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On commutative asynchronous nondeterministic automata *

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Abstract

In this paper, we deal with nondeterministic automata, in particular, commutative asynchronous ones. Our goal is to give their isomorphic representation under the serial product or equivalently, under the α_0-product. It turns out that this class does not contain any finite isomorphically complete system with respect to the α_0-product. On the other hand, we present an isomorphically complete system for this class which consists of one monotone nondeterministic automaton of three elements.

1 Introduction

The study of the compositions of nondeterministic (n.d. for short) automata was initiated in the work [3], where the isomorphically complete systems with respect to the general product were characterized. In [4] it is proved that the general and cube products of n.d. automata are equivalent regarding the isomorphically complete systems. A further result on this line can be found in [7], where the isomorphically complete systems of n.d. automata with respect to the α_0-product are characterized.

In this work, a particular class of n.d. automata, the class of all commutative asynchronous n.d. automata, is studied. The isomorphic representation of the deterministic commutative asynchronous automata was studied in [8], where it turned out that every commutative asynchronous automaton can be embedded into a quasi-direct power of a suitable two-state commutative asynchronous automaton. We show here that this is not valid for the n.d. case, and what is more, it is not valid neither under the stronger α_0-product. On the other hand, it is proved that

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every commutative asynchronous n.d. automaton can be embedded into a suitable α_0-power of a monotone n.d. automaton having three states.

The paper is organized as follows. First, in Section 2, we recall a few notions and notation and present some basic results necessary in the sequal. In Section 3, it is shown that there is no finite system of commutative asynchronous n.d. automata which is isomorphically complete for the class under consideration with respect to the α_0-product. Then we look for a finite isomorphically complete system in a larger class, namely, in the class of monotone n.d. automata, and we prove that every commutative asynchronous n.d. automaton can be embedded into a suitable α_0-power of a monotone n.d. automaton of three states.

2 Preliminaries

An automaton can be defined as an algebra $\mathbf{A} = (A, X)$ in which every input sign x is realized as a unary operation $x^{\mathbf{A}} : A \to A$. Then the n.d. automata can be introduced as generalized automata in which the unary operations are replaced by binary relations. Therefore, by *n.d. automaton* we mean a system $\mathbf{A} = (A, X)$, where A is a finite nonvoid set of *states*, X is a finite nonempty set of *input signs*, and every $x \in X$ is realized as a binary relation $x^{\mathbf{A}} (\subseteq A \times A)$ on A. For any $a \in A$ and $x \in X$, let $ax^{\mathbf{A}} = \{c : c \in A \text{ and } (a, c) \in x^{\mathbf{A}}\}$, *i.e.*, $ax^{\mathbf{A}}$ is the set of states into which $\mathbf{A}$ may enter from a by receiving the input sign x. For any $C \subseteq A$ and $x \in X$, we set $Cx^{\mathbf{A}} = \cup\{ax^{\mathbf{A}} : a \in C\}$. For a word $w \in X^*$, $Cw^{\mathbf{A}}$ can be defined inductively as follows:

(1) $C\varepsilon^{\mathbf{A}} = C$,

(2) $Cw^{\mathbf{A}} = (Cv^{\mathbf{A}})x^{\mathbf{A}}$ for $w = vx$, $v \in X^*$ and $x \in X$,

where ε denotes the empty word of X^*. An n.d. automaton is called *complete* if $ax^{\mathbf{A}} \neq \emptyset$, for all $a \in A$ and $x \in X$. Throughout this paper, by n.d. automaton we always mean a complete n.d. automaton. Let $\mathbf{A} = (A, X)$ be an n.d. automaton and $B \subseteq A$. Then one can define a subautomaton $\mathbf{B} = (B, X)$ of $\mathbf{A}$ by the realizations $x^{\mathbf{B}} = x^{\mathbf{A}} \cap (B \times B)$, $x \in X$. We note that a subautomaton of a complete n.d. automaton is not necessarily complete. Let $\mathbf{A} = (A, X)$ and $\mathbf{B} = (B, X)$ be two n.d. automata and μ a mapping of A onto B. The mapping μ is called a *homomorphism* of $\mathbf{A}$ onto $\mathbf{B}$ if $ax^{\mathbf{A}}\mu = a\mu x^{\mathbf{B}}$ is valid, for all $a \in A$ and $x \in X$. In this case, it is said that $\mathbf{B}$ is a *homomorphic image* of $\mathbf{A}$. If the homorphism μ is a one-to-one mapping, then it is called an *isomorphism* and in this case, it is said that $\mathbf{A}$ is *isomorphic* to $\mathbf{B}$. Furthermore, if $\mathbf{B}$ is isomorphic to some subautomaton of $\mathbf{A}$, then it is said that $\mathbf{B}$ *can be embedded* into $\mathbf{A}$.

