ by projection. The n-complete digraph $\mathcal{D} = (V, E)$ has *minimal number of edges* if for every n-complete digraph $\mathcal{D}' = (V', E')$, $|V| = |V'|$ implies $|E| \leq |E'|$.

2 Preliminary results

We start with the following technical result.

Lemma 2.1.(see [2]) *Given a finite group G, a positive integer $n > 1$, let us define for every distinct $i, j \in \{1, \ldots, n\}$ the functions $F^{(t)}_{i,j} : G^n \to G^n, t = 1, 2, 3$, $F^{(4)}_j : G^n \to G^n$, and $U_{i,j} : G^n \to G^n$ as follows.*

$$F^{(1)}_{i,j}(g_1, \ldots, g_n) = (g_1, \ldots, g_{j-1}, g_i g_j, g_{j+1}, \ldots, g_n),$$

$$F^{(2)}_{i,j}(g_1, \ldots, g_n) = (g_1, \ldots, g_{j-1}, g_i{}^{-1} g_j, g_{j+1}, \ldots, g_n),$$

$$F^{(3)}_{i,j}(g_1, \ldots, g_n) = (g_1, \ldots, g_{j-1}, g_i, g_{j+1}, \ldots, g_n),$$

$$F^{(4)}_{j}(g_1, \ldots, g_n) = (g_1, \ldots, g_{j-1}, g_j{}^{-1}, g_{j+1}, \ldots, g_n),$$

$$U_{i,j}(g_1, \ldots, g_n) = (g_1, \ldots, g_{i-1}, g_j, g_{i+1}, \ldots, g_{j-1}, g_i, g_{j+1}, \ldots, g_n).$$

Then for arbitrary, pairwise distinct $i, j, k \in \{1, \ldots, n\}$ we get

$$F^{(1)}_{i,j} = F^{(2)}_{i,k} \circ F^{(1)}_{k,j} \circ F^{(1)}_{i,k} \circ F^{(2)}_{k,j},$$

$$F_{i,j}^{(2)} = F_{i,k}^{(1)} \circ F_{k,j}^{(1)} \circ F_{i,k}^{(2)} \circ F_{k,j}^{(2)},$$
$$F_{i,j}^{(3)} = F_{k,j}^{(1)} \circ F_{i,k}^{(1)} \circ F_{k,j}^{(2)} \circ F_{i,k}^{(2)} \circ F_{k,j}^{(2)} \circ F_{k,j}^{(3)},$$
$$U_{i,j} = F_j^{(4)} \circ F_{i,j}^{(1)} \circ F_j^{(4)} \circ F_i^{(4)} \circ F_{j,i}^{(2)} \circ F_{i,j}^{(1)}.$$

□

Given a non-void set Y, a positive integer n, let $\mathcal{T}_Y$ denote the full transformation semigroup of all functions from Y to Y. In addition, for every subset $H \subseteq \mathcal{T}_Y$, let $< H >$ denote the subsemigroup of $\mathcal{T}_Y$ generated by H. Moreover, for any finite set X with $|X| > 1$ and positive integer $n > 1$, denote $\mathcal{T}_{X,n}$ the subsemigroup of all transformations of $\mathcal{T}_{X^n}$ having the form $F(x_1, \dots, x_n) = (x_{t(1)}, \dots x_{t(n)}), (x_1, \dots, x_n) \in X^n, t : \{1, \dots, n\} \to \{1, \dots, n\}$, and let

$$\begin{aligned}\Gamma_{X^n} \quad = \quad & \{F : X^n \to X^n \mid F(x_1, \dots, x_n) = (x_1, \dots, x_{i-1}, f(x_i, x_j), x_{i+1}, \dots, x_n), \\ & \text{where } f : X^2 \to X, i, j \in \{1, \dots, n\}, (x_1, \dots, x_n) \in X^n\},\end{aligned}$$

(It is understood that the case $i = j$ is allowed in the above definition of Γ_{X^n}.) Define the *elementary collapsing* $t_{j,k} : \{1, \dots, n\} \to \{1, \dots, n\}$ for $1 \le j \ne k \le n$,

$$t_{j,k}(i) = \begin{cases} j & \text{if } i = k \\ i & \text{otherwise} \end{cases}.$$

Moreover, as usual we say that $u_{j,k} : \{1, \dots, n\} \to \{1, \dots, n\}$ for $1 \le j \ne k \le n$ is a *transposition* if

$$u_{j,k}(i) = \begin{cases} j & \text{if } i = k \\ k & \text{if } i = j \\ i & \text{otherwise} \end{cases}.$$

Let $\mathcal{F}_{X^{n-1} \times \{d\}}$ be the semigroup of functions $\{F \in \mathcal{T}_{X^n} : F(x_1, \dots, x_n) \in X^{n-1} \times \{d\}, x_1, \dots, x_n \in X, F$ is really independent of its last variable$\}$.

Lemma 2.2. (see [2]) $\mathcal{F}_{X^{n-1} \times \{d\}} \subsetneq < \Gamma_{X^n} >$.

Proof. Fix arbitrary $c \ne d \in X$ and let $(c_1, \dots, c_{n-1}) \in X^{n-1}$,

$$F_{(c_1, \dots, c_{n-1})}(x_1, \dots, x_n) = \begin{cases} (x_1, \dots, x_n) & \text{if } (x_1, \dots, x_n) = (c_1, \dots, c_{n-1}, c), \\ (x_1, \dots, x_{n-1}, d) & \text{if } (x_1, \dots, x_n) \ne (c_1, \dots, c_{n-1}, c) \end{cases}$$

$((x_1, \dots, x_n) \in X^n)$. First we prove that $F_{(c_1, \dots, c_{n-1})} \in < \Gamma_{X^n} >$.

If $n = 2$, then our statement holds by definition. Otherwise, $n > 2$ and for every $b \in X$, define

$$F_b^{(0)}(x_1, \dots, x_n) = \begin{cases} (x_1, \dots, x_{n-1}, c) & \text{if } x_{n-1} = b, x_n = c, \\ (x_1, \dots, x_{n-1}, d) & \text{otherwise}, \end{cases}$$

where $(x_1, \ldots, x_n) \in X^n$.

For every $i \in \{1, \ldots, n-1\}, (c_i, \ldots, c_{n-1}) \in X^{n-i}$, let

$$F_{(c_i,\ldots,c_{n-1})}(x_1, \ldots, x_n) = \begin{cases} (x_1, \ldots, x_{n-1}, c) & \text{if } (x_i, \ldots, x_n) = (c_i, \ldots, c_{n-1}, c), \\ (x_1, \ldots, x_{n-1}, d) & \text{otherwise,} \end{cases}$$

where $x = (x_1, \ldots, x_n) \in X^n$). It is clear that $F_{(c_{n-1})} = F^{(0)}_{c_{n-1}}$. On the other hand, for every $i \in \{2, \ldots, n-1\}$, $F_{(c_{i-1},\ldots,c_{n-1})} = F_{(c_i,\ldots,c_{n-1})} \circ U_{i-1,n-1} \circ F^{(0)}_{c_{i-1}} \circ U_{i-1,n-1}$. Simultaneously, we have by definition that $F^{(0)}_{c_{i-1}} \in \Gamma_{X^n}$ holds for every $i \in \{2, \ldots, n-1\}$. Moreover, using Lemma 2.1, it can be shown easily $U_{i,j} \in <\Gamma_{X^n}>$. Thus we get our statement by induction.

