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Acts Over Completely 0-Simple Semigroups

Avdeyev A. Yu. * Kozhukhov I. B. †

The aim of this work is to describe, in the set-theoretical and group-theoretical terms, all the acts (automata) over completely 0-simple semigroups and also over completely simple and zero semigroups. As the consequence of this results we obtain a description of all the acts over rectangular groups, rectangular bands, right (or left) groups, and right (or left) zero semigroups. Moreover, we find all the subacts of some mentioned acts. Our results generalize the results of [3]. Theorem 1, Proposition 2 and Corollaries 9, 10, 11 of this work were published in [1]. We give them for the sake of completeness. Theorem 4 was announced by the second author in [7], Corollary 6 – by both authors in [2].

Recall that a *right act* (or *right operand*, or *S-set*) over a semigroup S is a set X with a mapping $X \times S \to X$ (the image of (x,s) we denote xs) such that the axiom $(xs)t = x(st)$ is held (for $x \in X$, $s,t \in S$) [6]. This notation coincides, in fact, with the notation "Moore's automaton" $V = (A,Q,\delta)$ where A is the input alphabet, Q is the set of the states, and δ is the transition function [8]. For the act X, we may assume that $Q = X$, A is the set of generators of S, and $\delta(x,s) = xs$. The S-set X is called *unitary* if S has a unity and $x \cdot 1 = x$ for all $x \in X$.

If the semigroup S has a simple structure, all the S-acts can be described. For example, an act X over the cyclic semigroup $S = \langle a \rangle$ is an unar (X,f) [9], i.e., the set X with the mapping $f : X \to X$; we have $x \cdot a^i = f^i(x)$. Ésik and Imreh [5] described the subdirectly irreducible commutative automata. Babcsányi and Nagy [3] obtained a description of the automata X over a right group S in case when the following conditions are satisfied:

$$XS = X, \tag{1}$$

$$\forall x,y \in X \forall s,t \in S \quad (xs = xt \Rightarrow ys = yt). \tag{2}$$

The condition (2) is called "state-independence". In this work we describe the right group acts (automata) in general case, i.e., without assuming (1), (2).

The notations and definitions of semigroup theory can be found in [4]. A completely 0-simple semigroup is signed by $\mathcal{M}^0(G,I,\Lambda,P)$, completely simple semigroup – by $\mathcal{M}(G,I,\Lambda,P)$. Here G is a group, I and Λ are sets, $P = \|p_{\lambda i}\|$ is a sandwich-matrix ($i \in I$, $\lambda \in \Lambda$, $p_{\lambda i} \in G \cup \{0\}$ or $p_{\lambda i} \in G$ resp.). The non-zero

*kv. 105, k. 200-б, 103305 Zelenograd, Russia.
†kv. 51, k. 1209, 103460 Zelenograd, Russia.

elements of $\mathcal{M}^0(G, I, \Lambda, P)$ have a form $(g)_{i\lambda}$ (where $g \in G$, $i \in I$, $\lambda \in \Lambda$) and their multiplication is defined by the rule

$$(g)_{i\lambda} \cdot (h)_{j\mu} = \begin{cases} (gp_{\lambda j}h)_{i\mu} & \text{if } p_{\lambda j} \neq 0, \\ 0 & \text{if } p_{\lambda j} = 0. \end{cases}$$

Let A be a set and θ an equivalence on A. Then A/θ is the set of θ-classes and $a\theta$ is the class of the element $a \in A$. An equivalence θ and a subset $B \subseteq A$ are called *compatible* if $|a\theta \cap B| = 1$ for every $a \in A$; in this case, the set B is called a *transversal* of θ. Let $\varphi : A \to A$ be a mapping. The *kernel* $\ker\varphi$ and the *image* $\mathrm{im}\varphi$ are defined as usual:

$$\ker\varphi = \{(a, b) | \varphi(a) = \varphi(b)\},$$

$$\mathrm{im}\varphi = \{\varphi(a) | a \in A\}.$$

If $\varphi^2 = \varphi$, $\ker\varphi$ and $\mathrm{im}\varphi$ are compatible, the opposite is false. If A is a right act over a semigroup S and $s \in S$, $\ker s$ and $\mathrm{im} s$ are the kernel and the image of the mapping $a \mapsto as$.