Let $\mathbf{A} = (A, X)$ be an n.d. automaton and Θ an equivalence relation on A. For every $a \in A$, let us denote by $\Theta(a)$ the equivalence class containing a, or equivalently, the set of the elements which are equivalent to a. Then we can construct a factor n.d. automaton $\mathbf{A}/\Theta$ as follows. For any $\Theta(a) \in A/\Theta$ and $x \in X$, let

$\Theta(a)x^{\mathbf{A}/\Theta} = \{\Theta(b) : \Theta(b) \in A/\Theta \text{ and } \Theta(a)x^{\mathbf{A}} \cap \Theta(b) \neq \emptyset\}$. It is worth noting that $\mathbf{A}/\Theta$ is not a homomorphic image of $\mathbf{A}$ in general. In what follows, we shall use particular equivalence relations. To define them, let A be an arbitrary nonempty set and a, b its two different elements. Then the equivalence relation $\Theta(a, b)$ is defined as follows. For every $u, v \in A$,

$$u\Theta(a,b)v \text{ if and only if } \{a,b\} = \{u,v\} \text{ or } u = v.$$

An n.d. automaton $\mathbf{A} = (A, X)$ is called *commutative* if $a(xy)^{\mathbf{A}} = a(yx)^{\mathbf{A}}$ is valid, for every $a \in A$ and $x, y \in X$. By the definition of the commutativity, one can easily prove the following fact.

Lemma 1. *If an n.d. automaton* $\mathbf{A}$ *is commutative and* $\mathbf{B}$ *is a homomorphic image of* $\mathbf{A}$*, then* $\mathbf{B}$ *is commutative as well.*

An n.d. automaton $\mathbf{A} = (A, X)$ is called *asynchronous* if for every $a \in A$ and $x \in X$, $b \in ax^{\mathbf{A}}$ implies $bx^{\mathbf{A}} = \{b\}$. In particular, if $a \in ax^{\mathbf{A}}$, then $ax^{\mathbf{A}} = \{a\}$. Since we recall this property in more times, we express it by the following remark.

Remark 1. *If* $\mathbf{A} = (A, X)$ *is an asynchronous n.d. automaton and* $a \in ax^{\mathbf{A}}$ *for some* $a \in A$ *and* $x \in X$*, then* $ax^{\mathbf{A}} = \{a\}$.

¿From the definition of the asynchronous n.d. automata the following fact follows immediately.

Lemma 2. *If an n.d. automaton* $\mathbf{A}$ *is asynchronous and* $\mathbf{B}$ *is a homomorphic image of* $\mathbf{A}$*, then* $\mathbf{B}$ *is also asynchronous.*

We shall study the commutative asynchronous n.d. automata. Let us denote by $\mathcal{K}_{nd}$ the class of all commutative asynchronous automata. Then, by Lemmas 1 and 2, we obtain the following observation.

Corollary 1. *If* $\mathbf{A} \in \mathcal{K}_{nd}$ *and* $\mathbf{B}$ *is a homomorphic image of* $\mathbf{A}$*, then* $\mathbf{B} \in \mathcal{K}_{nd}$.

An important property of the n.d. automata in $\mathcal{K}_{nd}$ is presented by the next assertion.