Now we consider a pair $(c_1, \ldots, c_{n-1})$, $(d_1, \ldots, d_{n-1}) \in X^{n-1}, d \in X$, $((c_1, \ldots, c_{n-1}) = (d_1, \ldots, d_{n-1})$ is allowed), and define

$$F^{(1)}_{(c_1,\ldots,c_{n-1}),(d_1,\ldots,d_{n-1})}(x) = \begin{cases} (c_1, \ldots, c_{n-1}, d) & \text{if } (x_1, \ldots, x_{n-1}) = (d_1, \ldots, \\ & \quad d_{n-1}), \\ (d_1, \ldots, d_{n-1}, d) & \text{if } (x_1, \ldots, x_{n-1}) = (c_1, \ldots, \\ & \quad c_{n-1}), \\ (x_1, \ldots, x_{n-1}, d) & \text{otherwise,} \end{cases}$$

$$F^{(2)}_{(c_1,\ldots,c_{n-1}),(d_1,\ldots,d_{n-1})}(x) = \begin{cases} (c_1, \ldots, c_{n-1}, d) & \text{if } (x_1, \ldots, x_{n-1}) = (d_1, \ldots, \\ & \quad d_{n-1}), \\ (x_1, \ldots, x_{n-1}, d) & \text{otherwise,} \end{cases}$$

where $x = (x_1, \ldots, x_n) \in X^n$.

Next we show $F^{(i)}_{(c_1,\ldots,c_{n-1}),(d_1,\ldots,d_{n-1})} \in <\Gamma_{X^n}>, i = 1, 2$.

We have $c \in X$ arbitrary with $c \neq d$ and set $F^{(3)}_c(x) = (x_1, \ldots, x_{n-1}, c)$, $F^{(3)}_d(x) = (x_1, \ldots, x_{n-1}, d)$, and

$$F^{(4)}_{(c_1,\ldots,c_{n-1}),(d_1,\ldots,d_{n-1})}(x) = \begin{cases} (c_1, \ldots, c_{n-1}, c) & \text{if } (x_1, \ldots, x_n) = (d_1, \ldots, \\ & \quad d_{n-1}, c), \\ (x_1, \ldots, x_{n-1}, d) & \text{otherwise} \end{cases}$$

where $x = (x_1, \ldots, x_n) \in X^n, c, d \in X, c \neq d$, moreover, consider $F_{(c_1,\ldots,c_{n-1})}$ as before. In addition, let

$$F^{(5)}(x_1, \ldots, x_n) = \begin{cases} (x_1, \ldots, x_{n-1}, c) & \text{if } x_n = d, \\ (x_1, \ldots, x_{n-1}, d) & \text{if } x_n = c, \\ (x_1, \ldots, x_{n-1}, x_n) & \text{otherwise,} \end{cases}$$

and let for every $a \in X$,

$$F^{(6)}_a(x_1, \ldots, x_n) = \begin{cases} (x_1, \ldots, x_{n-2}, a, x_n) & \text{if } x_n = c, \\ (x_1, \ldots, x_{n-1}, x_n) & \text{otherwise} \end{cases}$$

$((x_1, \ldots, x_n) \in X^n)$. It is clear that $F^{(3)}_c, F^{(3)}_d, F^{(5)}, F^{(6)}_a \in \Gamma_{X^n}$. Next we show that $F^{(4)}_{(c_1,\ldots,c_{n-1}),(d_1,\ldots,d_{n-1})} \in <\Gamma_{X^n}>$. Indeed, by an easy computation we get

$F^{(4)}_{(c_1,\ldots,c_{n-1}),(d_1,\ldots,d_{n-1})} = U_{n-2,n-1}\circ\ldots\circ U_{2,n-1}\circ U_{1,n-1}\circ F^{(6)}_{c_1}\circ U_{1,n-1}\circ F^{(6)}_{c_2}\circ U_{2,n-1}\circ \ldots\circ U_{n-3,n-1}\circ F^{(6)}_{c_{n-2}}\circ U_{n-2,n-1}\circ F^{(6)}_{c_{n-1}}\circ F_{(d_1,\ldots,d_{n-1})}$. On the other hand, by Lemma 2.1 we can see easily $U_{i,j} \in < \Gamma_{X^n} >$. But then $F^{(2)}_{(c_1,\ldots,c_{n-1}),(d_1,\ldots,d_{n-1})} = F^{(3)}_d \circ F^{(4)}_{(c_1,\ldots,c_{n-1}),(d_1,\ldots,d_{n-1})} \circ F^{(3)}_c$ implies $F^{(2)}_{(c_1,\ldots,c_{n-1}),(d_1,\ldots,d_{n-1})} \in < \Gamma_{X^n} >$. It remains to prove that $F^{(1)}_{(c_1,\ldots,c_{n-1}),(d_1,\ldots,d_{n-1})} \in < \Gamma_{X^n} >$. This connection, completing the proof, comes from $F^{(1)}_{(c_1,\ldots,c_{n-1}),(d_1,\ldots,d_{n-1})} = F^{(3)}_d \circ F^{(4)}_{(d_1,\ldots,d_{n-1}),(c_1,\ldots,c_{n-1})} \circ F^{(5)} \circ F^{(4)}_{(c_1,\ldots,c_{n-1}),(d_1,\ldots,d_{n-1})} \circ F^{(3)}_c$.

Finally, for every pair $(c_1,\ldots,c_{n-1}),(d_1,\ldots,d_{n-1}) \in X^{n-1}$, let us consider the mappings $F^{(1)}_{(c_1,\ldots,c_{n-1}),(d_1,\ldots,d_{n-1})}$, $F^{(2)}_{(c_1,\ldots,c_{n-1}),(d_1,\ldots,d_{n-1})}$ defined before. Observe that $F^{(1)}_{(c_1,\ldots,c_{n-1}),(d_1,\ldots,d_{n-1})}$ acts as a transposition in the permutation group over the set $X^{n-1}\times\{d\}$, while $F^{(2)}_{(c_1,\ldots,c_{n-1}),(d_1,\ldots,d_{n-1})}$ acts as an elementary collapsing in the transformation semigroup over the set $X^{n-1}\times\{d\}$. We have already proved that all of these transpositions and elementary collapsings are in $< \Gamma_{X^n} >$. Moreover, it is well-known that the set of all transpositions and elementary collapsings on a set generates all mappings on that set, so any map taking $X^{n-1}\times\{d\}$ to itself may be written as the restriction to $X^{n-1}\times\{d\}$ of a composite of the the above functions. A moment's reflections shows that the set of all these $F^{(1)}_{(c_1,\ldots,c_{n-1}),(d_1,\ldots,d_{n-1})}$, $F^{(2)}_{(c_1,\ldots,c_{n-1}),(d_1,\ldots,d_{n-1})}$ in fact generates all of $\mathcal{F}_{X^{n-1}\times\{d\}}$, since a function in the latter is uniquely determined by its 0 on $X^{n-1}\times\{d\}$. In addition, it is clear that $\Gamma_{X^n}\setminus\mathcal{F}_{X^{n-1}\times\{d\}}$ is non-void. This completes the proof. □

