

The Logic of Knights, Knaves, Normals and Mutes

L. Aszalós *

Abstract

R. M. Smullyan wrote in his book about islands, knights and knaves. The knights always tell the truth and the knaves are always lying. Instead of *say* we shall examine the *can say* modal operator. We show the soundness and the completeness of this logic.

1 Introduction

At first we introduce the characters of the puzzles. Then we describe a logical language suitable to formulate puzzles. Later we prove soundness and completeness of this logic and eventually we show some interesting properties of this logic.

In Smullyan's famous book [2] the knights always tell the truth. Consequently they cannot say false statements. Smullyan does not mention any taboo in his puzzles, so we can assume that the knights can say any true statement. For the knaves the opposite holds, so they can say any false statement and can not say any true statement. We can arrange our information in columns:

	can say false statements	can say no false statements
can say true statements		knights
can say no true statements	knaves	

Later Smullyan introduced a third type of islanders: the normals, who sometimes tell the truth and sometimes lie. If we put this type into the table, one entry will remain empty. To fill this gap we need a new type of islanders, who can not say anything; hence we call them *mutés*. So the complete table is the following:

	can say false statements	can say no false statements
can say true statements	normals	knights
can say no true statements	knaves	mutés

* University of Debrecen, 4010 Debrecen, PO Box 12, Hungary, e-mail: aszalos@math.klte.hu

2 Syntax

In the following we shall use the well-known definition of the syntax of propositional logic:

Definition 1 *Let be S a finite set of propositional letters. The set of propositional formulae is the smallest set \mathcal{F} such that*

1. $S \subset \mathcal{F}$.
2. If $A \in \mathcal{F}$ then $\neg A \in \mathcal{F}$.
3. If $A, B \in \mathcal{F}$ then $(A \wedge B)$, $(A \vee B)$ and $(A \supset B) \in \mathcal{F}$.

In the definition above the connectives are the usual: \neg (negation), \vee (disjunction), \wedge (conjunction) and \supset (implication). To formulate the puzzles we need to express that the person x can say true statements, the person x can say false statements and the person x can say the statement A . For this we introduce T_x , F_x and $S_x A$, respectively. Definition 1. is extended to

Definition 2 *Let be \mathcal{P} a finite set. The set of formulae is the smallest set \mathcal{F} that satisfies 1-3. and*

4. If $x \in \mathcal{P}$ then $T_x \in \mathcal{F}$ and $F_x \in \mathcal{F}$.
5. If $x \in \mathcal{P}$ and $A \in \mathcal{F}$ then $S_x A \in \mathcal{F}$.

In this definition \mathcal{P} is the set of persons, and elements of \mathcal{P} will be denoted by a, b, \dots . This definition allows the embedding of S_x in the formulae, so for example $S_a \neg S_b T_a$ is a legal formula, which means that a can say that b cannot say that a can say true statements.

3 Semantics

In the propositional logic the prime components are the propositional letters. In our logic the truth value of a formula can depend on the type of persons, so the formulae describing the type of persons are prime components too. Hence the definition of the valuation will be more complicated than usual:

Definition 3 *Let $\vartheta_S \subset S$, $\vartheta_T \subset \mathcal{P}$ and $\vartheta_F \subset \mathcal{P}$. The valuation $\vartheta = \langle \vartheta_S, \vartheta_T, \vartheta_F \rangle$ assigns a truth value to every formula. If the formula A is true in a valuation ϑ this is denoted by $\vartheta \models A$.*

- If $P \in S$, $\vartheta \models P$ iff $P \in \vartheta_S$
- $\vartheta \models T_x$ iff $x \in \vartheta_T$
- $\vartheta \models F_x$ iff $x \in \vartheta_F$

- $\vartheta \models \neg A$ iff $\vartheta \not\models A$
- $\vartheta \models A \wedge B$ iff $\vartheta \models A$ and $\vartheta \models B$
- $\vartheta \models A \vee B$ iff $\vartheta \models A$ or $\vartheta \models B$
- $\vartheta \models A \supset B$ iff $\vartheta \not\models A$ or $\vartheta \models B$
- $\vartheta \models S_x A$ iff ($\vartheta \models T_x$ and $\vartheta \models A$) or ($\vartheta \models F_x$ and $\vartheta \not\models A$)

A formula A is satisfiable if there exists a valuation ϑ such that $\vartheta \models A$ and a formula A is valid if at every valuation ϑ , $\vartheta \models A$.