Lemma 3. *If* $\mathbf{A} = (A, X) \in \mathcal{K}_{nd}$*, then its transition graph does not contain any directed cycle different from loop.*

Proof. Let $a \in aq^{\mathbf{A}}$ for some $a \in A$, $q \in X^+$ and let q be a minimum-length word with this property. Now, let us suppose that $|q| > 1$. Then $q = xp$ for some $x \in X$ and $p \in X^+$. By the commutativity of $\mathbf{A}$, $a \in ap^{\mathbf{A}}x^{\mathbf{A}}$. Therefore, there exists a state b such that $b \in ap^{\mathbf{A}}$ and $a \in bx^{\mathbf{A}}$. Let us distinguish now the following two cases depending on b.

Case 1. $a = b$. Then $a \in ax^{\mathbf{A}}$, and by Remark 1, $ax^{\mathbf{A}} = \{a\}$ contradicting the minimality of the word q.

Case 2. $a \neq b$. In this case, $a \in bx^{\mathbf{A}}$. Since $\mathbf{A}$ is an asynchronous n.d. automaton, $a \in bx^{\mathbf{A}}$ implies $ax^{\mathbf{A}} = \{a\}$ which contradicts the minimality of q again.

Consequently, the transition graph of $\mathbf{A}$ does not contain any directed cycle different from loop.

Let $\mathbf{A} = (A, X)$ be an arbitrary n.d. automaton. Let us define the *reachability relation* as follows. For a couple of states a, b, it is said that b is *reachable* from a, denoted by $a \leq b$, if there exists a word w such that $b \in aw^{\mathbf{A}}$. Obviously, that this relation is reflexive and transitive. In particular, if $\mathbf{A} \in \mathcal{K}_{nd}$, then by Lemma 3, this relation is antisymmetric, and thus, it is a partial ordering on A. Hence, we have the following statement.

Corollary 2. *For every* $\mathbf{A} \in \mathcal{K}_{nd}$*,* $(A, \leq)$ *is a partially ordered set.*

The more general composition, the general product of automata was introduced by V. M. Gluskov in [6]. This composition is extended to n.d. automata in [3]. Now, we recall this definition.

Let us consider the n.d. automata $\mathbf{A} = (X, A)$, $\mathbf{A}_j = (X_j, A_j)$, $j = 1, \ldots, k$, and let Φ be a family of mappings below

$$\varphi_j : A_1 \times \ldots \times A_k \times X \to X_j, \quad j = 1, \ldots, k.$$

It is said that $\mathbf{A}$ is the *general product* of $\mathbf{A}_j$ with respect to Φ if the following conditions are satisfied:

(1) $A = \prod_{j=1}^{k} A_j$,

(2) for any $(a_1, \ldots, a_k) \in \prod_{j=1}^{k} A_j$, and $x \in X$,

$$(a_1, \ldots, a_k)x^{\mathbf{A}} = a_1 x_1^{\mathbf{A}_1} \times \cdots \times a_k x_k^{\mathbf{A}_k},$$

where $x_j = \varphi_j(a_1, \ldots, a_k, x)$ for all $j \in \{1, \ldots, k\}$.

For the general product above we use the notation

$$\mathbf{A} = \prod_{j=1}^{k} \mathbf{A}_j(X, \Phi) .$$

The mappings φ_j, $j = 1, \ldots, k$ are called *feedback functions.*

Let $\mathcal{K}$ be a system of n.d. automata. $\mathcal{K}$ is *isomorphically complete* with respect to the general product if for any n.d. automaton $\mathbf{A}$, there exist automata $\mathbf{A}_j \in \mathcal{K}$, $j = 1, \ldots, k$, such that $\mathbf{A}$ can be embedded into a general product of $\mathbf{A}_j$, $j = 1, \ldots, k$.

Different compositions of automata can be obtain as a special case of the general product by using particular feedback functions. One of them is the serial composition of automata, where the automata form a chain and the input sign of a given automaton of the chain depends on the input sign received by the composition and the current states of the previos automata in the chain. The formal definition can be given as follows.

