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Equivalence of Mealy and Moore Automata

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Abstract

It is proved here that every Mealy automaton is a homomorphic image of a Moore automaton, and among these Moore automata (up to isomorphism) there exists a unique one which is a homomorphic image of the others. A unique simple Moore automaton **M** is constructed (up to isomorphism) in the set $MO(\mathbf{A})$ of all Moore automata equivalent to a Mealy automaton **A** such that **M** is a homomorphic image of every Moore automaton belonging to $MO(\mathbf{A})$. By the help of this construction, it can be decided in steps $|X|^k$ that automaton mappings inducing by states of a k-uniform finite Mealy [Moore] automaton are equal or not. The structures of simple k-uniform Mealy [Moore] automata are described by the results of [1]. It gives a possibility for us to get the k-uniform Mealy [Moore] automata from the simple k-uniform Mealy [Moore] automata. Based on these results, we give a construction for finite Mealy [Moore] automata.

1 Preliminaries

Let X be a nonempty set. A *Mealy automaton* (*over* X) is a system $\mathbf{A} = (A, X, Y, \delta, \lambda)$ consisting of a (nonempty) state set A, the input set X, a (nonempty) output set Y, a transition function $\delta : A \times X \to A$ and a surjective output function $\lambda : A \times X \to Y$.

A *Moore automaton* (*over* X) is a system $\mathbf{A} = (A, X, Y, \delta, \mu)$ consisting of a (nonempty) state set A, the input set X, a (nonempty) output set Y, a transition function $\delta : A \times X \to A$ and a surjective sign function $\mu : A \to Y$.

If A, X and Y are finite, the Mealy [Moore] automaton **A** is called *finite.*

For arbitrary Moore automaton $\mathbf{A} = (A, X, Y, \delta, \mu)$, the system $\mathbf{A}_\lambda = (A, X, Y, \delta, \lambda)$ with $\lambda = \mu\delta$ is a Mealy automaton over X. The Mealy automaton $\mathbf{A}_\lambda$ is called *the Mealy automaton associated with the Moore automaton* **A**. It is said that λ is *the output function of the Moore automaton* **A**. The Mealy automaton $\mathbf{A} = (A, X, Y, \delta, \lambda)$ fulfils the *Moore criterion* if

$$\delta(a_1, x_1) = \delta(a_2, x_2) \quad \Longrightarrow \quad \lambda(a_1, x_1) = \lambda(a_2, x_2)$$

for every $a_1, a_2 \in A$ and $x_1, x_2 \in X$. If $\mu : A \to Y$ is a surjective mapping such that $\lambda = \mu\delta$, the Moore automaton $\mathbf{A}_\mu = (A, X, Y, \delta, \mu)$ is called *a Moore*

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automaton associated with the Mealy automaton $\mathbf{A}$. Furthermore, we say that μ is *a sign function of the Mealy automaton* $\mathbf{A}$. We note that the output function λ is determined by restriction of μ to the subset $\delta(A,X) = \{\delta(a,x); a \in A, x \in X\}$ of A. Thus, the restrictions of all sign functions of the Mealy automaton $\mathbf{A}$ to $\delta(A,X)$ are equal. The Mealy automaton $\mathbf{A} = (A,X,Y,\delta,\lambda)$ is called *real* if there exist $a_1, a_2 \in A$ and $x_1, x_2 \in X$ such that

$$\delta(a_1,x_1) = \delta(a_2,x_2) \quad \text{and} \quad \lambda(a_1,x_1) \neq \lambda(a_2,x_2).$$

Let Z^* and Z^+ denote the free monoid and the free semigroup over a nonempty set Z, respectively. If $\mathbf{A} = (A,X,Y,\delta,\lambda)$ is a Mealy automaton, the functions δ and λ can be extended to $A \times X^*$ in the usual forms as follows:

$$\delta(a,e) = a, \quad \delta(a,px) = \delta(a,p)\delta(ap,x),$$

$$\lambda(a,e) = e, \quad \lambda(a,px) = \lambda(a,p)\lambda(ap,x),$$

where $a \in A$, $p \in X^+$, ap denotes the last letter of $\delta(a,p)$ and e denotes the empty word. ([5], [2]). If $\mathbf{A} = (A,X,Y,\delta,\mu)$ is a Moore automaton, the extension of δ is similar to the case when $\mathbf{A}$ is a Mealy automaton. The extension of μ to A^+ is given by

$$\mu(a_1a_2\ldots a_k) = \mu(a_1)\mu(a_2)\ldots\mu(a_k) \quad (a_1,a_2,\ldots,a_k \in A).$$

It means that if $\lambda = \mu\delta$, then

$$\lambda(a,p) = \mu(\delta(a,p)),$$

for all $a \in A$, $p \in X^+$. But $\lambda(a,e) = e$ and $\mu(\delta(a,e)) = \mu(a)$ for all $a \in A$.

The Mealy [Moore] automaton $\mathbf{A}' = (A',X,Y',\delta',\lambda'[\mu'])$ is a *subautomaton* of the Mealy [Moore] automaton $\mathbf{A}$ if $A' \subseteq A$, $Y' \subseteq Y$, δ' and λ' [μ'] are restrictions of δ and λ [μ] to $A' \times X$ [A'].

