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A note on the star-product*

Balázs Imreh †   Masami Ito ‡

Dedicated to Professor Ferenc Gécseg on the occasion of his 60th birthday

Abstract

In this paper, we compare the repesenting power of the star-product and the members of two product hierarchies, namely, the α_i-products, $i = 0, 1, \ldots$ and the ν_j-products, $j = 1, 2, \ldots$. In particular, it is proved that the star-product is not isomorphically (homomorphically) more general than any member of this two product families.

The family of the α_i-products, $i = 0, 1, \ldots$, is introduced by F. Gécseg in [3], and a systematic summarizing of the results concerning this product family can be found in the monography [4]. Another product family, the ν_j-products, $j = 1, 2, \ldots$ appears in [2]. Finally, a further product called star-product is studied by M. Tchuente in [8]. The comparison of the representing power of different compositions was initiated by F. Gécseg in [3]. Here, following the idea suggested by him, we compare the representing power of the star-product and the members of the two product families for both the isomorphic and homomorphic representations, and it is shown that the star-product is not isomorphically (homomorphically) more general than the members of the considered families. The inverse problems remain open.

The paper is organized as follows. First, we recall the basic notions and notation. Then, we compare the star-product with the α_i-products, finally, we compare the star-product with the ν_j-products.

By an *automaton* we mean a triplet $\mathbf{A} = (A, X, \delta)$ where A and X are finite nonempty sets, the set of the *states* and the set of the *input symbols*, respectively, and $\delta : A \times X \to A$ is the *transition function*. An automaton $\mathbf{A}$ can be also defined as an algebra $\mathbf{A} = (A, X)$ in which each input symbol is realized as the unary operation $x^{\mathbf{A}} : A \to A$, $a \to \delta(a, x)$. Using the latter definition, the notions such as *subautomaton*, *isomorphism*, and *homomorphism* can be defined in the usual way.

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†Department of Informatics, József Attila University, Árpád tér 2, H-6720 Szeged, Hungary

‡Department of Mathematics, Faculty of Science, Kyoto Sangyo University, Kyoto 603, Japan

To define the α_i-product, let $i \geq 0$ be an arbitrarily fixed integer. Let $\mathbf{A}_t = (A_t, X_t)$, $t = 1, \ldots, k$, be automata. Take a nonempty finite set X and a family of mappings $\varphi_t : A_1 \times \cdots \times A_{t+i-1} \times X \to X_t$, $t = 1, \ldots, k$. By the *α_i-product* $\mathbf{B} = \prod_{t=1}^k \mathbf{A}_t(X, \varphi)$ we mean the automata $(\prod_{t=1}^k A_t, X)$ defined by

$$(a_1, \ldots, a_k)x^{\mathbf{B}} = (a_1 x_1^{\mathbf{A}_1}, \ldots, a_k x_k^{\mathbf{A}_k})$$

where $x_t = \varphi_t(a_1, \ldots, a_{t+i-1}, x)$, $t = 1, \ldots, k$, for all $(a_1, \ldots, a_k) \in \prod_{t=1}^k A_t$ and $x \in X$. We note that φ_1 has the form $\varphi_1 : X \to X_1$ when $i = 0$.

A further product family, the ν_j-products, $j = 1, 2, \ldots$, was introduced in [2]. To define this kind of product, let j be an arbitrary positive integer. Then, the *ν_j-product* of automata can be defined in a similar way as the α_i-products, but the family of mappings has the following form:

$$\varphi_t : A_1 \times \cdots \times A_k \times X \to X_t, \quad t = 1, \ldots, k,$$

where each mapping depends on at most j state variables.

A summarizing on the comparisons of the members of the two product families under both isomorphic and homomorphic representations can be found in [1].