Let $\mathbf{A}_j = (A_j, X_j)$, $j = 1, \ldots, k$ be arbitrary n.d. automata. Moreover, let X be a finite nonvoid set and Φ is a family of mappings:

$$\varphi_j : A_1 \times \cdots \times A_{j-1} \times X \to X_j, \quad j = 1, \ldots, k.$$

An n.d. automaton $\mathbf{A} = (A, X)$ is called the *serial product* or α_0*-product* of the n.d. automata considered, if $A = \prod_{j=1}^{k} A_j$ and for every $(a_1, \ldots, a_k) \in \prod_{j=1}^{k} A_j$ and $x \in X$,

$$(a_1, \ldots, a_k)x^{\mathbf{A}} = a_1 x_1^{\mathbf{A}_1} \times \cdots \times a_k x_k^{\mathbf{A}_k}$$

is valid, where $x_j = \varphi_j(a_1, \ldots, a_{j-1}, x)$, $j = 1, \ldots, k$. If the component n.d. automata $\mathbf{A}_j$ are equal, say $\mathbf{A}_j = \mathbf{B}$, $j = 1, \ldots, k$, then it is said that the α_0-product $\mathbf{A}$ is an α_0*-power* of $\mathbf{B}$. In particular, if the mappings φ_j, $j = 1, \ldots, k$ are independent of the states, *i.e.*, they have the forms $\varphi_j : X \to X_j$, $j = 1, \ldots, k$, then $\mathbf{A}$ is called the *quasi-direct product* of the n.d. automata under consideration.

It has to be mentioned here that as generalizations of the serial product of automata a family of products, the α_i-product, $i = 0, 1, \ldots$, was introduced in [1] for the deterministic case and some nice results concerning the α_i-products can be found in the monography [2].

By the definition of the α_0-product, one can easily prove the following statement.

Lemma 4. *If for every t, $t = 1, \ldots, n$, the n.d. automata $\mathbf{A}_t$ can be embedded into an α_0-product of n.d. automata $\mathbf{A}_{tj}$, $j = 1, \ldots, k_t$, then any α_0-product of the n.d. automata $\mathbf{A}_t$, $t = 1, \ldots, n$ can be embedded into an α_0-product of the n.d. automata $\mathbf{A}_{tj}$, $t = 1, \ldots, n$; $j = 1, \ldots, k_t$.*

Finally, we define the notion of isomorphically complete systems of n.d. automata for the α_0-product. For this purpose, let $\mathcal{K}$ be an arbitrary class of n.d. automata. A system $\mathcal{M}$ of n.d. automata is called *isomorphically complete for $\mathcal{K}$ with respect to the α_0-product* if any n.d. automaton in $\mathcal{K}$ can be embedded into an α_0-product of n.d. automata in $\mathcal{M}$.

3 Isomorphic representation

In this section, the isomorphic representation of the automata in $\mathcal{K}_{nd}$ are studied. The next statement shows that contrary to the deterministic case, the class $\mathcal{K}_{nd}$ does not contain any finite isomorphically complete system for $\mathcal{K}_{nd}$ with respect to the general product.

Proposition 1. *There is no finite system $\mathcal{M} \subseteq \mathcal{K}_{nd}$ of n.d. automata which is isomorphically complete for $\mathcal{K}_{nd}$ with respect to the general product.*

Proof. In our proof we shall use some particular automata. Namely, for all $n \geq 3$, let us define the n.d. automaton $\mathbf{C}_n = (\{1,\ldots,n\},\{x_2,\ldots,x_{n-1}\})$ as follows. For every $i \in \{1,\ldots,n\}$ and $x_k \in \{x_2,\ldots,x_{n-1}\}$, let

$$ix_k^{\mathbf{C}_n} = \begin{cases} \{k,k+1,\ldots,n\} & \text{if } i < k, \\ \{i\} & \text{otherwise.} \end{cases}$$

From the definition of $\mathbf{C}_n$ it follows that $\mathbf{C}_n$ is an asynchronous n.d. automaton. Now, we prove that $\mathbf{C}_n$ is commutative. For this reason, let $i \in \{1,\ldots,n\}$ and $x_j, x_k \in \{x_2,\ldots,x_{n-1}\}$ be arbitrary elements with $j \neq k$. Without loss of generality, we may suppose that $j < k$. Then, for the case $k \leq i$, we have that $ix_j^{\mathbf{C}_n} x_k^{\mathbf{C}_n} = \{i\} = ix_k^{\mathbf{C}_n} x_j^{\mathbf{C}_n}$. If $j \leq i < k$, then

$$ix_j^{\mathbf{C}_n} x_k^{\mathbf{C}_n} = \{i\}x_k^{\mathbf{C}_n} = \{k,k+1,\ldots,n\} = ix_k^{\mathbf{C}_n} x_j^{\mathbf{C}_n}.$$