Let $\mathbf{A}_i = (A_i,X,Y,\delta_i,\lambda_i[\mu_i])$ $(i = 1,2)$ be arbitrary Mealy [Moore] automata over X. We say that a mapping $\varphi : A_1 \to A_2$ is a *homomorphism* of $\mathbf{A}_1$ into $\mathbf{A}_2$ if

$$\varphi(\delta_1(a,x)) = \delta_2(\varphi(a),x), \quad \lambda_1(a,x) = \lambda_2(\varphi(a),x) \quad [\mu_1(a) = \mu_2(\varphi(a))]$$

for all $a \in A$ and $x \in X$. It is easy to see that

$$\lambda_1(a,p) = \lambda_2(\varphi(a),p)$$

for all $p \in X^*$. The mapping $\varphi : A_1 \to A_2$ is called a *homomorphism* of a Moore automaton $\mathbf{A}_1$ into a Mealy automaton $\mathbf{A}_2$ if φ is a homomorphism of $(\mathbf{A}_1)_\lambda$ into $\mathbf{A}_2$. We note that every homomorphic image of a real Mealy automata is real, too.

Every state $a \in A$ of a Mealy automaton $\mathbf{A}$ induces a mapping $\alpha_a : X^* \to Y^*$ given by $\alpha_a(p) = \lambda(a,p)$ $(p \in X^*)$. The mapping $\alpha : X^* \to Y^*$ is called *automaton mapping* if there exist a Mealy automaton $\mathbf{A}$ and a state $a \in A$ such that $\alpha = \alpha_a$. The mapping $\alpha : X^* \to Y^*$ is an automaton mapping if and only if it preserves the length of words and the map of every prefix of a word is a prefix of the image word. The Mealy automata $\mathbf{A}$ and $\mathbf{B}$ are called *equivalent* if $\{\alpha_a; a \in A\} = \{\alpha_b; b \in B\}$. The Mealy automaton $\mathbf{A}$ and the Moore automaton $\mathbf{B}$ are *equivalent* if $\mathbf{A}$ and $\mathbf{B}_\lambda$ are equivalent. Similarly, the Moore automata $\mathbf{A}$ and $\mathbf{B}$ are *equivalent* if $\mathbf{A}_\lambda$ and $\mathbf{B}_\lambda$ are equivalent.

An equivalence relation ρ of state set A of a Mealy [Moore] automaton $\mathbf{A}$ is called a *congruence* on $\mathbf{A}$ if

$$(a,b) \in \rho \Longrightarrow (\delta(a,x), \delta(b,x)) \in \rho, \quad \lambda(a,x) = \lambda(b,x) \quad [\mu(a) = \mu(b)]$$

for all $a, b \in A$ and $x \in X$. The ρ-class of $\mathbf{A}$ containing the state a is denoted by $\rho[a]$. The greatest congruence on $\mathbf{A}$ is the relation $\rho_{\mathbf{A}}$ $[\pi_{\mathbf{A}}]$ defined by

$$(a,b) \in \rho_{\mathbf{A}}[\pi_{\mathbf{A}}] \quad \Longleftrightarrow \quad \lambda(a,p) = \lambda(b,p) \quad [\mu(\delta(a,p)) = \mu(\delta(b,p))]$$

for all $p \in X^*$. Denoting the identity relation on the state set A by ι_A, we say that $\mathbf{A}$ is *simple* if $\rho_{\mathbf{A}} = \iota_A$ $[\pi_{\mathbf{A}} = \iota_A]$, that is, $\mathbf{A}$ and $\mathbf{A}/\rho_{\mathbf{A}}$ $[\mathbf{A}/\pi_{\mathbf{A}}]$ are isomorphic.

Since every homomorphic image of a Mealy automaton $\mathbf{A}$ is equivalent to $\mathbf{A}$ ([5], [7]), therefore we can give the automaton mappings with simple Mealy automata. The Mealy automata $\mathbf{A}$ and $\mathbf{B}$ are equivalent if and only if $\mathbf{A}/\rho_{\mathbf{A}}$ and $\mathbf{B}/\rho_{\mathbf{B}}$ are isomorphic ([5]). Thus, simple Mealy automata are equivalent if and only if they are isomorphic. For every Mealy automaton $\mathbf{A}$, there exists a Moore automaton $\mathbf{B}$ such that $\mathbf{A}$ and $\mathbf{B}$ are equivalent ([4], [5], [6]). From this it follows that we can give the automaton mappings by simple Moore automata.

2 Moore automata equivalent to a Mealy automaton

For a Mealy automaton $\mathbf{A} = (A, X, Y, \delta, \lambda)$ over X, let us denote by $\mathbf{A}_Y = (A \times Y, X, Y, \delta_Y, \mu_Y)$ the Moore automaton over X for which

$$\delta_Y((a,y),x) = (\delta(a,x), \lambda(a,x)) \quad \text{and} \quad \mu_Y(a,y) = y \quad (a \in A, y \in Y, x \in X).$$

If $\lambda_Y = \mu_Y \delta_Y$, then

$$\lambda_Y((a,y),x) = \mu_Y(\delta_Y((a,y),x)) = \mu_Y(\delta(a,x), \lambda(a,x)) = \lambda(a,x)$$

for every $a \in A, y \in Y, x \in X$, and hence, $\mathbf{A}_Y$ is equivalent to $\mathbf{A}$.