Next we show

Lemma 2.3. *Given a finite group G, a pair of relatively prime integers m,n with $1\le m<n$, let us define for every $\ell\in\{1,\ldots,n\}$, the transformations $T^{(0)}_i : G^n\to G^n$, $T^{(k)}_i : G^n\to G^n, k=1,2,3,4$ as follows.*

$$T^{(0)}(g_1,\ldots,g_n) = (g_n,g_1,\ldots,g_{n-1}),$$

$$T^{(1)}_\ell(g_1,\ldots,g_n) = (g_n,g_1,\ldots,g_{\ell-2},g_{\ell-m-1(\bmod n)}g_{\ell-1},g_\ell,\ldots,g_{n-1}),$$

$$T^{(2)}_\ell(g_1,\ldots,g_n) = (g_n,g_1,\ldots,g_{\ell-2},g^{-1}_{\ell-m-1(\bmod n)}g_{\ell-1},g_\ell,\ldots,g_{n-1}),$$

$$T^{(3)}_\ell(g_1,\ldots,g_n) = (g_n,g_1,\ldots,g_{\ell-2},g_{\ell-m-1(\bmod n)},g_\ell,\ldots,g_{n-1}),$$

$$T^{(4)}_\ell(g_1,\ldots,g_n) = (g_n,g_1,\ldots,g_{\ell-2},g^{-1}_{\ell-1},g_\ell,\ldots,g_{n-1}).$$

Then for any fixed $\ell \in \{1,\dots,n\}$, $\mathcal{T}_{G,n} \subsetneq < \{T^{(0)}, T^{(k)}_\ell : k = 1,2,3,4\} >$.

Proof. For every $i \in \{1,\dots,n\}$, $k \in \{1,\dots,4\}$, $T^{(k)}_i = (T^{(0)})^{n+i-\ell} \circ T^{(k)}_\ell \circ (T^{(0)})^{n+\ell-i}$. Thus we shall show only $\mathcal{T}_{G,n} \subsetneq < \{T^{(0)}, T^{(k)}_\ell : \ell \in \{1,\dots,n\}, k = 1,2,3,4 >$. It is clear that using the notions in Lemma 2.1, by the simple fact that every permutation is a composite of transpositions, and moreover, transformations can be generated by permutations and elementary collapsings, we obtain $\mathcal{T}_{G,n} \subseteq < \{F^{(3)}_{i,j}, U_{i,j} : i,j \in \{1,\dots,n\} >$. On the other hand, $< \{T^{(0)}, T^{(k)}_\ell : k = 1,2,3,4, \ell = 1,\dots,n\} > \setminus \mathcal{T}_{G,n} \neq \emptyset$ is clear. Thus, it is enough to prove that for every $i,j \in \{1,\dots,n\}$, $F^{(3)}_{i,j}, U_{i,j} \in < \{T^{(0)}, T^{(k)}_\ell : k = 1,2,3,4, \ell = 1,\dots,n\} >$. Using $F^{(d)}_{i+(j-1)m-1(\bmod n), i+jm-1(\bmod n)} = (T^{(0)})^{n-1} \circ T^{(d)}_{i+jm(\bmod n)}$, $d = 1,2,3, i \in \{1,\dots,n\}, j = 0,1,\dots$, by an inductive application of Lemma 2.1, we have $F^{(d)}_{i-m-1(\bmod n), i+jm-1(\bmod n)} \in < \{T^{(0)}, T^{(k)}_\ell : k = 1,2,3,4, \ell = 1,\dots,n\} >$ $(i \in \{1,\dots,n,\}, j = 0,1\dots)$.

Therefore, because m and n are relatively prime, we receive $F^{(d)}_{i,j} \in < \{T^{(0)}, T^{(k)}_\ell : k = 1,2,3,4,\ \ell = 1,\dots,n\} >$ $(d = 1,2,3, i,j \in \{1,\dots,n\})$.

Moreover, we also have $F^{(4)}_{i-1} = (T^{(0)})^{n-1} \circ T^{(4)}_i$, $i \in \{1,\dots,n\}$. Hence, applying Lemma 2.1 again, we obtain $U_{i,j} \in < \{T^{(0)}, T^{(k)}_\ell : k = 1,\dots,4, \ell = 1,\dots,n\} >$, $i,j \in \{1,\dots,n\}$ and thus, having $F^{(3)}_{i,j} \subseteq < \{T^{(0)}, T^{(k)}_\ell : k = 1,2,3,4\} >$ $(i,j \in \{1,\dots,n\})$, the proof is complete. □

We shall use the following

Lemma 2.4. (see [1], [2]) *Given a positive integer* n, *let* $G = < g >$ *denote a finite non-trivial cyclic group with a generator* $g \in G$. *There exists an arrangement* $a_1,\dots,a_m$ $(m = |G|^n)$ *of the elements in the* n^{th} *direct power* G^n *of* G *such that for every* $i = 1,\dots,m-1$ *there is a* $j \in \{1,\dots,n\}$ *with* $a_{i+1} \in \{(g_1,\dots,g_{j-1}, g_jg^{-1}, g_{j+1},\dots,g_n), (g_1,\dots,g_{j-1}, g_jg, g_{j+1},\dots,g_n)\}$, *whenever* $a_i = (g_1,\dots,g_n)$ $(\in G^n)$. □

Now we are ready to prove the following key lemma.

Lemma 2.5. *For any fixed* $\ell \in \{1,\dots,n\}$, $\mathcal{T}_{X^n}$ *is generated by the union of* $< \{T^{(0)}, T^{(k)}_\ell : k = 1,\dots,4\} >$ *and the set of all functions* $F : X^n \to X^n$ *having the form* $F(x_1,\dots,x_n) = (x_1,\dots,x_{\ell-1}, f(x_1,\dots,x_n), x_{\ell+1},\dots,x_n)$, $f : X^n \to X$, *where* $x_1,\dots,x_n \in X$.

Proof. We can take out of consideration the trivial case $|X| = 1$. Thus we assume $|X| > 1$.