The element z of an S-act X is called a *zero* if $zs = z$ for all $s \in S$. Of course, an act X may have no zero. If the act X has an unique zero, we denote it by 0. Let (X_α) be a family of the S-acts X_α. Then $\coprod\limits_\alpha X_\alpha$ is the *coproduct* (or *disjoint union*) of the acts X_α.

Let G be a group and H be a subgroup of G, not necessarily normal. Denote by G/H the set of the classes Hg where $g \in G$. The set G/H is an unitary right G-act with respect to the action $*$ where $Hg * g' = Hgg'$. Every unitary right act over the group G is obviously a disjoint union of *orbits* xG of the elements of X. It can be easily verified that every orbit is isomorphic (as a right G-act) to an act of form G/H for some subgroup H of G. Thus, we have the obvious assertion:

Lemma 1. *If G is a group and X is an unitary right G-act, then $X \cong \coprod\limits_\alpha (G/H_\alpha)$ where (H_α) is a family of subgroups of G.*

Recall some definitions of the semigroup theory.

Zero semigroup is a semigroup S with 0 such that $ab = 0$ for all $a, b \in S$.

Left zero semigroup (L) is a semigroup satisfying the identity $xy = x$.

Right zero semigroup (R) is a semigroup with identity $xy = y$.

Rectangular band ($L \times R$) is a semigroup determined by the identities $x^2 = x$, $xyz = xz$. It is known [4] that the rectangular band is isomorphic to a direct product of the left zero semigroup and the right zero one. Moreover, the rectangular band is isomorphic to the Rees matrix semigroup $\mathcal{M}(\{e\}, I, \Lambda, P)$ where $p_{\lambda i} = e$ for all $\lambda \in \Lambda$, $i \in I$.

Left group ($L \times G$) is a direct product of a group and a left zero semigroup.

Right group ($R \times G$) is a direct product of a group and a right zero semigroup.

We shall describe all the acts over zero semigroups. Let A be a set which is a disjoint union of some subsets A_α, i.e., $A = \cup\{A_\alpha | \alpha \in \Gamma\}$, B_α ($\alpha \in \Gamma$) is some subset of A_α, and b_α ($\alpha \in \Gamma$) is some element of B_α. Further, let S be a non-empty set and let φ_s, $s \in S$, be a family of mappings $\varphi_s : A \to A$ such that $\varphi_s(A_\alpha) \subseteq B_\alpha$ and $\varphi_s(B_\alpha) = \{b_\alpha\}$ for all $\alpha \in \Gamma$. Moreover, assume that there exists an element $\theta \in S$ such that $\varphi_\theta(A_\alpha) = \{b_\alpha\}$ for all $\alpha \in \Gamma$. If we put $st = \theta$ for all $s, t \in S$, then S turns a zero semigroup (with zero θ). Define the action of the semigroup S on the set A as follows: $as = \varphi_s(a)$ ($a \in A$, $s \in S$).

Theorem 2. *The set A is a right act over the zero semigroup S. Conversely, every right act over a zero semigroup can be obtained by this way.*

Proof. At the first we check that A is a right S-act. Indeed, let $a \in A$ and $s, t \in S$. Then $a \in A_\alpha$ for some $\alpha \in \Gamma$. We have $(as)t = \varphi_t(\varphi_s(a)) \in \varphi_t(B_\alpha) = \{b_\alpha\}$, i.e., $(as)t = b_\alpha$. Moreover, $a(st) = a\theta = \varphi_\theta(a) = b_\alpha$. Thus, $(as)t = a(st)$.