In ϑ model the sets of knights, knaves, normals and mutes are $\vartheta_T \cap \overline{\vartheta_F}$, $\overline{\vartheta_T} \cap \vartheta_F$, $\vartheta_T \cap \vartheta_F$ and $\overline{\vartheta_T} \cap \overline{\vartheta_F}$, respectively.

4 Sequent Calculus

In our proofs we shall use the sequent calculus described for example in [1, §48]. We shall use the notations and definitions of this book, but we shall give informally the basic definitions for those who are unfamiliar with this topic. We do not need the last four rules about quantifiers [1, p. 289], but we need two other rules about *can say*

$$\frac{\Gamma, T_x, A \longrightarrow \Theta ; \Gamma, F_x \longrightarrow A, \Theta}{\Gamma, S_x A \longrightarrow \Theta} \quad \text{and} \quad \frac{\Gamma, A \longrightarrow T_x, \Theta ; \Gamma, \longrightarrow A, F_x, \Theta}{\Gamma \longrightarrow S_x A, \Theta}$$

We say a sequent $\Gamma \longrightarrow \Theta$ is *falsifiable*, if there exists a valuation such that all formulae of Γ are true and all formulae of Θ are false.

5 Soundness

Theorem 4 For each of 12 rules: The sequent written below the line is falsifiable iff the sequent or at least one of the two sequents written above the line is falsifiable.

Proof. For the first 10 rules this was proven in [1], so we prove the claim only for rules of *can say*.

If $\Gamma, S_x A \longrightarrow \Theta$ is falsifiable then there exists a ϑ such that all formulae of Γ and $S_x A$ are true and all formulae of Θ are false in ϑ . If $S_x A$ is true then by definition either A and T_x are true or A is false and F_x is true. In the first case $\Gamma, T_x, A \longrightarrow \Theta$, in the other case $\Gamma, F_x \longrightarrow A, \Theta$ is falsifiable. To prove it in other direction

- 1) If $\Gamma, T_x, A \longrightarrow \Theta$ is falsifiable, then there exists a ϑ such that all formulae of Γ , T_x and A are true and all formulae of Θ are false in ϑ and by definition $S_x A$ is true in ϑ so $\Gamma, S_x A \longrightarrow \Theta$ is falsifiable.

- 2) If $\Gamma, F_x \rightarrow A, \Theta$ is falsifiable, then there exists a \mathcal{V} such that all formulae of Γ and F_x are true and all formulae of Θ and A are false in \mathcal{V} and by definition $S_x A$ is true in \mathcal{V} so $\Gamma, S_x A \rightarrow \Theta$ is falsifiable.

It is easy to check that $\mathcal{V} \not\models S_x A$ iff $(\mathcal{V} \models A \text{ and } \mathcal{V} \not\models T_x)$ or $(\mathcal{V} \not\models A \text{ and } \mathcal{V} \not\models F_x)$, and the proof about other rule is similar. \square

The axioms of the sequent calculus are $\Gamma, A \rightarrow A, \Theta$. This kind of sequent is not falsifiable, so it is valid. We can prove a formula A in the sequent calculus if we can construct a tree according to the rules such that each path ends in an axiom. Since all axioms are valid, we can go upside-down on the tree line by line and by the lemma above (which states that if the sequents above the line are valid then the sequent below the line is valid, too) all the sequents in the tree are valid; hence A is too. This proves the following theorem:

Theorem 5 *Each provable formula is valid.*

6 Completeness.

We want to prove that any valid formula is provable. At first we shall show that any proof-tree is finite. To do this we define a function:

Definition 6 *On the rank of a formula we understand a natural number such that*

- Rank of propositional letters are 0.
- If $x \in \mathcal{P}$ then the ranks of T_x and F_x are 0, too.
- If the rank of A is n , then rank of $\neg A$ and $S_x A$ are $n + 1$.
- If the rank of A is n and the rank of B is m then the rank of $A \wedge B$, $A \vee B$ and $A \supset B$ are $m + n + 1$, so the ranks of subformulae are smaller than the rank of the formulae.