To define the star-product, let the automata $\mathbf{A}_t = (A_t, X_t)$, $t = 1, \ldots, k$, be given. Take a nonemty finite set X and a family of mappings $\varphi_1 : A_1 \times \cdots \times A_k \times X \to X_1$ and $\varphi_t : A_1 \times A_t \times X \to X_t$, $t = 2, \ldots, k$. By a *star-product* $\mathbf{A} = (A, X) = \prod_{t=1}^k \mathbf{A}_t(X, \varphi)$ (see *e.g.* [6] or [8]) we mean the automaton $(\prod_{t=1}^k A_t, X)$ where

$$(a_1, \ldots, a_k)x^{\mathbf{A}} = (a x_1^{\mathbf{A}_1}, \ldots, a_k x_k^{\mathbf{A}_k})$$

with $x_1 = \varphi_1(a_1, \ldots, a_k, x)$ and $x_t = \varphi_t(a_1, a_t, x)$, $t = 2, \ldots, k$, for all $x \in X$ and $(a_1, \ldots, a_k) \in \prod_{s=1}^k A_s$.

Now, let β denote one of the products defined above. If in a β-product each component automaton is equal to a given automaton, then this β-product is called a *β-power*. For an arbitrary set $\mathcal{K}$ of automata, let us denote by

$S_\beta(\mathcal{K})$ the β-products of automata from $\mathcal{K}$,

$H(\mathcal{K})$ the homomorphic images of automata from $\mathcal{K}$,

$I(\mathcal{K})$ the isomorphic images of automata from $\mathcal{K}$,

$S(\mathcal{K})$ the subautoma of automata from $\mathcal{K}$.

The class $\mathcal{K}$ is called *isomorphically (homomorphically) complete with respect to the β-product* if $ISP_\beta(\mathcal{K})$ $(HSP_\beta(\mathcal{K}))$ is the class of all automata.

Let γ denote one of the products introduced in this paper and which differs from β. It is said that β is *isomorphically (homomorphically) more general than the γ-product* if $ISP_\gamma(\mathcal{K}) \subseteq ISP_\beta(\mathcal{K})$ $(HSP_\gamma(\mathcal{K}) \subseteq HSP_\beta(\mathcal{K}))$ is valid for every set $\mathcal{K}$ of automata, moreover, there exists a set $\mathcal{K}_0$ of automata such that $ISP_\gamma(\mathcal{K}_0) \subset ISP_\beta(\mathcal{K}_0)$ $(HSP_\gamma(\mathcal{K}_0) \subset HSP_\beta(\mathcal{K}_0))$.

Now, we are ready to present our results. Regarding the α_i-products and isomorphic representation, the following statement is valid.

Theorem 1. *The star-product is not isomorphically more general than the α_i-product, for all i, $i = 0, 1 \ldots$.*

Proof. For an arbitrary positive integer n, let us define the automaton $\mathbf{I}_n = (\{1, \ldots, n\}, \{x\})$ by $nx^{\mathbf{I}_n} = n$ and $jx^{\mathbf{I}_n} = j+1$, $j = 1, \ldots, n-1$. Furthermore, let the automaton $\mathbf{E} = (\{0,1\}, \{x, y\})$ be defined by $0y^{\mathbf{E}} = 0$, $0x^{\mathbf{E}} = 1x^{\mathbf{E}} = 1y^{\mathbf{E}} = 1$. It is known (*cf.* [4] or [7]) that $\mathcal{K} = \{\mathbf{E}\}$ is isomorphically complete for the class of nilpotent automata with respect to the α_0-product, *i.e.*, every nilpotent automaton can be embedded isomorphically into an α_0-power of $\mathbf{E}$. Hence, $\mathbf{I}_n \in ISP_{\alpha_0}(\{\mathbf{E}\})$, for every positive integer n.