Lemma 1 *If the Mealy automaton $\mathbf{A}'$ is a homomorphic [isomorphic] image of the Mealy automaton $\mathbf{A}$, then $\mathbf{A}'_Y$ is a homomorphic [isomorphic] image of $\mathbf{A}_Y$.*

Proof. If φ is a homomorphism [isomorphism] of $\mathbf{A}$ onto $\mathbf{A}'$, the mapping $\psi : A \times Y \to A' \times Y$, such that

$$\psi(a,y) = (\varphi(a), y) \quad (a \in A, \; y \in Y),$$

is a homomorphism [isomorphism] of $\mathbf{A}_Y$ onto $\mathbf{A}'_Y$.

Consider the subautomata $\mathbf{M} = (M, X, Y, \delta'_Y, \mu'_Y)$ of $\mathbf{A}_Y$ where for every $a \in A$ there exists $y \in Y$ such that $(a, y) \in M$. Let $M(\mathbf{A})$ be the set of all such subautomata $\mathbf{M}$.

Lemma 2 *The Mealy automaton $\boldsymbol{A}$ is a homomorphic image of every automaton $\boldsymbol{M}$ in $M(\boldsymbol{A})$.*

Proof. It is easy to see that the mapping $\varphi : M \to A$, defined by $\varphi(a, y) = a$ $(a \in A)$, is a homomorphism of $\mathbf{M}_\lambda$ onto $\mathbf{A}$.

Theorem 1 *The Mealy automaton $\boldsymbol{A}_1 = (A_1, X, Y, \delta_1, \lambda_1)$ is a homomorphic image of a Moore automaton $\boldsymbol{A}_2 = (A_2, X, Y, \delta_2, \mu_2)$ if and only if there exists a homomorphic image of $\boldsymbol{A}_2$ in $M(\boldsymbol{A}_1)$.*

Proof. First, we note that every automaton $\mathbf{M} \in M(\mathbf{A}_1)$ is a Moore automaton. By Lemma 2, if there exists a homomorphic image of $\mathbf{A}_2$ in $M(\mathbf{A}_1)$, then $\mathbf{A}_1$ is a homomorphic image of $\mathbf{A}_2$.

Conversely, assume that φ is a homomorphism of the Moore automaton $\mathbf{A}_2$ onto the Mealy automaton $\mathbf{A}_1$. It is evident that by the state set $M = \{(\varphi(b), \mu_2(b)); b \in A_2\}$,

$$\mathbf{M} = (M, X, Y, \delta'_Y, \mu'_Y) \in M(\mathbf{A}_1).$$

We show that the mapping $\psi : A_2 \to M$, defined by

$$\psi(b) = (\varphi(b), \mu_2(b)) \quad (b \in A_2),$$

is a homomorphism of $\mathbf{A}_2$ onto $\mathbf{M}$. It is obvious that the mapping ψ is surjective. For every $b \in A_2$ and $x \in X$

$$\psi(\delta_2(b,x)) = (\varphi(\delta_2(b,x)), \mu_2(\delta_2(b,x))) = (\delta_1(\varphi(b),x), \lambda_2(b,x)) =$$

$$= (\delta_1(\varphi(b),x), \lambda_1(\varphi(b),x)) = \delta'_Y((\varphi(b), \mu_2(b)), x) = \delta'_Y(\psi(b), x),$$

$$\mu_2(b) = \mu'_Y(\varphi(b), \mu_2(b)) = \mu'_Y(\psi(b)).$$

Therefore, ψ is a homomorphism.

Theorem 2 *For every Mealy automaton $\boldsymbol{A}$ (up to isomorphism) there exists a unique automaton $\boldsymbol{M} \in M(\boldsymbol{A})$ which is a homomorphic image of any automaton in $M(\boldsymbol{A})$.*

Proof. First, we give the automaton $\mathbf{M}$. If $A \neq \delta(A, X)$, let κ be a mapping of $A \setminus \delta(A, X)$ into Y. For all $a \in A$, consider the sets $Y_a \subseteq Y$ such that

$$\lambda(b,x) \in Y_a \iff \delta(b,x) = a \quad (b \in A, x \in X).$$

We define the sets M_a $(a \in A)$ as follows. If $a \in \delta(A, X)$, let $M_a = \{(a, y); y \in Y_a\}$, and if $a \notin \delta(A, X)$, let $M_a = \{(a, \kappa(a))\}$. Let $M = \cup_{a \in A} M_a$. Then $\mathbf{M} =$

$(M, X, Y, \delta_Y, \mu_Y) \in M(\mathbf{A})$. Let $M'(\mathbf{A})$ be the set of all such automata $\mathbf{M}$. If $A = \delta(A, X)$, then $|M'(\mathbf{A})| = 1$. We show that if $A \neq \delta(A, X)$, then all automata in $M'(\mathbf{A})$ are isomorphic. Assume that κ_i $(i = 1, 2)$ are arbitrary mappings of $A \setminus \delta(A, X)$ into Y and the automaton $\mathbf{M}_i \in M'(\mathbf{A})$ is defined by the mapping κ_i. It can be easily verified that the mapping $\varphi : M_1 \to M_2$, defined by

$$\varphi(a, y) = \begin{cases} (a, y) & \text{if } y \neq \kappa_1(a); \\ (a, \kappa_2(a)) & \text{if } y = \kappa_1(a), \end{cases}$$

is an isomorphism of $\mathbf{M}_1$ onto $\mathbf{M}_2$.