Conversely, let A be an arbitrary right act over the zero semigroup S and θ is the zero of S. Introduce the equivalence σ on S putting $a\sigma b \Leftrightarrow a\theta = b\theta$. The equivalence determines the partition $A = \cup\{A_\alpha | \alpha \in \Gamma\}$. Check that $A_\alpha s \subseteq A_\alpha$ for all $\alpha \in \Gamma$, $s \in S$. Indeed, let $a \in A_\alpha$, $s \in S$. As $(as)\theta = a(s\theta) = a\theta$, then $(as, a) \in \sigma$. Therefore, $as \in A_\alpha$. Thus, $A_\alpha s \subseteq A_\alpha$. Put $B_\alpha = A_\alpha S$ for any $\alpha \in \Gamma$. If $a, b \in A_\alpha$ and $(a, b) \in \sigma$, we have $a\theta = b\theta$, therefore $|A_\alpha \theta| = 1$, and hence $A_\alpha \theta = \{b_\alpha\}$ for some b_α. Define, for any $s \in S$, the mapping $\varphi_s : A \to A$ putting $\varphi_s(a) = as$ for $a \in A$. Then $\varphi_s(A_\alpha) \subseteq B_\alpha$ and $\varphi_s(B_\alpha) = \{b_\alpha\}$. The theorem is proved.

The following proposition gives a description of all subacts of the act over a zero semigroup. The statements can be easily checked, and the proofs are omitted.

Proposition 3. *Let $A = \cup\{A_\alpha | \alpha \in \Gamma\}$ be a right act over the zero semigroup S, and $B_\alpha = A_\alpha S$ for $\alpha \in \Gamma$, and $\{b_\alpha\} = B_\alpha S$. If $\Delta \subseteq \Gamma$ is a non-empty subset and $A'_\delta \subseteq A_\delta$ (for $\delta \in \Delta$) such that $A'_\delta s \subseteq A'_\delta$ for all $s \in S$, then the act $\cup\{A'_\delta | \delta \in \Delta\}$ is a subact of A. Conversely, every subact of A can be obtained by this way.*

Now we shall consider the case of the completely 0-simple semigroup $S = \mathcal{M}^0(G, I, \Lambda, P)$. We may assume without loss of generality that $1 \in I \cap \Lambda$ and $p_{11} = e$ where e is the unity of the group G. The following theorem describes all the acts over such semigroups. We require here that $0 \cdot s = x \cdot 0 = 0$ for all $s \in S$, $x \in X$ where X is a right S-act, and 0 denotes the zero of S and the zero of X. The assumption of the existence of zero does not restrict the generality because of the fact that every act can be complemented by zero.

Theorem 4. *Let $S = \mathcal{M}^0(G, I, \Lambda, P)$ be a completely simple semigroup and X be a set with some element 0 (it is called conditionally as zero). Further, let (H_α) be a family of subgroups of the group G, $Q = \bigsqcup_\alpha (G/H_\alpha)$ is the coproduct of the right G-acts, and $Q^0 = Q \sqcup 0$. Finally, let us suppose that, for $i \in I$ and $\lambda \in \Lambda$, the* *mappings* $\kappa_\lambda : Q^0 \to X$ *and* $\pi_i : X \to Q^0$ *are defined such that*

$$\kappa_\lambda(0) = 0; \quad \pi_i(0) = 0; \tag{3}$$

$$\pi_i(\kappa_\lambda(q)) = q * p_{\lambda i} \quad \textit{for all} \quad q \in Q^0. \tag{4}$$

Put, for $x \in X$ *and* $s = (g)_{i\lambda} \in S$,

$$x \cdot s = x \cdot (g)_{i\lambda} = \kappa_\lambda(\pi_i(x) * g) \quad \textit{and} \quad x \cdot 0 = 0. \tag{5}$$

Then X *is a right* S*-act with zero. Conversely, every right act with zero over a completely 0-simple semigroup can be obtained by this way.*

Proof. At the first, we shall check that the set X satisfying the written conditions is really a right S-act. Clearly, it is sufficient to prove that

$$(x \cdot (g)_{i\lambda}) \cdot (h)_{j\mu} = x \cdot ((g)_{i\lambda} \cdot (h)_{j\mu}). \tag{6}$$

We have

$$(x \cdot (g)_{i\lambda}) \cdot (h)_{j\mu} = \kappa_\lambda(\pi_i(x) * g) \cdot (h)_{j\mu} =$$
$$\kappa_\mu(\pi_j(\kappa_\lambda(\pi_i(x) * g)) * h) = \kappa_\mu(\pi_i(x) * g * p_{\lambda j} * h) =$$
$$= \begin{cases} 0, & \text{if } p_{\lambda j} = 0, \\ (gp_{\lambda j}h)_{i\mu} & \text{if } p_{\lambda j} \neq 0. \end{cases}$$

This implies (6).