It is clear that without loss of generality we may suppose $\ell = 1$. On the other hand, using Lemma 2.3, $\{U_{i,j} : i,j \in \{1,\dots,n\}\} \subsetneq < \{T^{(0)}, T^{(k)}_\ell : k = 1,\dots,4 >$. Thus it is enough to prove that the union of $\{U_{i,j} : i,j \in \{1,\dots,n\}\}$ and the set of all functions $F : X^n \to X^n$ having the form $F(x_1, \dots, x_n) = (f(x_1, \dots, x_n), x_2, \dots, x_n)$, generates $\mathcal{T}_{X^n}$.

For every pair $i \in \{1, \dots, n\}$, $f : X^n \to X$, define the function $F_{i,f} : X^n \to X^n$ with $F_{i,f}(x_1, \dots, x_n) = (x_1, \dots, x_{i-1}, f(x_1, \dots, x_n), x_{i+1}, \dots, x_n)\,(x_1, \dots, x_n \in X)$. Thus, by letting $f' = f \circ U_{i,j}$, we have $F_{j,f} = U_{i,j} \circ F_{i,f'} \circ U_{i,j}$. So for every pair $i \in \{1, \dots, n\}$, $f : X^n \to X$, $F_{i,f} \in < \mathcal{T}_{X,n} \cup \{F : X^n \to X^n \mid F(x_1, \dots, x_n) = (f(x_1, \dots, x_n), x_2, x_3, \dots, x_n),\ f : X^n \to X, x_1, \dots, x_n \in X\} >$.

Let us identify X with a non-trivial finite cyclic group with generating element $g \in X$. Thus we also have that for any $c_1, \dots, c_n \in X$, ${F^{(1)}}_{\epsilon,j,(c_1,\dots,c_n)}$, ${F^{(2)}}_{\epsilon,j,(c_1,\dots,c_n)} \in < \mathcal{T}_{X,n} \cup \{F : X^n \to X^n \mid F(x_1, \dots, x_n) = (f(x_1, \dots, x_n), x_2, x_3, \dots, x_n), f : X^n \to X, x = (x_1, \dots, x_n) \in X^n\} >$, whenever $\epsilon \in \{1, -1\}$,

$${F^{(1)}}_{\epsilon,j,(c_1,\dots,c_n)}(x) =$$

$$\begin{cases} (c_1, \dots, c_n) & \text{if } x = (c_1, \dots, c_{j-1}, c_j g^\epsilon, c_{j+1}, \dots, c_n), \\ (c_1, \dots, c_{j-1}, c_j g^\epsilon, c_{j+1}, \dots, c_n) & \text{if } x = (c_1, \dots, c_n), \\ x & \text{otherwise,} \end{cases}$$

$${F^{(2)}}_{\epsilon,j,(c_1,\dots,c_n)}(x) = \begin{cases} (c_1, \dots, c_{j-1}, c_j g^\epsilon, c_{j+1}, \dots, c_n) & \text{if } x = (c_1, \dots, c_n), \\ x & \text{otherwise,} \end{cases}$$

where $x = (x_1, \dots, x_n) \in X^n$. On the other hand, by Lemma 2.4, there exists an arrangement $a_1, \dots, a_m$ of X^n, such that for every $k = 1, \dots, m-1$, $p_k \in \{{F^{(1)}}_{\epsilon,j,(c_1,\dots,c_n)} : \epsilon \in \{-1, 1\}, j \in \{1, \dots, n\}, c_1, \dots, c_n \in X\}$, $t_k \in \{{F^{(2)}}_{\epsilon,j,(c_1,\dots,c_n)} : \epsilon \in \{-1, 1\}, j \in \{1, \dots, n\}, c_1, \dots, c_n \in X\}$, where

$$p_k(a_\ell) = \begin{cases} a_{k+1} & \text{if } \ell = k, \\ a_k & \text{if } \ell = k+1, \\ a_\ell & \text{otherwise,} \end{cases}$$

$$t_k(a_\ell) = \begin{cases} a_{k+1} & \text{if } \ell = k, \\ a_\ell & \text{otherwise.} \end{cases}$$

But then $p_1, \dots, p_{m-1}$ is a set of transpositions such that $\{p_1, \dots, p_{m-1}\}$ generates all permutations over X^n. And simultaneously, $t_1, \dots, t_{m-1}$ is a set of elementary collapsings over X^n. Thus by the well-known fact that for every $j = 1, \dots, m-1$, $\{p_1, \dots, p_{m-1}, t_j,\}$ generates all transformations over X^n, the proof is complete. □

3 Main Results

First we show the next statement.

Theorem 3.1. *Given a positive integer $n > 1$, $\mathcal{D} = (V, E)$ with $V = \{1, \dots, n\}$ is an n-complete digraph with minimal number of edges if and only if there exists a permutation $p : \{1, \dots, n\} \to \{1, \dots, n\}$ such that $E = \{(p(i), p(j)) : i, j \in \{1, \dots, n\}, p(j) = p(i + 1 \bmod n)\} \cup \{(p(i), p(1)) : i \in \{1, \dots, n\}\}$.*

Proof. We may assume without loss of generality that the permutation p is the identity. Then it is clear that for an arbitrary $m \in \{1, \ldots, n\}$, the functions $T^{(0)}, T_{\ell}^{(k)} : k = 1, 2, 3, 4$ defined in Lemma 2.3 are compatible with $\mathcal{D}$. Suppose that m is 3 such that it is relatively prime to n. Then the sufficiency of this statement is a direct consequence of Lemma 2.5. To the necessity first we show the existence of $j \in V$ with $\{(i, j) : i \in V\} \subseteq E$, whenever $\mathcal{D}$ is n-complete.

Let $T : X^n \to X^n$ such that $|\{T(x_1, \ldots, x_n) : x_1, \ldots, x_n \in X\}| = |X^n| - 1$. First we show that for every $F_1, \ldots, F_m \in \mathcal{T}_{X^n}$, $T = F_1 \circ \ldots \circ F_m$ implies the existence of an index i 0 the property $|\{F_i(x_1, \ldots, x_n) : x_1, \ldots, x_n \in X\}| = |X^n| - 1$. Of course, if $F_1, \ldots, F_m$ are injective then $T = F_1 \circ \ldots \circ F_m$ should be also injective, a contradiction. On the other hand, $T = F_1 \circ \ldots \circ F_m$ implies $|\{F(x_1, \ldots, x_n) : x_1, \ldots, x_n \in X\}| \leq \min\{|\{F_i(x_1, \ldots, x_n) : x_1, \ldots, x_n \in X\}| : i = 1, \ldots, m\}$. Therefore, we obtain our assumption regarding the existence of an index i preserving the property $|\{F_i(x_1, \ldots, x_n) : x_1, \ldots, x_n \in X\}| = |X^n| - 1$.

Now we identify the elements of X in a fixed but arbitrary way with the elements of $\{1, \ldots, |X|\}$ and consider X^n as a subset of the n^{th} direct power of integers. For every $(a_{1,1}, \ldots, a_{1,n}), \ldots, (a_{m,1}, \ldots, a_{m,n}) \in X^n$, let $\sum\{(a_{i,1}, \ldots, a_{i,n}) : i = 1, \ldots, m\} = (\sum_{i=1}^{m} a_{i,1}, \ldots, \sum_{i=1}^{m} a_{i,n})$. Let $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_n) \in X^n$ denote distinct elements with $|F_i^{-1}(a)| = 0$ and $|F_i^{-1}(b)| = 2$. And let $j \in \{1, \ldots, n\}$ be an index with $a_j \neq b_j$.