The rank of a sequent and the rank of the set of formulae are the sum of the ranks of its formulae.

Lemma 7 *The rank of a sequent above the line is smaller than the rank of the sequent below the line.*

Proof. Let us show this only for one of the new rules. For the others the proof is similar. If the rank of Γ , Θ and $S_x A$ are n , m and l , respectively, then the rank of $\Gamma, S_x A \rightarrow \Theta$, $\Gamma, T_x, A \rightarrow \Theta$ and $\Gamma, F_x \rightarrow A, \Theta$ are $n + m + l$, $n + m + l - 1$ and $n + m + l - 1$, respectively. \square

When we construct a proof-tree then in each step we reduce the rank of the sequents. This can be done finitely many times, because the rank of the original formula was finite. This proves the following lemma.

Lemma 8 *Every proof-tree is finite.*

Theorem 9 *Every valid formula is provable.*

Proof. Let us assume that there is a valid formula A which is not provable. Take a maximal proof ending with the given formula A . By the lemma above its proof-tree is finite and since the formula is not provable, one path of the tree does not end with an axiom and no rule can be applied here, so this node contains only prime components, namely predicate letters, formulae of types T_x and F_x . The sequent is not axiom, so the two sets of formulae of this sequent are disjunct, hence falsifiable, and we only need to assign the value true to each formula to the left of the arrow (antecedent) and the value false to each formula to the right of the arrow (succedent). By theorem 4. the sequent below this is falsifiable, too, and repeating the process we get the original formula falsifiable, but we assumed that it was valid. We get a contradiction because we assumed that this formula was unprovable. \square

7 A puzzle and some properties.

It is hard to typeset the proof-trees in the original form so we shall use a different notation. We typeset

$$\frac{\Gamma, A \rightarrow \Theta; \Gamma, B \rightarrow \Theta}{\Gamma, A \vee B \rightarrow \Theta}$$

as

$$\begin{array}{c} \boxed{\Gamma, B \rightarrow \Theta} \\ \boxed{\Gamma, A \rightarrow \Theta} \\ \Gamma, A \vee B \rightarrow \Theta \end{array}$$

so the two paths are boxed and positioned vertically.

This logic is not as nice as the logic of belief or logic of knowledge. For example, we do not have here the two common properties **T** and **4**. Fig. 1. contains the proofs. The problematic paths are denoted by a star.

Smullyan did not examined this logic, so no puzzles are for it. After Smullyan it is hard to invent new puzzles but we shall try it: *We met three islander: A, B and C. A said that B cannot say that C is a knight. B said that C cannot say that A is a knight. C said that A cannot say that B is a knight. Let us prove that at least one of them isn't a knight.* We formulate this puzzle by the following formula: $S_a \neg S_b (T_c \wedge \neg F_c) \wedge S_b \neg S_c (T_a \wedge \neg F_a) \wedge S_c \neg S_a (T_b \wedge \neg F_b) \supset \neg (T_a \wedge \neg F_a) \vee \neg (T_b \wedge \neg F_b) \vee \neg (T_c \wedge \neg F_c)$. Without the first step our proof is in Fig. 2.

We have seen in Fig. 1. that two properties are lacking. This is also true for many of usual properties, but the formula $S_x(A \supset B) \supset (S_x A \supset S_x B)$, also known as **K**, is still valid. We prove this in Fig. 3.