Now we show that for every $\mathbf{B} \in M(\mathbf{A})$, there is an $\mathbf{M} \in M'(\mathbf{A})$ such that $\mathbf{M}$ is a homomorphic image of $\mathbf{B}$. We define the following partition of the state set B:

$$B_a = \{(a, y); (a, y) \in B\} \quad (a \in A).$$

Take an automaton $\mathbf{M} \in M'(\mathbf{A})$ such that $M_a \subseteq B_a$ $(a \in A)$. By the definition of $M'(\mathbf{A})$, one can see that there exists such an automaton $\mathbf{M}$. Let ψ be an arbitrary mapping of B onto M for which

$$\{\psi(b); b \in B_a\} = M_a \quad \text{and} \quad \forall b \in M_a : \psi(b) = b.$$

It is clear that ψ is a homomorphism of $\mathbf{B}$ onto $\mathbf{M}$.

Lemma 3 ([7]) Let $\mathbf{A}$ be a Mealy automaton and $ME(\mathbf{A})$ be the set of all Mealy automata equivalent to $\mathbf{A}$. Then (up to isomorphism) there exists a unique simple Mealy automaton in $ME(\mathbf{A})$ which is a homomorphic image of every automaton in $ME(\mathbf{A})$.

We have a similar statement for Moore automata which are equivalent to a Mealy automaton.

Theorem 3 *Let $\mathbf{A}$ be a Mealy automaton and $MO(\mathbf{A})$ be the set of all Moore automata which are equivalent to $\mathbf{A}$. Then (up to isomorphism) there exists a unique simple Moore automaton in $MO(\mathbf{A})$ which is a homomorphic image of each automaton in $MO(\mathbf{A})$.*

Proof. Let $\mathbf{A}_0$ denote a simple Mealy automaton in $ME(\mathbf{A})$ which is homomorphic image of any automaton in $ME(\mathbf{A})$. By Lemma 3, such an automaton exists. Moreover, by Theorem 2, (up to isomorphism) there is a unique Moore automaton $\mathbf{M}_0 \in M(\mathbf{A}_0)$ which is homomorphic image of any automaton in $M(\mathbf{A}_0)$. Using the last fact, it can be seen that $\mathbf{M}_0$ is a simple Moore automaton.

Now, let $\mathbf{B}$ be an arbitrary Moore automaton equivalent to $\mathbf{A}$. We prove that $\mathbf{M}_0$ is a homomorphic image of $\mathbf{B}$. Since $\mathbf{B}$ is equivalent $\mathbf{A}$, $\mathbf{B}_\lambda \in ME(\mathbf{A})$, and hence, $\mathbf{A}_0$ is a homomorphic image of $\mathbf{B}$. This implies, by Theorem 1, that there is an $\mathbf{M} \in M(\mathbf{A}_0)$ such that $\mathbf{M}$ is a homomorphic image of $\mathbf{B}$, and therefore, $\mathbf{M}_0$ is a homomorphic image of $\mathbf{B}$ as well.

3 Uniform automata

Let $\mathbf{A} = (A, X, Y, \delta, \lambda[\mu])$ be a Mealy [Moore] automaton over X. Denote by $|p|$ the length of the word $p \in X^*$. Let $X^k = \{p \in X^*; |p| = k\}$ and $X(k) = \{p \in X^*; |p| \leq k\}$. For every nonnegative integer k, we define the equivalence relation η_k on A as follows:

$$(a,b) \in \eta_k \iff \lambda(a,p) = \lambda(b,p) \; [\mu(\delta(a,p)) = \mu(\delta(b,p))]$$

for all $p \in X(k)$. We note that if $\mathbf{A}$ is a Mealy automaton, the relation η_0 is the universal relation on A and η_1 is the output-equivalence of $\mathbf{A}$ ([2]). If $\mathbf{A}$ is a Moore automaton, η_0 is the sign-equivalence of $\mathbf{A}$ ([3]).

Lemma 4 *If a and b are arbitrary states of a Mealy [Moore] automaton $\mathbf{A} = (A, X, Y, \delta, \lambda[\mu])$, then*

$$(a,b) \in \eta_k \iff \lambda(a,p) = \lambda(b,p) \quad [\mu(a) = \mu(b), \; \lambda(a,p) = \lambda(b,p)]$$

for all $p \in X^k$.

Proof. If $(a,b) \in \eta_k$, the statement follows from the definition of η_k.

Conversely, assume that if $\mathbf{A}$ is a Mealy automaton, $\lambda(a,p) = \lambda(b,p)$, and if $\mathbf{A}$ is a Moore automaton, then $\mu(a) = \mu(b)$, $\lambda(a,p) = \lambda(b,p)$ holds for every $p \in X^k$. Take arbitrary words $q, r \in X^*$ such that $|q| \leq k$ and $|r| = k - |q|$. Then

$$\lambda(a,q)\lambda(aq,r) = \lambda(a,qr) = \lambda(b,qr) = \lambda(b,q)\lambda(bq,r).$$

Thus, $\lambda(a,q) = \lambda(b,q)$, which implies our statement.