Prove that $|X|$ does not divide $pr_j(\sum\{F_i(x_1, \ldots, x_n) : x_1, \ldots, x_n \in X\})$. Indeed, then $pr_j(\sum\{F_i(x_1, \ldots, x_n) : x_1, \ldots, x_n \in X\}) = pr_j(\sum\{(x_1, \ldots, x_n) : x_1, \ldots, x_n \in X\}) + b_j - a_j) = |X^{n-1}|(\sum_{k=0}^{|X|-1} k) + b_j - a_j$. Of course, by this equality we received that $|X|$ does not divide $pr_j(\sum\{F_i(x_1, \ldots, x_n) : x_1, \ldots, x_n \in X\})$.

Suppose that for every $j \in V$ there exists an $i \in V$ with $(i, j) \notin E$. Consider the set $\mathcal{D}_X$ of all functions of the form $X^n \to X^n$ which are compatible with $\mathcal{D}$. Now we show that for every $F \in \mathcal{D}_X$, $|X|$ divides $pr_j(\sum\{F(x_1, \ldots, x_n) : x_1, \ldots, x_n \in X\})$, implying $F_i \notin \mathcal{D}_X$.

By $F \in \mathcal{D}_X$ we have that for an appropriate $\ell \in \{1, \ldots, n\}$, $pr_j(F(x_1, \ldots, x_n)) = pr_j(F(x_1, \ldots, x_{\ell-1}, x'_{\ell}, x_{\ell+1}, \ldots, x_n))$ ($(x_1, \ldots, x_n) \in X^n, x'_{\ell} \in X, \ell = j$ is allowed). Therefore, for an arbitrary fixed $c \in X$, $pr_j(\sum\{F(x_1, \ldots, x_n) : x_1, \ldots, x_n \in X\}) = |X| pr_j(\sum\{F(x_1, \ldots, x_{\ell-1}, c, x_{\ell+1}, \ldots, x_n)) : x_1, \ldots, x_{\ell-1}, x_{\ell+1}, \ldots, x_n \in X\}$. But then $|X|$ divides $pr_j(\sum\{F(x_1, \ldots, x_n) : x_1, \ldots, x_n \in X\})$ for every $j = 1, \ldots, n$. Hence we get $F_i \notin \mathcal{D}_X$. Consequently, there exists a $T \in \mathcal{T}_{X^n}$ whith $T \notin < \mathcal{D}_X >$. This ends the proof of the existence of $j \in V$ with $\{(i, j) : i \in V\} \subseteq E$, whenever $\mathcal{D}$ is n-complete. Then we are ready if we can prove the existence of a permutation $p : \{1, \ldots, n\} \to \{1, \ldots, n\}$ having $\{(p(i), p(j)) : i, j \in \{1, \ldots, n\}, p(j) = p(i) + 1 (\text{mod} n)\} \subseteq E$.

Consider the mapping $T^{(0)} : X^n \to X^n$ defined by $T^{(0)}(x_1, \ldots, x_n) = (x_n, x_1, \ldots, x_{n-1})$ $(x_1, \ldots, x_n \in X)$. To complete the proof of our theorem, we will show $T^{(0)} \notin \mathcal{D}_X$ if there exists no such a permutation p.

It is also clear that an n-complete digraph $\mathcal{D}$, having n vertices, should be strongly connected. Therefore, all vertices have (non-loop) incoming edges. Thus, by the minimality of $|E|$, we get $|E \setminus \{(i, j) : i \in V\}| = n - 1$. Simultaneously, the strongly connectivity of $\mathcal{D}$ implies $\{j\} \times (V \setminus \{j\}) \cap E \neq \emptyset$ (where $j \in V$ with $\{(i,j) : i \in V\} \subseteq E$). On the other hand, if there exists no permutation p having the above discussed property, then by the strongly connectivity of $\mathcal{D}, V \times \{j\} \subseteq E$ and $|E \setminus \{(i,j) : i \in V\}| = n-1$, we can prove $|\{j\} \times (V \setminus \{j\}) \cap E| \geq 2$, implying the existence of two distinct vertices $i_1, i_2 \in V$ with $\{(\ell, i_r) : r = 1,2, \ell \in V\}$ $\cap E = \{(j,i_1),(j,i_2)\}$.

It is enough to prove that in this case $T^{(0)} \notin \mathcal{D}_X$. Clearly, $F_1 \in \mathcal{D}_X$ implies the existence of functions $f_k : X \to X,\ k = 1,2$ with $pr_{i_k}(F_1(x_1,\ldots,x_n)) = f_k(x_j)$. Therefore, the cardinality of $\{(y_1,y_2) : y_k = pr_{i_k}(F_1(x_1,\ldots,x_n)), k = 1,2,\ x_1,\ldots, x_n \in X\}$ is not greater than $|X|$. In a similar way, for every $F_1,\ldots,F_m \in \mathcal{D}_X, m > 1$ there exist functions $f_k : X \to X, k = 1,2$ such that $pr_{i_k}(F_1 \circ \ldots \circ F_m(x_1,\ldots,x_n)) = f_k(pr_j(F_2 \circ \ldots \circ F_m(x_1,\ldots,x_m)))$ implying that the cardinality of $\{(y_1,y_2) : y_k = pr_{i_k}(F_1 \circ \ldots \circ F_m(x_1,\ldots,x_n)), k = 1,2, x_1,\ldots,x_n \in X\}$ is not greater than $|X|$. On the other side, the cardinality of $\{(y_1,y_2) : y_k = pr_{i_k}(T^{(0)}(x_1,\ldots,x_n)), k = 1,2,\ x_1,\ldots,x_n \in X\}$ is $|X|^2$ yielding to $T^{(0)} \notin \mathcal{D}_X$. The proof is complete. □

Now we prove the following characterization.

Theorem 3.2. *Given a positive integer $n > 1$, $\mathcal{D} = (V,E)$ with $V = \{1,\ldots,m\}$, $m > n$ is an n-complete digraph with minimal number of edges if and only if there exists a permutation $p : \{1,\ldots,m\} \mapsto \{1,\ldots,m\}$ such that $E = \{(p(i),p(j)) : p(i),p(j) \in \{1,\ldots,n+1\}, p(j) = p(i+1 \bmod n+1)\} \cup \{(p(i'),p(j'))\}$, where $i',j' \in \{1,\ldots,n+1\}, |j'-i'| \neq 1$, moreover, $|j'-i'|-1$ and $n+1$ are relatively prime. NB: The case $i' = j'$ is not excluded. Moreover, if there are more than $n+1$ vertices then all except for $n+1$ are isolated.*

Proof. To the sufficiency it is enough to prove for any $n > 2$ the n-completeness of $\mathcal{D} = (\{1,\ldots,n+1\}, \{(i,i+1(\bmod n+1)) : i \in \{1,\ldots,n+1\}\} \cup \{(1,r)\}$, where $r \in \{1,\ldots,n+1\}, r \neq 2$, and in addition, $r-2$ and $n+1$ are relative primes.