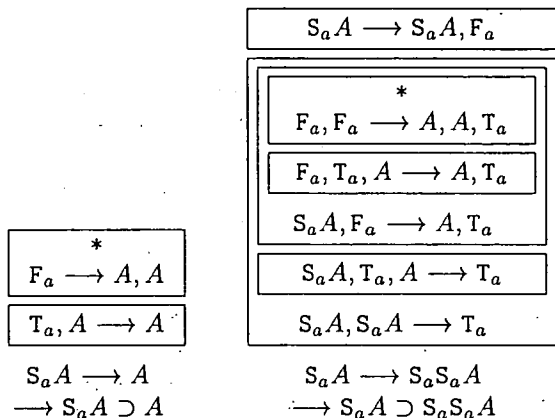
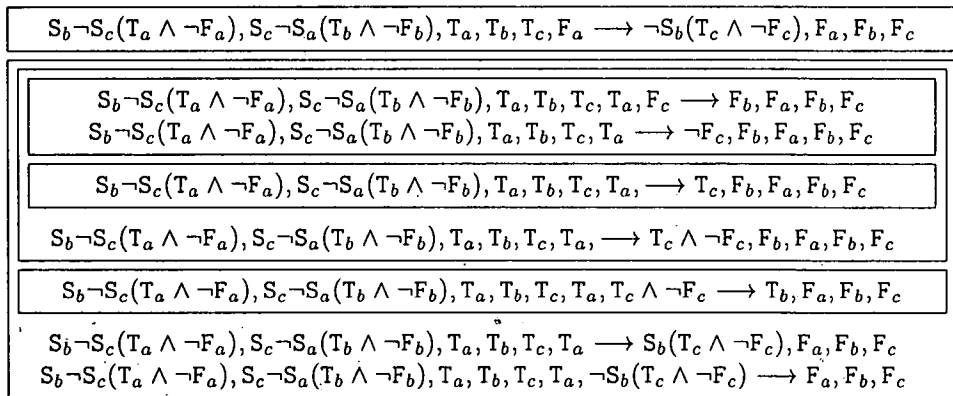
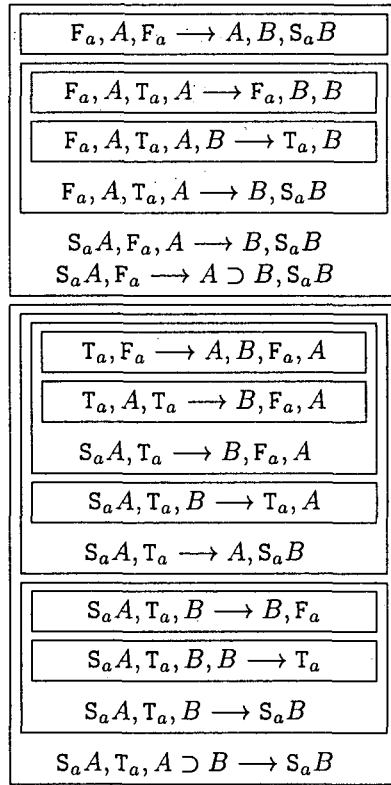


Fig. 1.



$S_a \neg S_b (T_c \wedge \neg F_c), S_b \neg S_c (T_a \wedge \neg F_a), S_c \neg S_a (T_b \wedge \neg F_b), T_a, T_b, T_c \rightarrow F_a, F_b, F_c$
 $S_a \neg S_b (T_c \wedge \neg F_c); S_b \neg S_c (T_a \wedge \neg F_a), S_c \neg S_a (T_b \wedge \neg F_b), T_a, \neg F_a, T_b, \neg F_b, T_c, \neg F_c \rightarrow$

Fig. 2.



$$\begin{aligned}
 & S_a(A \supset B), S_a A \rightarrow S_a B \\
 & S_a(A \supset B) \rightarrow S_a A \supset S_a B \\
 & \rightarrow S_a(A \supset B) \supset (S_a A \supset S_a B)
 \end{aligned}$$

Fig. 3.

8 Acknowledgements

The author is grateful to K. Pásztor-Varga and A. Kron for helpful comments and useful discussions. We are also want to pay tribute to A. G. Dragalin who guided us before his death in 1998.

References

- [1] S. C. Kleene. *Mathematical Logic*. John Wiley & Sons, Inc., 1967.
- [2] R. M. Smullyan. *What is the name of this book? (The riddle of Dracula and other logical puzzles)*. Prentice Hall, Inc., 1978.

Received October, 1999