The Mealy [Moore] automaton $\mathbf{A}$ is called *k-uniform* if $\eta_k = \rho_{\mathbf{A}}[\pi_{\mathbf{A}}]$. The k-uniform Mealy [Moore] automata are $(k+1)$-uniform. Every subautomaton of a k-uniform Mealy [Moore] automaton is k-uniform, too. An arbitrary homomorphic image of a Mealy [Moore] automaton is k-uniform if and only if it is k-uniform. The Mealy [Moore] automaton is said to be *uniform* if there exists a positive integer k such that it is k-uniform. Every finite Mealy [Moore] automaton is k-uniform for some positive integer k. Let α_a and α_b be automaton mappings induced by states a and b of a k-uniform finite Mealy [Moore] automaton $\mathbf{A} = (A, X, Y, \delta, \lambda[\mu])$, respectively. If $\alpha_a(p) = \alpha_b(p)$ for every $p \in X^k$, then $\alpha_a = \alpha_b$. Thus, it can be decided in $|X|^k$ steps whether two automaton mappings of this kind are equal or not.

Theorem 4 *If the Moore automaton $\mathbf{A} = (A, X, Y, \delta, \mu)$ is k-uniform, the Mealy automaton $\mathbf{A}_\lambda$ is (k+1)-uniform.*

Proof. We note that $\mathbf{A}_\lambda$ is (k+1)-uniform if and only if $\rho_{\mathbf{A}_\lambda} = \zeta_{k+1}$, where $(a,b) \in \zeta_{k+1}$ $(a, b \in A)$ if and only if $\lambda(a,p)) = \lambda(b,p))$ for all $p \in X(k+1)$.

Let the Moore automaton $\mathbf{A} = (A, X, Y, \delta, \mu)$ be k-uniform, that is, $\eta_k = \pi_{\mathbf{A}}$. Assume that $(a, b) \in \zeta_{k+1}$. Then,

$$\mu(\delta(a, x)) = \lambda(a, x) = \lambda(b, x) = \mu(\delta(b, x)),$$

$$\mu(\delta(\delta(a, x), q)) = \lambda(\delta(a, x), q) = \lambda(\delta(b, x), q) = \mu(\delta(\delta(b, x), q))$$

for every $x \in X$, $q \in X^k$. Thus, by Lemma 4, $(\delta(a, x), \delta(b, x)) \in \eta_k = \pi_{\mathbf{A}}$. This yields that

$$\lambda(\delta(a, x), r) = \mu(\delta(\delta(a, x), r)) = \mu(\delta(\delta(b, x), r)) = \lambda(\delta(b, x), r)$$

for all $r \in X^+$. Therefore, $(\delta(a, x), \delta(b, x)) \in \zeta_{k+1}$, that is, ζ_{k+1} is a congruence on $\mathbf{A}_\lambda$. Thus, $\zeta_{k+1} = \rho_{\mathbf{A}_\lambda}$. From this we get that $\mathbf{A}_\lambda$ is (k+1)-uniform.

Theorem 5 *The Mealy [Moore] automaton* $\mathbf{A} = (A, X, Y, \delta, \lambda[\mu])$ *is k-uniform if and only if* $\eta_k = \eta_{k+1}$.

Proof. Assume that the Mealy [Moore] automaton $\mathbf{A}$ is k-uniform, that is, $\eta_k = \rho_{\mathbf{A}}$. Since $\eta_{k+1} \subseteq \eta_k$ and $\cap_{k=0}^{\infty} \eta_k = \rho_{\mathbf{A}}[\pi_{\mathbf{A}}]$, therefore $\eta_k = \eta_{k+1}$.

Conversely, assume that $\eta_k = \eta_{k+1}$. If $\mathbf{A}$ is a Mealy automaton, η_0 is the universal relation on A. If $\eta_0 = \eta_1$, the relation η_1 is a congruence on $\mathbf{A}$. It yields that $\eta_0 = \eta_1 = \rho_{\mathbf{A}}$. Furthermore let us assume that $\mathbf{A}$ is a Mealy automaton and $1 \leq k$. Let $(a, b) \in \eta_k$. Since $\eta_k = \eta_{k+1}$, then $(a, b) \in \eta_{k+1}$. By Lemma 4, $\lambda(a, xp) = \lambda(b, xp)$ for every $x \in X$ and $p \in X^k$. From this it follows that

$$\lambda(\delta(a, x), p) = \lambda(\delta(b, x), p).$$

Moreover, if $\mathbf{A}$ is a Moore automaton,

$$\mu(\delta(a, x)) = \lambda(a, x) = \lambda(b, x) = \mu(\delta(b, x)),$$

that is, $(\delta(a, x), \delta(b, x)) \in \eta_k$. This results in that η_k is a congruence on $\mathbf{A}$, and so $\eta_k = \rho_{\mathbf{A}}[\pi_{\mathbf{A}}]$. Hence, $\mathbf{A}$ is k-uniform.