Consider the set $\mathcal{D}_X$ of all functions of the form $X^{n+1} \to X^{n+1}$ which are compatible with $\mathcal{D}$. By definition, we obtain $\{T^{(0)}, T_\ell^{(k)} : k = 1,\ldots,4,\} \subsetneq \mathcal{D}_X$, where $T^{(0)}, T_\ell^{(k)}, k = 1,\ldots,4$ are defined as in Lemma 2.3 (taking m of the lemma to be $r-2$). Identifying X with a finite group and using Lemma 2.3, then we get $\mathcal{T}_{X,n} \subsetneq < \mathcal{D}_X >$, too. On the other hand, we have by definition $\{F : X^{n+1} \to X^{n+1} \mid F(x_1,\ldots,x_{n+1}) = (x_{n+1},x_1,\ldots,x_{r-1},f(x_1,x_{r-1(\bmod n+1)}),x_{r+1},\ldots,x_n),$ $f : X^2 \to X, i \in \{1,\ldots,n+1\},\ (x_1,\ \ldots,x_{n+1}) \in X^{n+1}\} \in \mathcal{D}_X$. But then $\{F : X^{n+1} \to X^{n+1} \mid F(x_1,\ldots,x_{n+1}) = (x_1,\ldots,x_{i-1},\ f(x_i,x_{i+1(\bmod n+1)}),x_{i+1},\ldots, x_{n+1}), f : X^2 \to X, i \in \{1,\ldots,n+1\},\ (x_1,\ldots,x_{n+1}) \in X^{n+1}\} \cup \mathcal{T}_{X,n+1} \subseteq \mathcal{D}_X$ resulting $\Gamma_{X^n} \subseteq \mathcal{D}_X$. Applying Lemma 2.2, this shows the n-completeness of $\mathcal{D}$.

Using the obvious fact that n-complete digraph should have a strongly connected n-complete subdigraph, by our minimality conditions, we will consider digraphs which have a strongly connected subdigraph and all vertices outside of this digraph are isolated. Thus, the sufficiency of our statement implies that by our minimality conditions, we can restrict our investigations to the strongly connected n-complete digraphs having not more than $n+2$ edges. (We can take out of consideration the isolated vertices.) If we have $n+1$ vertices and fewer than $n+1$ edges then our digraph is not strongly connected. On the other hand, if we consider a strongly connected digraph $\mathcal{D}$ with $n+1$ vertices and $n+1$ edges, i.e., a cycle having $n+1$ length, then for every $F \in< \mathcal{D}_X >$, there exist $k \in \{1, \ldots, n+1\}$, $f_i : X \to X, i = 1, \ldots, n+1$ with $F(x_1, \ldots, x_n) = (f_1(x_k), f_2(x_{k+1(\bmod (n+1))}), \ldots, f_{n+1}(x_{n+k(\bmod n+1)})$ $(x_1, \ldots, x_n \in X)$. Therefore, for any $1 \leq i_1 < i_2 < \ldots < i_m \leq n+1$, $pr_{i_1,\ldots,i_m}(F(x_1, \ldots, x_{n+1}))$ $= (f_{i_1}(x_{i_1+k(\bmod n+1)}), \ldots, f_{i_m}(x_{i_m+k(\bmod n+1)}))$, $(x_1, \ldots, x_{n+1} \in X)$ which obviously shows that this type of digraphs can not be n-complete.

Therefore, to the necessity of our statement, we can consider only strongly connected digraphs having $n+1$ vertices and $n+2$ edges.

By the strongly connectivity of $\mathcal{D}$ we may suppose that $\mathcal{D} = (V, E)$, with $|V| = n+1, |E| = n+2$, has a cycle $\mathcal{C} = (V', E')$ with k length for some $1 \leq k \leq n+1$, where $V' = \{v_1, \ldots, v_k\}(\subsetneq V)$, $E' = \{(v_i, v_{i+1(\bmod k)}) \mid i = 1, \ldots, k\}(\subsetneq E)$.

Using the strongly connectivity of $\mathcal{D}$ again, for every $V' \subsetneq V$ there are distinct (v_i, v_j), $(v_s, v_t) \in E$ with $v_i, v_t \in V', v_j, v_s \in V \setminus V'$. Therefore, by an induction we get the structure of $\mathcal{D}$ in the following manner.

If $k < n+1$ then $V = \{v_1, \ldots, v_k, v_{k+1}, \ldots, v_{n+1}\}, E = E' \cup \{(v_{k+i-1}, v_{k+i}) \mid i = 1, \ldots, n-k+1\} \cup \{(v_{n+1}, v_\ell)\}$, where $\ell \in \{1, \ldots, k\}$ is arbitrarily fixed.

If $k = n+1$ then, of course, $V = V'$, and $E = E' \cup \{(v_{n+1}, v_\ell)\}$ for some $\ell \in \{2, \ldots, n+1\}$.

To complete the case $k = n+1$, first we study digraphs having the form $\mathcal{D} = (\{v_1, \ldots, v_{n+1}\}, \{(v_i, v_{i+1(\bmod n+1)}) : i \in \{1, \ldots, n+1\}\} \cup \{(v_1, v_\ell)\}$, where $\ell \in \{1, \ldots, n+1\}, \ell \neq 2$, such that $\ell - 2(\bmod\ n+1)$ and $n+1$ are not relative primes. Then $n+1$ has a divisor $d > 1$ such that for any mapping $F \in \mathcal{D}_X$, $F(x_1, \ldots, x_{n+1}) = (f_1(x_{i_{1,1}}, \ldots, x_{i_{1,j_1}}), \ldots, f_{n+1}(x_{i_{n+1,1}}, \ldots, x_{i_{n+1,j_{n+1}}}))$, where for every $w \in \{1, \ldots, n+1\}, u, v \in \{1, \ldots, j_w\}$, $i_{w,u} \equiv i_{w,v}(\bmod\ d), i_{w,u} \equiv w - 1(\bmod\ d)$ $(x_1, \ldots, x_n \in X)$. These hold for compatible maps, i.e. if $w \neq r$ then f_w depends only on x_{w-1}, otherwise $w = r$ and f_w depends only on x_{r-1} and x_1. It is also clear that every composition of such functions preserves this property. Therefore, for every $F \in< \mathcal{D}_X >$ and $i \in \{1, \ldots, n+1\}$, $pr_i(F)$ depends on proper divisor of $n+1$ many variables which is fewer than n. Therefore, digraphs having this like structures are not n-complete.