Now, we will show that if $\mathbf{I}_n \in ISP_{\text{star}}(\{\mathbf{E}\})$, then $n \leq 4$. For this purpose, let us suppose that $\mathbf{I}_n$ can be embedded isomorphically into a star-power $\mathbf{A} = \prod_{t=1}^{s} \mathbf{E}_t(\{x\}, \varphi)$ where $\mathbf{E}_t$ denotes the t-th copy of $\mathbf{E}$. Without loss of generality, we may assume that s is minimal with this property, *i.e.*, if $\mathbf{I}_n$ is isomorphic to a subautomaton of a star-power of $\mathbf{E}$, then the number of the factors of the star-power considered is at least s. Let μ be a suitable isomorphism and $\mu(j) = (a_{j1}, \ldots, a_{js})$, $j = 1, \ldots, n$. If $a_{1t} = 1$ for some $t \in \{1, \ldots, s\}$, then $a_{jt} = 1$, $j = 1, \ldots, n$, by the definition of $\mathbf{E}$. Then, $\mathbf{E}_t$ can be omitted from the star-product $\mathbf{A}$ which contradicts the minimality of s. Consequently, we may assume that $a_{1t} = 0$, $t = 1, \ldots, s$. Furthermore, let us observe that if $a_{jr} = a_{jt}$, $j = 1, \ldots, n$, for some integers $r \neq t \in \{1, \ldots, s\}$, then omitting one of the automata $\mathbf{E}_r$ and $\mathbf{E}_t$, the automaton $\mathbf{I}_n$ can be embedded isomorphically into the remaining star-product which contradicts the minimality of s. Therefore, we may assume that the vectors $(a_{1t}, a_{2t}, \ldots, a_{nt})^T$, $t = 1, \ldots, s$, are pairwise different and none of them is equal to the n-dimensional zero vector. Now, let us classify the automata $\mathbf{E}_2, \mathbf{E}_3, \ldots, \mathbf{E}_s$ into the classes M_1 and M_2 dependig on the values of the mappings $\varphi_2, \varphi_3, \ldots, \varphi_s$ as follows:

$$M_1 = \{\mathbf{E}_t : 2 \leq t \leq s \ \ \& \ \ \varphi_t(0,0,x) = x\},$$

$$M_2 = \{\mathbf{E}_t : 2 \leq t \leq s \ \ \& \ \ \varphi_t(0,0,x) = y\}.$$

If $\mathbf{E}_t \in M_1$, then let us observe that $(a_{1t}, \ldots, a_{nt})^T = (0, 1, \ldots, 1)$, and hence, the star-product $\mathbf{A}$ may contain at most one element from M_1, by the minimality of s. If $\mathbf{E}_t \in M_2$, then there are two possibilities for $\varphi_t(1,0,x)$. If $\varphi_t(1,0,x) = y$, then $(a_{1t}, \ldots, a_{nt})^T = (0, \ldots, 0)$, and therefore, $\varphi_t(1,0,x) = x$ must hold, by the minimality of s. But in this case, $a_{1t} = a_{2t} = \ldots a_{it} = 0$, $a_{i+1,t} = \ldots = a_{nt} = 1$ for some $i > 2$, provided that $a_{11} = a_{21} = \ldots = a_{i-2,1} = 0$ and $a_{i-1,1} = \ldots = a_{n1} = 1$. Hence, the star-product $\mathbf{A}$ may contain only one automaton from M_2. Consequently, $s \leq 3$. Now, it is easy to see that starting from the 3-dimensional zero vector, $(0,0,0)(xx)^{\mathbf{A}} = (1,1,1)$ if $a_{21} = 1$ and $(0,0,0)(xxx)^{\mathbf{A}} = (1,1,1)$ if $a_{21} = 0$ and $a_{31} = 1$, furthermore, $a_{21} = a_{31} = 0$ is impossible. Thus, $n \leq 4$ which yields that $ISP_{\alpha_0}(\{\mathbf{E}\}) \not\subseteq ISP_{\text{star}}(\{\mathbf{E}\})$. On the other hand, $ISP_{\alpha_0}(\{\mathbf{E}\}) \subseteq ISP_{\alpha_i}(\{\mathbf{E}\})$, for all i, $i = 0, 1, \ldots$, and hence, $ISP_{\alpha_i}(\{\mathbf{E}\}) \not\subseteq ISP_{\mathrm{star}}(\{\mathbf{E}\})$ is valid for every nonnegative integer i. This completes the proof of our statement.