Finally, if $i < j$, then

$$ix_j^{\mathbf{C}_n} x_k^{\mathbf{C}_n} = \{j,j+1,\ldots,n\}x_k^{\mathbf{C}_n} = \{k,k+1,\ldots,n\} = ix_k^{\mathbf{C}_n} = ix_k^{\mathbf{C}_n} x_j^{\mathbf{C}_n}.$$

These observations lead to the commutativity of $\mathbf{C}_n$. Consequently, $\mathbf{C}_n \in \mathcal{K}_{nd}$, for all integer $n \geq 3$.

For proving the statement, contrary, let us suppose that $\mathcal{M} \subseteq \mathcal{K}_{nd}$ is a finite isomorphically complete system for $\mathcal{K}_{nd}$ with respect to the general product. Then there exists an integer n such that $|A| < n$ is valid for every n.d. automaton $\mathbf{A} = (A, X) \in \mathcal{M}$. Since $\mathcal{M}$ is an isomorphically complete system for $\mathcal{K}_{nd}$ with respect to the general product and $\mathbf{C}_n \in \mathcal{K}_{nd}$, there are n.d. automata $\mathbf{A}_t \in \mathcal{M}$, $t = 1,\ldots,k$ such that $\mathbf{C}_n$ can be embedded into a general product $\prod_{t=1}^{k} \mathbf{A}_t(\{x_2,\ldots,x_{n-1}\},\Phi)$. Let μ denote a suitable isomorphism of $\mathbf{C}_n$ into the general product considered and let

$$i\mu = (a_{i1}, a_{i2}, \ldots, a_{ik}), \quad i = 1,\ldots,n.$$

Denote by r an integer for which $a_{n-1,r} \neq a_{nr}$. Such an integer exists. We shall now prove that the states $a_{1r}, a_{2r}, \ldots, a_{nr}$ are pairwise different. First, let us consider the state $a_{n-2,r}$. Since μ is an isomorphism, $a_{n-2,r}\varphi_r(a_{n-2,1},\ldots,a_{n-2,k},x_{n-1})^{\mathbf{A}_r} \cap \{a_{n-1,r}, a_{n,r}\} = \{a_{n-1,r}, a_{n,r}\}$. Thus, by $a_{n-1,r} \neq a_{nr}$ and Remark 1, we obtain that $a_{n-2,r} \notin \{a_{n-1,r}, a_{n,r}\}$. Therefore, $a_{n-2,r}, a_{n-1,r}, a_{nr}$ are pairwise different. Now, if for some integer $2 \leq i \leq n-2$, the elements $a_{n-i,r}, a_{n-i+1,r}, \ldots, a_{nr}$ are pairwise different, then in a similar way as above, we get that

$$a_{n-i-1,r}\varphi_r(a_{n-i-1,1},\ldots,a_{n-i-1,k},x_{n-i})^{\mathbf{A}_r} \supseteq \{a_{n-i,r}, a_{n-i+1,r}, \ldots, a_{nr}\}.$$

This inclusion and Remark 1 yield that $a_{n-i-1,r} \notin \{a_{n-i,r}, \ldots, a_{nr}\}$, and therefore, the elements $a_{n-i-1,r}, a_{n-i,r}, \ldots, a_{nr}$ are pairwise different. From these observations it follows immediately that the elements $a_{1r}, a_{2r}, \ldots, a_{nr}$ are pairwise different. This implies that $n \leq |A_r|$ contradicting the definition of n. Consequently, there is no finite system $\mathcal{M} \subseteq \mathcal{K}_{nd}$ so that $\mathcal{M}$ is isomorphically complete for $\mathcal{K}_{nd}$ with respect to the general product.

Since the α_0-product is a particular case of the general product, we get the following observation.