Now, let X be a right S-act with zero. Put $Y = X \cdot (e)_{11}$. Define an action of the group G on the set Y as follows: $y * g = y \cdot (g)_{11}$ for $y \in Y$, $g \in G$. Because of condition $p_{11} = e$ we have $(y * g) * h = (y \cdot (g)_{11}) \cdot (h)_{11} = y \cdot ((g)_{11} \cdot (h)_{11}) = y \cdot (gh)_{11} = y * gh$. Moreover, $y * e = (x \cdot (e)_{11}) \cdot (e)_{11} = x \cdot (e)_{11} = y$. Therefore, Y is a unitary right G-act with zero. It follows from Lemma 1 that there exists a family of subgroups $H_\alpha \subseteq G$ and an isomorphism $\theta : Y \to Q^0 = \bigsqcup\limits_\alpha (G/H_\alpha) \bigsqcup 0$ of right G-acts.

Construct, for every $\lambda \in \Lambda$, the mapping $\kappa_\lambda : Q^0 \to X$ putting $\tau_\lambda(x) = x \cdot (e)_{1\lambda}$ and $\kappa_\lambda(q) = \tau_\lambda(\theta^{-1}(q))$ where $x \in X$, $q \in Q^0$. Then construct, for $i \in I$, the mapping $\pi_i : X \to Q^0$ putting $\pi_i(x) = \theta(x \cdot (e)_{i1})$ where $x \in X$. If $i \in I$, $\lambda \in \Lambda$, $q \in Q^0$, and $p_{\lambda i} \neq 0$, we obtain $\pi_i(\kappa_\lambda(q)) = \theta(\kappa_\lambda(q) \cdot (e)_{i1}) = \theta(\tau_\lambda(\theta^{-1}(q)) \cdot (e)_{i1}) = \theta(\theta^{-1}(q) \cdot (e)_{1\lambda} \cdot (e)_{i1}) = \theta(\theta^{-1}(q) \cdot (p_{\lambda i})_{11}) = \theta(\theta^{-1}(q) * p_{\lambda i}) = \theta(\theta^{-1}(q * p_{\lambda i})) = q * p_{\lambda i}$. If $p_{\lambda i} = 0$, we obtain $\pi_i(\kappa_\lambda(q)) = \theta(\theta^{-1}(q) \cdot (e)_{1\lambda} \cdot (e)_{i1}) = \theta(\theta^{-1}(q) \cdot 0) = 0 = q * 0 = q * p_{\lambda i}$. Therefore, the equality (4) is satisfied in any case.

Finally, we verify the equality (5). We have $\kappa_\lambda(\pi_i(x) * g) = \kappa_\lambda(\theta(x \cdot (e)_{i1}) * g) = \kappa_\lambda(\theta((x \cdot (e)_{i1}) * g)) = \kappa_\lambda(\theta(x \cdot (e)_{i1} \cdot (g)_{11})) = \kappa_\lambda(\theta(x \cdot (g)_{i1})) = \tau_\lambda(\theta^{-1}(\theta(x \cdot (g)_{i1}))) = \tau_\lambda(x \cdot (g)_{i1}) = x \cdot (g)_{i1} \cdot (e)_{1\lambda} = x \cdot (g)_{i\lambda}$. The theorem is proved.

Now we consider the case of the completely simple semigroup $S = \mathcal{M}(G, I, \Lambda, P)$. As before, we assume that $1 \in I \cap \Lambda$ and $p_{11} = e$ where e is the unity of the group G. Moreover, as the matrix P has only non-zero elements, then we may assume (without loss of generality) that some column and some row consists only of unities. Let $p_{\lambda 1} = p_{1i} = e$ for all $i \in I, \lambda \in \Lambda$. The description of the acts over completely simple semigroup is given by the following theorem.