Remark 1. It is an open problem whether there exists a nonnegative integer i such that the α_i-product is isomorphically more general than the star-product.

Regarding the homomorphic representation, the following statement is valid.

Theorem 2. *The star-product is not homomorphically more general than the α_i-product, for all i, $i = 0, 1, \ldots$.*

Proof. For arbitrary integers $k > 0$ and $l \geq 0$, let the automaton $\mathbf{J}_{k,l} = (\{1, \ldots, k, k+1, \ldots, k+l\}, \{x\})$ be defined by $(k+l)x^{\mathbf{J}_{k,l}} = k$ and $jx^{\mathbf{J}_{k,l}} = j+1$, $j = 1, \ldots, k+l-1$. Then, it can be easily seen that $\mathbf{I}_n \in HSP_{\mathrm{star}}(\{\mathbf{E}\})$ if and only if $\mathbf{J}_{k,l} \in ISP_{\mathrm{star}}(\{\mathbf{E}\})$ for some $k \geq n$ and $l \geq 0$. By the proof of Theorem 1, $\mathbf{J}_{k,l} \in ISP_{\mathrm{star}}(\{\mathbf{E}\})$ implies $k \leq 4$. Therefore, $\mathbf{I}_n \in HSP_{\mathrm{star}}(\{\mathbf{E}\})$ yields that $n \leq 4$. On the other hand, $ISP_{\alpha_0}(\{\mathbf{E}\}) \subseteq HSP_{\alpha_0}(\{\mathbf{E}\})$, and thus, $\mathbf{I}_n \in HSP_{\alpha_0}(\{\mathbf{E}\})$ for every positive integer n. Now, we can obtain the validity of Theorem 2 in a similar way as above.

Remark 2. It is an open problem whether there is a nonnegative integer i such that the α_i-product is homomorphically more general than the star-product.

Regarding the ν_j-products, we have the following assertion for the isomorphic representation.

Theorem 3. *The star-product is not isomorphically more general than the ν_j-product, for all j, $j = 1, 2, \ldots$.*

Proof. For an arbitrary positive integer n, let us define the automaton $\mathbf{C}_n = (\{0, 1, \ldots, n-1\}, \{x\})$ by $ix^{\mathbf{C}_n} = i+1 \pmod{n}$. $\mathbf{C}_n$ is called a *counter of length n*. Furthermore, let $\mathbf{B} = (\{0, 1\}, \{x, y\})$ be defined by $0x^{\mathbf{B}} = 1x^{\mathbf{B}} = 1$, $1y^{\mathbf{B}} = 0y^{\mathbf{B}} = 0$. We show that $\mathbf{C}_n \in ISP_{\nu_1}(\{\mathbf{B}\})$ is valid for every positive integer n. For this purpose, let n be an arbitrary positive integer. Let us form the ν_1-power $\mathbf{B}^n(\{x\}, \varphi)$ as follows. For every $(a_1, \ldots, a_n) \in \{0, 1\}^n$ and $t \in \{2, \ldots, n\}$, let

$$\varphi_t(a_1, \ldots, a_n, x) = \varphi_t(a_{t-1}, x) = \begin{cases} x & \text{if } a_{t-1} = 1, \\ y & \text{otherwise,} \end{cases}$$

and

$$\varphi_1(a_1, \ldots, a_n, x) = \varphi_1(a_n, x) = \begin{cases} x & \text{if } a_n = 1, \\ y & \text{otherwise.} \end{cases}$$