It is remained to study the case $k < n+1$. Then $V = \{v_1, \ldots, v_k, v_{k+1}, \ldots, v_{n+1}\}$, $E = E' \cup \{(v_{k+i-1}, v_{k+i}) : i = 1, \ldots, n-k+1\} \cup \{(v_{n+1}, v_\ell)\}$, where $\ell \in \{1, \ldots, k\}$ is arbitrarily fixed. Of course, if $k = 1$ or $\ell = 1$ then we have one of the cases discussed previously. Thus we assume $k, \ell \neq 1$.

Given a set X with $|X| \geq 2$, let $\mathcal{M}_X = \{F : X^n \to X^n : |X^n| - 1 \leq |\{F(x_1, \ldots, x_n) : (x_1, \ldots, x_n) \in X^n\}|(\leq |X^n|)\}$. Clearly, then for every $F : X^n \to X^n$, $F \in< \mathcal{M}_X >$.

To complete our proof, now we show that there exists a network $\mathcal{D}' = (V', E')$ with $|V'| = n, E' = V' \times V' \setminus \{(v_i, v_i) : v_i \in V'\}$ such that for every pair $F \in< \mathcal{D}_X >$, $H \subsetneq \{1, \ldots, n+1\}$, $|H| = n$, the existence of $pr_H(F)$ implies $pr_H(F) \in< \mathcal{D}'_X >$ whenever $pr_H(F) \in \mathcal{M}_X$ (where $\mathcal{D}'_X$ denotes the set of all functions of the form

$F : X^n \mapsto X^n$ to be compatible with $\mathcal{D}'$).

Observe that for every $F_m \in \mathcal{D}_X$ there are $f_j : X \to X, j = 1, \ldots, \ell-1, \ell+1, \ldots, n+1, f_\ell : X^2 \to X$ with $F_m(x_1, \ldots, x_n) = (f_1(x_k), f_2(x_1), \ldots, f_{\ell-1}(x_{\ell-2}), f_\ell(x_{\ell-1}, x_{n+1}), f_{\ell+1}(x_\ell), \ldots, f_{n+1}(x_n))((x_1, \ldots, x_{n+1}) \in X^{n+1})$. Therefore, $H = \{1, \ldots, n+1\} \setminus \{i\}$, $i \in \{1, \ldots, n+1\} \setminus \{\ell-1, n+1\}$ and $F = F_1 \circ \ldots \circ F_m, F_1, \ldots, F_m \in \mathcal{D}_X$ implies $|\{pr_H(F)(x_1, \ldots, x_n) : (x_1, \ldots, x_n) \in X^n\}| \leq |X^{n-1}|$. Hence, in this case $pr_H(F) \notin \mathcal{M}_X$. Thus we may assume $H = \{1, \ldots, n+1\} \setminus \{i\}$, $i \in \{\ell-1, n+1\}$. In addition, it is clear that by the structure of $\mathcal{D}$, for every $T \in \mathcal{D}_X$, $cp_1(T)$ and $cp_{k+1}(T)$ may really depend only on the same k^{th} variable of F.

Let $F = F_1 \circ \ldots \circ F_m$ with $F_1, \ldots, F_m \in \mathcal{D}_X$, such that, $pr_H(F) \in \mathcal{M}_X$ exists for a suitable $H = \{1, \ldots, i-1, i+1, \ldots, n+1\}$, $i \in \{\ell-1, n+1\}$.

First we suppose $m = 1$. Consider functions $f_j : X \to X, j \in \{1, \ldots, \ell-1, \ell+1, \ldots, n+1\}, f_\ell : X^2 \to X$ with $F(x_1, \ldots, x_{n+1}) = (f_1(x_k), f_2(x_1), \ldots, f_{\ell-1}(x_{\ell-2}), f_\ell(x_{\ell-1}, x_{n+1}), f_{\ell+1}(x_\ell), \ldots, f_{n+1}(x_n))$ $(x_1, \ldots, x_{n+1}) \in X^{n+1})$. Clearly, then $i \in \{1, k+1\}$ also holds provided $pr_H(F) \in \mathcal{M}_X$.

Suppose $i = 1$. Then in consequence of $i \in \{\ell-1, n+1\}$, we have $\ell = 2$. Clearly, then f_2 really may not depend on its first variable, i.e. there exists a $g : X \to X$ with $f_2(x_1, x_2) = g(x_2)$ $(x_1, x_2 \in X)$. Construct the function $T : X^n \to X^n$ with $T(x_1, \ldots, x_n) = (g(x_n), f_3(x_1), \ldots, f_{n+1}(x_{n-1}))$ $((x_1, \ldots, x_n) \in X^n)$. Then we get $pr_H(F) = T$. On the other side, $T \in \mathcal{D}'_X$ is also obvious.

Suppose $i = k+1$. By $i \in \{\ell-1, n+1\}$ and $\ell \leq k$, this implies $k = n$. On the other side, then f_ℓ really may not depend on its second variable, i.e. there exists a $g : X \to X$ with $f_\ell(x_1, x_2) = g(x_1)$ $(x_1, x_2 \in X)$: Let $T : X^n \to X^n$ with $T(x_1, \ldots, x_n) = (f_1(x_n), f_2(x_1), \ldots, f_{\ell-1}(x_{\ell-2}), g(x_{\ell-1}), f_{\ell+1}(x_\ell), \ldots, f_n(x_{n-1})$ $((x_1, \ldots, x_n) \in X^n)$. It is obvious that $T \in \mathcal{D}'_X$ and $pr_H(F) = T$.

Now we turn to the case $m > 1$. Then first we define the mappings $F'_1, \ldots, F'_m \in \mathcal{D}_X$ in the following way. For every $r = 1, \ldots, m$, define functions $f_r : X \mapsto X, g_r : X \mapsto X$ with $f_r(x) = pr_1(F_r(x_1, \ldots, x_{k-1}, x, x_{k+1}, \ldots, x_{n+1}))$, $g_r(x) = pr_{k+1}(F_r(x_1, \ldots, x_{k-1}, x, x_{k+1}, \ldots, x_{n+1}))$, $x_1, \ldots, x_{k-1}, x, x_{k+1}, \ldots, x_{n+1} \in X$. ($F_r \in \mathcal{D}_X$ implies that f_r and g_r are well-defined.) In addition, let for every $r = 1, \ldots, m$, $pr_j(F'_r(x_1, \ldots, x_{n+1})) =$

$$\begin{cases} f_1(x_k) & \text{if } r = 1 \text{ and } j = 1, \\ g_1(x_k) & \text{if } r = 1 \text{ and } j = k+1, \\ x_k & \text{if } r > 1 \text{ and } j \in \{1, k+1\}, \\ pr_j(F_m(x_1, \ldots, x_{n+1})) & \text{if } r = m \text{ and } j \in \{2, \ldots, k, k+2, \ldots, n+1\}, \\ pr_j(F_r(f_{r+1}(x_1), x_2, \ldots, x_k, & \\ \quad g_{r+1}(x_{k+1}), x_{k+2}, \ldots, x_{n+1})) & \text{otherwise} \end{cases}$$

$(x_1, \ldots, x_{n+1} \in X)$. By an easy computation we get $F_1 \circ \ldots \circ F_m = F'_1 \circ \ldots \circ F'_m$. Define for a fixed $c \in X$, $m = 2$, $pr_j(F''_r(x_1, \ldots, x_n))$

$$= \begin{cases} pr_j(F'_r(x_1, \ldots, x_{i-1}, c, x_i, \ldots, x_n)) & \text{if } 1 \leq j < i, \\ pr_{j+1}(F'_r(x_1, \ldots, x_{i-1}, c, x_i, \ldots, x_n)) & \text{if } i \leq j \leq n \end{cases}$$

$(x_1, \ldots, x_{n+1} \in X, r = 1, 2)$.