Theorem 5. *Let X be a set, $S = \mathcal{M}(G, I, \Lambda, P)$ be a completely simple semigroup, (H_α) be a family of subgroups of G, and $Q = \bigsqcup_\alpha (G/H_\alpha)$ be the coproduct of G-acts. Suppose that, for every $i \in I$, an equivalence σ_i on X is given, for every $\lambda \in \Lambda$, a subset $X_\lambda \subseteq X$ is given, for $i \in I$, the mappings $\pi_i : X \to Q$, $\kappa_\lambda : Q \to X$ are given. Suppose that the following conditions hold (for $i \in I$, $\lambda \in \Lambda$, $x \in X$, $q \in Q$):*

$$\ker \pi_i = \sigma_i, \tag{7}$$

$$\operatorname{im} \kappa_\lambda = X_\lambda, \tag{8}$$

$$|X_\lambda \cap x\sigma_i| = 1, \tag{9}$$

$$(\pi_i \kappa_\lambda)(q) = q * p_{\lambda i}. \tag{10}$$

Put

$$x \cdot (g)_{i\lambda} = \kappa_\lambda(\pi_i(x) * g) \tag{11}$$

for $x \in X$, $(g)_{i\lambda} \in S$. Then X turns a right S-act. Conversely, every right act over the completely simple semigroup $S = \mathcal{M}(G, I, \Lambda, P)$ can be obtained by this way.

Proof. As is seen in the proof of THeorem 4, from the conditions (10) and (11), it can be shown that X is a right S-act.

Now we assume that X is an arbitrary right S-act. Put $e_i = (e)_{i1}$, $e_{i\lambda} = (p_{\lambda i}^{-1})_{i\lambda}$ for $i \in I$, $\lambda \in \Lambda$. Clearly, e_i and $e_{i\lambda}$ are idempotents. It is easy to check that $e_{i\lambda} e_i = e_i$, $e_i e_{i\lambda} = e_{i\lambda}$, $e_{1\lambda} e_{i\lambda} = e_{1\lambda}$ and $e_{i\lambda} e_{1\lambda}$. Put $X_\lambda = X e_{i\lambda}$. Then $X_\lambda = X e_{1\lambda} = X e_{1\lambda} e_{i\lambda} \subseteq X e_{i\lambda} = X_\lambda$. Then $X_\lambda = X e_{i\lambda}$ for any i.

For every $i \in I$, we put $\sigma_i = \{(x, y) \in X \times X | x e_i = y e_i\}$. Prove that

$$\forall x, y \in X \quad \forall \lambda \in \Lambda \quad (x e_i = y e_i \Leftrightarrow x e_{i\lambda} = y e_{i\lambda}). \tag{12}$$

Indeed, $x e_i = y e_i$ implies $x e_{i\lambda} = x e_i e_{i\lambda} = y e_i e_{i\lambda}$ and similarly $x e_{i\lambda} = y e_{i\lambda}$ implies $x e_i = y e_i$. Therefore the (12) holds.

We shall prove the property (9), i.e., every σ_i-class intersects with every X_λ in one element (in other words, X_λ is a set of representatives of σ_i). Let $x \in X$. Then from the above facts, we have that $x e_{i\lambda} \in X_\lambda$ and $(x e_{i\lambda}) e_i = x e_i$, so that $x e_{i\lambda} \in X_\lambda \cap x\sigma_i$. Then $X_\lambda \cap x\sigma_i \neq \emptyset$ (notice: if $s^2 = s$ and $x \in Xs$, then $xs = x$, since $x = us = us^2 = (us)s = xs$). Let $x, y \in X_\lambda$ with $(x, y) \in \sigma_i$. Then again from the above facts, we have $X = x e_{i\lambda} = y e_{i\lambda} = y$. Thus $X_\lambda \cap x\sigma_i = \{x e_{i\lambda}\}$ for every $x \in X$.

For $i \in I$ and $\lambda \in \Lambda$, let π_i and κ_λ be as in the proof of Theorem 4. Then we can similarly show that the conditions (10) and (11) hold. This completes the proof.