Then, it is easy to prove that $\mathbf{C}_n$ can be embedded isomorphically into the defined ν_1-power, $\mathbf{B}^n(\{x\}, \varphi)$. For example, μ is an appropriate isomorphism where μ is defined by

$$\mu(0) = (1, 0, 0, \ldots, 0, 0)$$

$$\mu(1) = (0, 1, 0, \ldots, 1, 1)$$

$$\vdots \qquad\qquad \vdots$$

$$\mu(n-1) = (0, 0, 0, \ldots, 0, 1)$$

Now, we show that if $\mathbf{C}_n \in ISP_{\text{star}}(\{\mathbf{B}\})$, then $n \leq 2^{2^5-1}$. For this purpose, let us suppose that $\mathbf{C}_n$ can be embedded isomorphically into a star-power $\mathbf{A} = \prod_{t=1}^{s} \mathbf{B}_t(\{x\}, \varphi)$ where $\mathbf{B}_t$ denotes the t-th copy of $\mathbf{B}$. We may assume that s is minimal with this property. Let μ be a suitable isomorphism and $\mu(i) = (a_{i1}, \ldots, a_{is})$, $i = 0, 1, \ldots n-1$. Let us classify the automata $\mathbf{B}_2, \mathbf{B}_3, \ldots, \mathbf{B}_s$, into the classes M_{r,z_1,z_2,z_3,z_4}, $r \in \{0,1\}$, $z_l \in \{x, y\}$, $l = 1, \ldots, 4$ where

$$M_{r,z_1,z_2,z_3,z_4} = \{\mathbf{B}_t : 2 \leq t \leq s \ \ \& \ \ r = a_{0,t} \ \ \& \ \ \varphi_t(0,0,x) = z_1 \ \ \&$$
$$\varphi_t(0,1,x) = z_2 \ \ \& \ \ \varphi_t(1,0,x) = z_3 \ \ \& \ \ \varphi_t(1,1,x) = z_4\}$$

It is easy to show that if $\mathbf{B}_u$ and $\mathbf{B}_v$ are the same class for some integers $u, v \in \{2, 3, \ldots, s\}$, then one of them can be omitted from the star-product considered which contradicts the minimality of s. Consequently, $\mathbf{A}$ may contain at most one factor from each class. Then, $s \leq 2^5 + 1$, and therefore, $n \leq 2^{2^5+1}$. From this it follows that $ISP_{\nu_1}(\{\mathbf{B}\}) \not\subseteq ISP_{\text{star}}(\{\mathbf{B}\})$. On the other hand, by the definition of the ν_j-products, $ISP_{\nu_1}(\{\mathbf{B}\}) \subseteq ISP_{\nu_j}(\{\mathbf{B}\})$, for all j, $j = 1, 2, \ldots$, and hence, $ISP_{\nu_j}(\{\mathbf{B}\}) \not\subseteq ISP_{\text{star}}(\{\mathbf{B}\})$ is valid for every positive integer j.

Remark 3. It remains an open problem whether there exists a positive integer j such that the ν_j-product is isomorphically more general than the star-product.

For the homomorphic representation, we can conclude the following assertion.

Theorem 4. *The star-product is not homomorphically more general than the ν_j-product, for all j, $j = 1, 2, \ldots$.*

Proof. Since $ISP_{\nu_1}(\{\mathbf{B}\}) \subseteq HSP_{\nu_1}(\{\mathbf{B}\})$, we have $\mathbf{C}_n \in HSP_{\nu_1}(\{\mathbf{B}\})$, for every positive integer n. On the other hand, it is easy to see that $\mathbf{C}_n \in HSP_{\text{star}}(\{\mathbf{B}\})$ if and only if $\mathbf{C}_m \in ISP_{\text{star}}(\{\mathbf{B}\})$ for some multiple m of n. Thus, by the proof of Theorem 3, we obtain that $\mathbf{C}_n \in HSP_{\text{star}}(\{\mathbf{B}\})$ implies $n \leq 2^{2^5+1}$. This results in $HSP_{\nu_1}(\{\mathbf{B}\}) \not\subseteq HSP_{\text{star}}(\{\mathbf{B}\})$, and then, by the definition of the ν_j-product, we obtain the validity of Theorem 4.

Remark 4. It is an open problem whether there is a positive integer j such that the ν_j-product is homomorphically more general than the star-product.

Conjecture. We think so that the problems presented as open ones have negative answers, and thus, the star-product is incomparable with the members of the considered product families with respect to both the isomorphic and homomorphic representations. The proof of this conjecture may need further deep investigations.

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