Lemma 5 *If a and b are arbitrary states of a Mealy [Moore] automaton* $\mathbf{A} = (A, X, Y, \delta, \lambda[\mu])$, *then*

$$(a, b) \in \eta_{k+1} \iff (a, b) \in \eta_k \quad \textit{and} \quad (\delta(a, x), \delta(b, x)) \in \eta_k, \quad \textit{for all} \quad x \in X.$$

Proof. Assume that $(a, b) \in \eta_{k+1}$. Since $\eta_{k+1} \subseteq \eta_k$, then $(a, b) \in \eta_k$. By Lemma 4, $\lambda(a, xp) = \lambda(b, xp)$ for every $x \in X$ and $p \in X^k$. But

$$\lambda(a, x)\lambda(\delta(a, x), p) = \lambda(a, xp) = \lambda(b, xp) = \lambda(b, x)\lambda(\delta(b, x), p),$$

and so

$$\lambda(\delta(a, x), p) = \lambda(\delta(b, x), p).$$

Moreover, if **A** is a Moore automaton,

$$\mu(\delta(a,x)) = \lambda(a,x) = \lambda(b,x) = \mu(\delta(b,x)).$$

By Lemma 4, this yields that $(\delta(a,x), \delta(b,x)) \in \eta_k$.

Conversely, assume that $(a,b) \in \eta_k$ and $(\delta(a,x), \delta(b,x)) \in \eta_k$ for every $x \in X$. If $x \in X$ and $q \in X^k$, then $\lambda(a,x) = \lambda(b,x)$ and $\lambda(\delta(a,x),q) = \lambda(\delta(b,x),q)$. From this it follows that

$$\lambda(a,xq) = \lambda(a,x)\lambda(\delta(a,x)q) = \lambda(b,x)\lambda(\delta(b,x),q) = \lambda(b,xq).$$

Moreover, if **A** is a Moore automaton, $\mu(a) = \mu(b)$. By Lemma 4, $(a,b) \in \eta_{k+1}$.

Theorem 6 *For every Mealy automaton* $\mathbf{A} = (A, X, Y, \delta, \lambda)$, $\mathbf{A}_Y$ *is k-uniform [simple] if and only if* **A** *is k-uniform [simple].*

Proof. If $a \in A$, $y \in Y$ and $p \in X^+$, then $\mu_Y(\delta_Y((a,y),p)) = \lambda(a,p)$.

We note that $\mathbf{A}_Y$ is k-uniform if $\pi_{\mathbf{A}_Y} = \zeta_k$, where ζ_k is an equivalence relation on $A \times Y$ for which

$$((a,y_1),(b,y_2)) \in \zeta_k \iff \mu_Y(\delta_Y((a,y_1),p)) = \mu_Y(\delta_Y((b,y_2),p))$$

for all $p \in X(k)$.

Assume that the Mealy automaton **A** is k-uniform. Consider two arbitrary elements (a,y_1) and (b,y_2) of $A \times Y$ with $((a,y_1),(b,y_2)) \in \zeta_k$. Then

$$y_1 = \mu_Y(a,y_1) = \mu_Y(b,y_2) = y_2,$$

$$\lambda(a,p) = \mu_Y(\delta_Y((a,y_1),p)) = \mu_Y(\delta_Y((b,y_2),p)) = \lambda(b,p)$$

for all $p \in X^k$. By Lemma 4, this implies $(a,b) \in \eta_k = \rho_{\mathbf{A}}$. By Theorem 5, $(a,b) \in \eta_{k+1}$, that is,

$$\mu_Y(\delta_Y((a,y_1),p)) = \lambda(a,p) = \lambda(b,p) = \mu_Y(\delta_Y((b,y_2),p))$$

for all $p \in X^{k+1}$ which results in $(a,b) \in \zeta_{k+1}$. Thus, $\zeta_k = \zeta_{k+1}$. By Theorem 5, $\mathbf{A}_Y$ is k-uniform.

Conversely, assume that $\mathbf{A}_Y$ is k-uniform. Let $(a,b) \in \eta_k$. If $y \in Y$, then $((a,y),(b,y)) \in \zeta_k = \pi_{\mathbf{A}_Y}$. By Theorem 5, $((a,y),(b,y) \in \zeta_{k+1}$, and thus $(a,b) \in \eta_{k+1}$. Therefore, $\eta_k = \eta_{k+1}$, that is, **A** is k-uniform.

We can prove, in a similar way, that $\mathbf{A}_Y$ is simple if and only if **A** is simple (see Lemma 2 in [1]).

By Theorem 6 and Lemma 2, every k-uniform Mealy automaton is equivalent to a k-uniform Moore automaton. By Theorem 3, among these Moore automata (up to isomorphism) there exists a unique simple k-uniform Moore automaton which is a homomorphic image of these Moore automata, that is, the cardinality of its state set is the least among these Moore automata.

4 Simple uniform automata

In this part of the paper, we describe the structure of the simple uniform Mealy [Moore] automata using the results of paper [1].

Lemma 6 (Lemma 3 in [1]) Every subautomaton of a simple Mealy [Moore] automaton $\mathbf{A}$ over X is simple and the subautomata of $\mathbf{A}$ are isomorphic if and only if they are equal.