Corollary 6. *Let G be a group, X, L, R be sets, (H_α) be a family of subgroups of G, and $Q = \coprod\limits_\alpha (G/H_\alpha)$ be the coproduct of the right G-acts. Assume that the following objects are given:*
the equivalences σ_l on X for all $l \in L$,
the subsets $X_r \subseteq X$ for all $r \in R$,
the mappings $\pi_l : X \to Q$, $\kappa_r : Q \to X$ for all $l \in L$, $r \in R$
such that the following conditions hold (for $x \in X$, $l \in L$, $r \in R$):

$$\ker \pi_l = \sigma_l, \quad \operatorname{im} \kappa_r = X_r, \quad |X_r \cap x\sigma_l| = 1, \quad \pi_l \kappa_r = \mathrm{id}_Q.$$

Let $S = L \times G \times R$. Define the multiplication on S by the rule

$$(l, g, r) \cdot (l', g', r') = (l, gg', r')$$

and the action of S on X by the rule

$$x \cdot (l, g, r) = \kappa_r(\pi_l(x) * g).$$

Then S is a rectangular group and X is a right S-act. Conversely, every right act over a rectangular group can be obtained by this way.

Corollary 7. *Let G be a group, X and R be sets, (H_α) be a family of subgroups of G, and $Q = \coprod\limits_\alpha (G/H_\alpha)$ be the coproduct of the right G-acts. Assume that the following objects are given:*
the equivalence σ on X,
the subsets $X_r \subseteq X$ for all $r \in R$,
the mappings $\pi : X \to Q$, $\kappa_r : Q \to X$ for all $r \in R$
such that the following conditions hold (for $x \in X$, $r \in R$):

$$\ker \pi = \sigma, \quad \operatorname{im} \kappa_r = X_r, \quad |X_r \cap x\sigma| = 1, \quad \pi \kappa_r = \mathrm{id}_Q.$$

Let $S = G \times R$. Define the multiplication on S by the rule

$$(g, r) \cdot (g', r') = (gg', r')$$

and the action of S on X by the rule

$$x \cdot (g, r) = \kappa_r(\pi(x) * g).$$

Then S is a right group and X is an S-act. Conversely, every right act over a right group can be obtained by this way.

Remark. This corollary gives a description of all acts over the right groups, the "state-independence" and the condition (1) are not necessarily satisfied. Let us see what will be obtained in case when this conditions (1), (2) are fulfilled.

Let $S = G \times R$ and X be an S-act with the properties (1), (2). At first we notice that X_r is a unitary right G-act with respect to the operation $a * g = a \cdot (g, r)$ for $a \in X_r$, $g \in G$. Indeed, $u * g = a \cdot (g, r) = \kappa_r(\pi(a) * g) \in \mathrm{im}\kappa_r = X_r$, therefore $X_r * G \subseteq X_r$. Further, $a * e = \kappa_r(\pi(a) * e) = \kappa_r(\pi(a)) = a$. Finally, $a * (g_1 g_2) = a \cdot (g_1 g_2, r) = a \cdot ((g_1, r) \cdot (g_2, r)) = (a \cdot (g_1, r)) \cdot (g_2, r) = (a * g_1) * g_2$.

Now we notice that $X_r \cong Q$ as the right G-acts. Indeed, as $\pi\kappa_r = \mathrm{id}_Q$, then κ_r is an injection. It implies that κ_r is a bijection from Q onto $\mathrm{im}\kappa_r = X_r$. Moreover, $\kappa_r(q) * g = \kappa_r(q) \cdot (g, r) = \kappa_r(\pi(\kappa_r(q)) * g) = \kappa_r(q * g)$. Thus, κ_r is an isomorphism of X_r and Q.

The condition (1) implies that $X = \cup\{X_r | r \in R\}$. Check that X_r are disjoint. Let $X_r \cap X_{r'} \neq \emptyset$, and $a \in X_r \cap X_{r'}$. Then $a = \kappa_r(q) = \kappa_{r'}(q')$ for some $q, q' \in Q$. As $\pi(a) = \pi\kappa_r(q) = q$ and similarly $\pi(a) = q'$, then $q = q'$. Further, $a \cdot (e, r) = \kappa_r(\pi(a) * e) = \kappa_r \pi(a) = \kappa_r(q) = a$ and similarly $a \cdot (e, r') = a$. Because of the property (2) we have $x \cdot (e, r) = x \cdot (e, r')$ for all $x \in X$, i.e., $\kappa_r(\pi(x)) = \kappa_{r'}(\pi(x))$. Since π is surjective, we have $\kappa_r = \kappa_{r'}$, and hence $X_r = X_{r'}$. Thus, *X is a disjoint union of the pairwise isomorphic G-acts X_r*. This is the main result of [3].