Similarly, for a fixed $c \in X$ and $m = 3$, let $pr_j(F_r''(x_1, \ldots, x_n))$

$$= \begin{cases} pr_j(F_1'(x_k, x_2, \ldots, x_{\ell-2}, \\ x_n, x_{\ell-1}, \ldots, x_{n-1}, x_1)) & \text{if } i = \ell - 1, r = 1 \text{ and } \\ & 1 \leq j < \ell - 1, \\ pr_{j+1}(F_1'(x_k, x_2, \ldots, x_{\ell-2}, \\ x_n, x_{\ell-1}, \ldots, x_{n-1}, x_1)) & \text{if } i = \ell - 1, r = 1 \text{ and } \\ & \ell - 1 \leq j \leq n, \\ pr_j(F_1'(x_{k+1}, x_2, \ldots, x_n)) & \text{if } i = n + 1, r = 1 \text{ and } \\ & 1 \leq j \leq n, \\ pr_{n+1}(F_2'(x_1, \ldots, x_{i-1}, c, x_i, \ldots, x_n)) & \text{if } r = 2 \text{ and } j = 1, \\ pr_j(F_2'(x_1, \ldots, x_{i-1}, c, x_i, \ldots, x_n)) & \text{if } r = 2 \text{ and } 1 < j < i, \\ pr_{j+1}(F_2'(x_1, \ldots, x_{i-1}, c, x_i, \ldots, x_n)) & \text{if } r = 2 \text{ and } i \leq j \leq n, \\ pr_j(F_3'(x_1, \ldots, x_{i-1}, c, x_i, \ldots, x_n)) & \text{if } r = 3 \text{ and } 1 \leq j < i, \\ pr_{j+1}(F_3'(x_1, \ldots, x_{i-1}, c, x_i, \ldots, x_n)) & \text{if } r = 3 \text{ and } i \leq j \leq n, \end{cases}$$

$(x_1, \ldots, x_{n+1} \in X, r = 1, 2, 3)$.

In addition, let for a fixed $c \in X$ and $m > 3$, $pr_j(F_r''(x_1, \ldots, x_n))$

$$= \begin{cases} pr_j(F_r'(x_k, x_2, \ldots, x_{\ell-2}, \\ x_n, x_{\ell-1}, \ldots, x_{n-1}, x_1)) & \text{if } i = \ell - 1, r = 1, 1 \leq j < \ell - 1, \\ & \text{or } i = \ell - 1, 1 < r \leq m - 2 \text{ and } \\ & 1 < j < \ell - 1, \\ pr_{j+1}(F_r'(x_k, x_2, \ldots, x_{\ell-2}, \\ x_n, x_{\ell-1}, \ldots, x_{n-1}, x_1)) & \text{if } i = \ell - 1, r = 1, \ell - 1 \leq j \leq n, \\ & \text{or } i = \ell - 1, 1 < r \leq m - 2 \text{ and } \\ & \ell - 1 \leq j \leq n, \\ pr_{n+1}(F_r'(x_k, x_2, \ldots, x_{\ell-2}, \\ x_n, x_{\ell-1}, \ldots, x_{n-1}, x_1)) & \text{if } i = \ell - 1, 1 < r \leq m - 2 \text{ and } \\ & j = 1, \\ pr_{n+1}(F_r'(x_{k+1}, x_2, \ldots, x_n)) & \text{if } i = n + 1, 1 < r \leq m - 2 \text{ and } \\ & j = 1, \\ pr_j(F_r'(x_{k+1}, x_2, \ldots, x_n)) & \text{if } i = n + 1, r = 1, 1 \leq j \leq n, \\ & \text{or } i = n + 1, 1 < r \leq m - 2 \text{ and } \\ & 1 < j \leq n, \\ pr_{n+1}(F_{m-1}'(x_1, \ldots, x_{i-1}, c, x_i, \ldots, x_n)) & \text{if } r = m - 1 \text{ and } j = 1, \\ pr_j(F_{m-1}'(x_1, \ldots, x_{i-1}, c, x_i, \ldots, x_n)) & \text{if } r = m - 1 \text{ and } 1 < j < i, \\ pr_{j+1}(F_{m-1}'(x_1, \ldots, x_{i-1}, c, x_i, \ldots, x_n)) & \text{if } r = m - 1 \text{ and } i \leq j \leq n, \\ pr_j(F_m'(x_1, \ldots, x_{i-1}, c, x_i, \ldots, x_n)) & \text{if } r = m \text{ and } 1 \leq j < i, \\ pr_{j+1}(F_m'(x_1, \ldots, x_{i-1}, c, x_i, \ldots, x_n)) & \text{if } r = m \text{ and } i \leq j \leq n, \end{cases}$$

$(x_1, \ldots, x_{n+1} \in X, r \in \{1, \ldots, n\}))$.

We remark that, of course, for every $j = 2, \ldots, m$, the value of $F_j'' \circ \ldots \circ F_m''(x_1, \ldots, x_m)$ $(x_1, \ldots, x_n \in X)$ may depend of the value of (the above fixed) $c \in X$. But the value of $F_1'' \circ \ldots \circ F_m''(x_1, \ldots, x_m)$ $(x_1, \ldots, x_n \in X)$ may not depend on the value of $c \in X$ in question, because $F_H = F_1'' \circ \ldots \circ F_m''$ by definition.

(Remember that the existence of $pr_H(F)$ $(= pr_H(F_1 \circ \ldots \circ F_m), m > 1)$ is supposed with $H = \{1, \ldots, i-1, i+1, \ldots, n+1\}$ for a fixed $i \in \{\ell - 1, n+1\}$.)

By an elementary computation we can prove $F_1'', \ldots, F_m'' \in \mathcal{D}'_X$. Applying Theorem 3.1, $\mathcal{D}'$ may not be n-complete because it is not centralized. Therefore, there exists a $T \in \mathcal{M}_X$ with $T \notin <\mathcal{D}'_X>$. But then for every $F \in <\mathcal{D}_X>$, $H = \{1, \ldots, n+1\}$, $|H| = n$, $pr_H(F) \neq T$. Therefore, $\mathcal{D}$ can not be n-complete.

This ends the proof. □

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