Denote the set of mappings $\alpha^{(i)} : X^i \to Y$ by $\mathcal{A}^{(i)}$ for every integer $i > 0$. Consider the set $\mathcal{A} = \prod_{i=1}^{\infty} \mathcal{A}^{(i)}$. Let

$$\alpha = (\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(i)}, \ldots) \quad (\alpha^{(i)} \in \mathcal{A}^{(i)}),$$

$$\alpha_x^{(i)}(x_1, x_2, \ldots, x_i) = \alpha^{(i+1)}(x, x_1, x_2, \ldots, x_i) \quad (x, x_1, x_2, \ldots, x_i \in X),$$

$$\alpha_x = (\alpha_x^{(1)}, \alpha_x^{(2)}, \ldots, \alpha_x^{(i)}, \ldots).$$

Assume that $\alpha_e = \alpha$ and let $\alpha_{px} = (\alpha_p)_x$ for every $p \in X^*$ and $x \in X$. Define the Mealy automaton $\underline{\mathcal{A}} = (\mathcal{A}, X, Y, \delta, \lambda)$ with transition and output functions:

$$\delta(\alpha, x) = \alpha_x, \quad \lambda(\alpha, x) = \alpha^{(1)}(x) \quad (\alpha \in \mathcal{A},\ x \in X).$$

Theorem 7 (Theorem 4 in [1]) The Mealy automaton $\underline{\mathcal{A}}$ is simple. A Mealy automaton $\mathbf{A} = (A, X, Y', \delta, \lambda)$ over X is simple if and only if it is isomorphic to a subautomaton of $\underline{\mathcal{A}}$, where $Y' \subseteq Y$.

Theorem 8 (Theorem 5 in [1]) The Moore automaton $\underline{\mathcal{A}}_Y$ is simple and $\underline{\mathcal{A}}$ is a homomorpic image of $\underline{\mathcal{A}}_Y$. A Moore automaton $\mathbf{A} = (A, X, Y', \delta, \mu)$ $(Y' \subseteq Y)$ over X is simple if and only if it is isomorphic to a subautomaton of $\underline{\mathcal{A}}_Y$.

Consider the set $\mathcal{A}_k = \prod_{i=1}^{k} \mathcal{A}^{(i)}$ and a mapping $g : \mathcal{A}_k \to \mathcal{A}^{(k+1)}$. Let

$$\alpha_k = (\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(k)}) \quad (\alpha^{(i)} \in \mathcal{A}^{(i)}),$$

$$\alpha_{k,g,x} = (\alpha_x^{(1)}, \alpha_x^{(2)}, \ldots, \alpha_x^{(k)}),$$

where $\alpha^{(k+1)} = g(\alpha_k)$.

We define the Mealy automaton $\underline{\mathcal{A}}_{k,g} = (\mathcal{A}_k, X, Y, \delta, \lambda)$ with the following transition and output functions:

$$\delta(\alpha_k, x) = \alpha_{k,g,x}, \quad \lambda(\alpha_k, x) = \alpha^{(1)}(x) \quad (\alpha_k \in \mathcal{A}_k,\ x \in X).$$

Consider a nonempty set $H_0 \subseteq \mathcal{A}_k$. It is evident that

$$H_j = \{\alpha_{k,g,x}; \alpha_k \in H_{j-1}, x \in X\} \subseteq \mathcal{A}_k \ (j = 1, 2, \ldots).$$

If $H^{(j)} = H_0 \cup H_1 \cup \ldots \cup H_j$ for every nonnegative integer j, then $\mathbf{H}^{(j)}$ is a subautomaton of $\underline{\mathcal{A}}_{k,g}$ if and only if $H^{(j+1)} \subseteq H^{(j)}$. We note that if X and Y are finite sets, then there exists a nonnegative integer j such that $H^{(j+1)} \subseteq H^{(j)}$.

Theorem 9 *A Mealy automaton* **A** *over* X *is simple k-uniform if and only if there exists a mapping* $g : \mathcal{A}_k \to \mathcal{A}^{(k+1)}$ *such that* **A** *is isomorphic to some subautomaton of* $\underline{\mathcal{A}}_{k,g}$.

Proof. As in the proof of Theorem 7, we can show that the Mealy automaton $\underline{\mathcal{A}}_{k,g}$ is simple. By Lemma 6, every subautomaton of $\underline{\mathcal{A}}_{k,g}$ is simple. On the other hand, it is easy to verify that the subautomata of $\underline{\mathcal{A}}_{k,g}$ are k-uniform.

Therefore, by Theorem 7, it is sufficient to show that every k-uniform subautomaton of $\underline{\mathcal{A}}$ is isomorphic to an automaton $\mathbf{H}^{(j)}$. Let $\underline{\mathcal{A}}' = (\mathcal{A}', X, Y', \delta_{\mathcal{A}'}, \lambda_{\mathcal{A}'})$ be a k-uniform subautomaton of $\underline{\mathcal{A}}$. Let

$$\alpha_k = (\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(k)})$$

for every $\alpha = (\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(k)}, \ldots) \in \mathcal{A}'$. Define a mapping $g : \mathcal{A}_k \to \mathcal{A}^{(k+1)}$ such that $g(\alpha_k) = \alpha^{(k+1)}$ for every $\alpha \in \mathcal{A}'$. Let $H_0 = \{\alpha_k; \alpha \in \mathcal{A}'\}$. Since $\underline{\mathcal{A}}'$ is a subautomaton of $\underline{\mathcal{A}}$, then $H_1 \subseteq H_0$. Thus, $\mathbf{H}^{(0)}$ is a subautomaton of $\underline{\mathcal{A}}_{k,g}$. The mapping $\varphi : \mathcal{A}' \to H_0$, for which $\varphi(\alpha) = \alpha_k$ $(\alpha \in \mathcal{A}')$, is an isomorphism of $\underline{\mathcal{A}}'$ onto $\mathbf{H}^{(0)}$.