Corollary 8. *Let G be a group, X and L be sets, (H_α) be a family of subgroups of G, and $Q = \coprod_\alpha (G/H_\alpha)$ be the coproduct of the right G-acts. Assume that the following objects are given:*

the equivalences σ_l on X for all $l \in L$,

the subset $Y \subseteq X$,

the mappings $\pi_l : X \to Q$, $\kappa : Q \to X$ for all $l \in L$,

such that the following conditions hold (for $x \in X$, $l \in L$):

$$\ker \pi_l = \sigma_l, \quad \mathrm{im}\, \kappa = Y, \quad |Y \cap x\sigma_l| = 1, \quad \pi_l \kappa = \mathrm{id}_Q.$$

Let $S = L \times G$. Define the multiplication on S by the rule

$$(l, g) \cdot (l', g') = (l, gg')$$

and the action of S on X by the rule

$$x \cdot (l, g) = \kappa(\pi_l(x) * g).$$

Then S is a left group and X is a right S-act. Conversely, every right act over a left group can be obtained by this way.

Corollary 9 [1]. *Let X, L, R be sets. Assume that the following objects are given:*

the equivalences σ_l on X for all $l \in L$,

the subsets $X_r \subseteq X$ for all $r \in R$.

Also assume that the following conditions hold, for any $r, r' \in R$, $l, l' \in L$, $x \in X$:

$$|X_r \cap x\sigma_l| = 1, \tag{13}$$

$$\forall a \in X_r \quad \forall b \in X_{r'} \quad (a, b) \in \sigma_l \Leftrightarrow (a, b) \in \sigma_{l'}. \tag{14}$$

Define the multiplication on the set $S = L \times R$ *by the rule*

$$(l,r) \cdot (l',r') = (l,r')$$

and the action of S *on* X *by the rule*

$$a \cdot (l,r) = b \quad \text{where} \quad a\sigma_l \cap X_r = \{b\}.$$

Then S *is a rectangular band and* X *is a right* S*-act. Conversely, every right act over a rectangular band can be obtained by this way.*

Proof. We give the proof another than the proof of [1]. Clearly, the formulated rule determines a rectangular band. We shall prove that X is a right S-act. Indeed, let $a \in X$, $a \cdot (l,r) = b$, and $b \cdot (l',r') = c$. Then $a\sigma_l \cap X_r = \{b\}$ and $b\sigma_{l'} \cap X_{r'} = \{c\}$. We see that $(b,c) \in \sigma_{l'}$, therefore, because of the (14), $(b,c) \in \sigma_l$. As $(a,b) \in \sigma_l$, then $(a,c) \in \sigma_l$. Since $c \in X_{r'}$, $a \cdot (l,r') = c$. Thus, $a \cdot ((l,r) \cdot (l',r')) = a \cdot (l,r') = c = b \cdot (l',r') = (a \cdot (l,r)) \cdot (l',r')$. We see that X is a right S-act.

Further, we need to prove that the sets X_r and the equivalences σ_l of Corollary 6 satisfy to (14). Indeed, let $(a,b) \in \sigma_l$ where $a \in X_r$, $b \in X_{r'}$. Then we have $a = \kappa_r(q)$, $b = \kappa_{r'}(q')$ for some $q, q' \in Q$. As $(a,b) \in \sigma_l$, then $\pi_l(a) = \pi_l(b)$. We have $\pi_l(a) = \pi_l(\kappa_r(q)) = q$ and similarly $\pi_{l'}(a) = q$, $\pi_l(b) = \pi_{l'}(b) = q'$. As $\pi_l(a) = \pi_l(b)$, then $q = q'$. It implies $(a,b) \in \sigma_{l'}$.

We want to show that the Corollary 6 coincides with the Corollary 9 in case when $G = \{1\}$. Indeed, we may take $Q = X_{r_0}$ where $r_0 \in R$ is a fixed element and put $\pi_l(x) = y$ when $x\sigma_l \cap X_{r_0} = \{y\}$. Also we put $\kappa_r(q) = x$ when $q\sigma_l \cap X_r = \{x\}$ (the correctness, i.e., independence on l follows from (13) and (14): as $q \in X_{r_0}$, then $(q,x) \in \sigma_l \Leftrightarrow (q,x) \in \sigma_{l'} \Leftrightarrow q\sigma_{l'} \cap X_r = \{x\}$). It remains to show that $\pi_l\kappa_r = \mathrm{id}_Q$. Let $q \in Q$, $\kappa_r(q) = x$, and $\pi_l(x) = q'$. Then $q\sigma_l \cap X_r = \{x\}$ and $x\sigma_l \cap X_r = \{q'\}$. We have $(q',x) \in \sigma_l$. It follows that $q, q' \in X_{r_0} \cap x\sigma_l$. The condition (13) implies $q = q'$.

Corollary 10. *Let* X *and* S *be sets,* σ *be an equivalence on* X*, and* (X_s)*,* $s \in S$ *be a family of subsets of the set* X *such that* $|X_s \cap a\sigma| = 1$ *for all* $s \in S$*,* $a \in X$*. Define the multiplication on the set* S *by the rule* $st = t$ *for all* $s,t \in S$*, and define the action of* S *on* X *by the rule*

$$as = b \Leftrightarrow X_s \cap a\sigma = \{b\}.$$

Then S *is a right zero semigroup, and* X *is a right* S*-act. Conversely, every right act over a right zero semigroup can be obtained by this way.*

Corollary 11. *Let* X *and* S *be sets,* Y *be a non-empty subset of* X*, and* (σ_s)*,* $s \in S$ *be a family of the equivalences on* X *such that* $|Y \cap a\sigma_s| = 1$ *for all* $s \in S$*,* $a \in X$*. Define the multiplication on the set* S *by the rule* $st = s$ *for all* $s,t \in S$*, and define the action of* S *on* X *by the rule*

$$as = b \Leftrightarrow Y \cap a\sigma_s = \{b\}.$$

Then S is a left zero semigroup, and X is a right S-act. Conversely, every right act over a right zero semigroup can be obtained by this way.

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References

[1] Avdeyev A.Yu., Kozukhhov I.B. Acts over semigroups of simple structure. The 6th Conf. of Moscow State Social Univ. "Mathematical Methods and Applications". Abstr. of reports, p. 103-107. Moscow, 1999 (in Russian).

[2] Avdeyev A.Yu., Kozhukhov I.B. Acts over the rectangular groups. 12th Intern. Conf. "Problems of Theoretical Cybernetics" (Nizhny Novgorod, Russia, 1999). Abstr. of reports, p. 5. Moscow, 1999 (in Russian).

[3] Babcsányi I., Nagy A. Right group-type automata. Acta Cybernetica, 1995, 12, 131-136.

[4] Clifford A.H., Preston G.B. The Algebraic Theory of Semigroups. Amer. Math. Soc., Providence, I(1961), II(1967).

[5] Ésik Z., Imreh B. Subdirectly irreducible commutative automata. Acta cybernetica, 1981, 5, 251-260.

[6] Kilp M., Knauer U., Mikhalev A.V. Monoids, acts and categories. W. de Gruyter, Berlin – New York, 2000.

[7] Kozhukhov I.B. Acts over completely simple semigroups. Intern. Seminar "Universal Algebra and Its Applications" due to memory of Prof. Skornjakov L.A. (Volgograd, Russia, 1999). Abstr. of reports, p. 35-36. Volgograd, 1999 (in Russian).

[8] Kudryavtsev V.B., Podkolzin A.S. Introduction into Theory of Abstract Automata. Moscow, Moscow State Univ., 1985 (in Russian).

[9] Skornjakov L.A. Unars. Colloq. Math. Soc. János Bolyai. 29. Universal Algebra, Esztergom (Hungary), 1977, p. 735-742.

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