Every finite Mealy [Moore] automaton is k-uniform for some nonnegative integer k. Thus, we get easily the following theorem from Theorem 9.

Theorem 10 *A finite Mealy automaton* **A** *over* X *is simple if and only if there exist a nonnegative integer* k *and a mapping* $g : \mathcal{A}_k \to \mathcal{A}^{(k+1)}$ *for which* **A** *is isomorphic to some subautomaton of* $\underline{\mathcal{A}}_{k,g}$.

By Theorems 6, 9 and 10, the following two theorems are true.

Theorem 11 *A Moore automaton* **A** *over* X *is simple k-uniform if and only if it is isomorphic to some subautomaton of* $(\underline{\mathcal{A}}_{k,g})_Y$.

Theorem 12 *A finite Moore automaton* **A** *over* X *is simple if and only if there exists a nonnegative integer* k *for which it is isomorphic to some subautomaton of* $(\underline{\mathcal{A}}_{k,g})_Y$.

Let $\underline{\mathcal{C}} = (\mathcal{C}, X, Y', \delta_{\mathcal{C}}, \lambda_{\mathcal{C}})$ be a subautomaton of the automaton $\underline{\mathcal{A}}$. Consider a family of nonempty sets U_α $(\alpha \in \mathcal{C})$ such that $U_\alpha \cap U_\beta = \emptyset$ if $\alpha \neq \beta$. Let $U_C = \cup_{\alpha \in \mathcal{C}} U_\alpha$. For all $x \in X$ and $\alpha \in \mathcal{C}$, let $\varphi_{\alpha,x}$ be a mapping of U_α into U_{α_x}. Define the functions $\delta_{U_C}(a, x) = \varphi_{\alpha,x}(a)$ and $\lambda_{U_C}(a, x) = \alpha^{(1)}(x)$ for all $a \in U_\alpha$, $\alpha \in \mathcal{C}$ and $x \in X$. It can be easily verified that $\mathbf{U}_C = (U_C, X, Y', \delta_{U_C}, \lambda_{U_C})$ is a Mealy automaton ([2]).

Lemma 7 *Every Mealy automaton* $\mathbf{A} = (A, X, Y', \delta, \lambda)$ $(Y' \subseteq Y)$ *equals an automaton* $\mathbf{U}_C$.

Proof. By Theorem 7, there exists an isomorphism φ of $\mathbf{A}/\rho_{\mathbf{A}}$ onto a subautomaton $\underline{C}$ of $\underline{A}$. Assume that $\varphi(\rho_{\mathbf{A}}[a]) = \alpha_a$, $U_{\alpha_a} = \rho_{\mathbf{A}}[a]$ $(a \in A)$, $\varphi_{\alpha_a,x} = \delta(a,x)$ and $U_C = \cup_{a\in A} U_{\alpha_a}$. Since

$$\lambda(a,x) = \lambda_{A/\rho_{\mathbf{A}}}(\rho_{\mathbf{A}}[a],x) = \lambda_{\underline{C}}(\alpha_a,x) = \alpha_a^{(1)}(x) = \lambda_{U_C}(a,x),$$

therefore $\mathbf{A} = \mathbf{U}_C$.

Theorem 13 *The automaton U_C is k-uniform if and only if $\underline{C}$ is simple k-uniform.*

Proof. It is evident that if the automaton $\mathbf{U}_C$ is k-uniform, then $\underline{C}$ is simple k-uniform.

Conversely, assume that the automaton $\underline{C}$ is simple k-uniform. Assume that $(a,b) \in \eta_k$ for some $a \in U_\alpha$ and $b \in U_\beta$. Then, by Lemma 4, for every $p \in X^k$

$$\lambda_C(\alpha,p) = \lambda_{U_C}(a,p) = \lambda_{U_C}(b,p) = \lambda_C(\beta,p).$$

But $\underline{C}$ is simple k-uniform, thus $\alpha = \beta$, that is, $a,b \in U_\alpha$. It means that $ap, bp \in U_{\alpha_p}$. Then, for all $x \in X$,

$$\lambda_{U_C}(a,px) = \lambda_{U_C}(a,p)\lambda_{U_C}(ap,x) = \lambda_{U_C}(b,p)\lambda_{U_C}(bp,x) = \lambda_{U_C}(b,px),$$

that is $(a,b) \in \eta_{k+1}$. By Theorem 5, $\mathbf{U}$ is a k-uniform automaton.

By Theorems 6 and 13, we get the following theorem:

Theorem 14 *The automaton $(U_C)_Y$ is k-uniform if and only if $\underline{C}_Y$ is simple k-uniform.*

By Theorems 10 and 12, we give a construction for finite simple Mealy and Moore automata. Thus, by using Theorems 13 and 14, we can give all finite Mealy and Moore automata.

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