Corollary 3. *There is no finite system $\mathcal{M} \subseteq \mathcal{K}_{nd}$ of n.d. automata which is isomorphically complete for $\mathcal{K}_{nd}$ with respect to the α_0-product.*

Corollary 3 shows that there is a significant difference between the isomorphic representations of deterministic and n.d. automata. The class of all deterministic automata denoted by $\mathcal{L}_d$ does not contain any finite system which is isomorphically complete for $\mathcal{L}_d$ with respect to the α_0-product (see [5]). On the other hand, the class of all n.d. automata denoted by $\mathcal{L}_{nd}$ contains finite isomorphically complete system for $\mathcal{L}_{nd}$ with respect to the α_0-product (*cf.* [7]). Therefore, this pair of classes is an example for the case when the deterministic class does not contain any finite isomorphically complete system while the n.d. class contains a finite isomorphically complete system with respect to the α_0-product. The pair of the classes of the commutative asynchronous deterministic and n.d. automata denoted by $\mathcal{K}_d$ and $\mathcal{K}_{nd}$, respectively, is an example for the opposite case. Indeed, in [8], it is proved that $\mathcal{K}_d$ contains finite isomorphically complete systems for $\mathcal{K}_d$ with respect to the quasi-direct product. Since the quasi-direct product is a particular case of the α_0-product, this result yields that this class contains some finite isomorphically complete systems for $\mathcal{K}_d$ with respect to the α_0-product. On the other hand, by Proposition 1, it is not valid for the class $\mathcal{K}_{nd}$. Consequently, the pair of classes $\mathcal{K}_d$ and $\mathcal{K}_{nd}$ is an example for the case when the determinstic class conatins a finite isomorphically system while the n.d. class does not do it.

Of course there are finite isomorphically complete systems for $\mathcal{K}_{nd}$ with respect to the α_0-product, but they are not contained in $\mathcal{K}_{nd}$. Proposition 2 shows that there are finite isomorphically complete systems for $\mathcal{K}_{nd}$ with respect to the α_0-product such that they contain monotone n.d. automata in that sense that the transition graphs of these automata do not contain any directed cycle different from a loop. Moreover, it turns out that there exists such an isomorphically complete system for $\mathcal{K}_{nd}$ with respect to the α_0-product which consists of a monotone n.d. automaton having three states.

The n.d. automaton what we need is denoted by $\mathbf{B} = (\{0,1,2\}, \{x,y,u,v\})$ and it is defined as follows:

$$
\begin{aligned}
&0x^{\mathbf{B}} = \{0,1,2\},\ ix^{\mathbf{B}} = \{i\},\ i = 1,2,\\
&0y^{\mathbf{B}} = \{0,1\},\ iy^{\mathbf{B}} = \{i\},\ i = 1,2,\\
&0u^{\mathbf{B}} = \{0,2\},\ iu^{\mathbf{B}} = \{i\},\ i = 1,2,\\
&iv^{\mathbf{B}} = \{2\},\ i = 0,1,2.
\end{aligned}
$$

It is easy to check that $\mathbf{B}$ is monotone, *i.e.*, its transition graph does not contain any directed cycle different from loop.

Proposition 2. *Any system $\mathcal{M}$, containig such an n.d. automaton* $\mathbf{A}$ *that* $\mathbf{B}$ *can be embedded into an* α_0*-product of* $\mathbf{A}$ *with a single factor, is isomorphically complete for* $\mathcal{K}_{nd}$ *with respect to the* α_0*-product.*

Proof. By Lemma 4, it is sufficient to prove that any n.d. automaton from $\mathcal{K}_{nd}$ can be embedded into a suitable α_0-power of $\mathbf{B}$.

We shall prove this statement by induction on the number of states of the n.d. automata. It is worth noting that for every positive integer n, $\mathcal{K}_{nd}$ contains automata having n states.

One can easily check that if $\mathbf{A} \in \mathcal{K}_{nd}$ and $|A| \leq 2$, then $\mathbf{A}$ can be embedded into an α_0-product of $\mathbf{B}$ with a single factor. Now, let $n \geq 2$ be an arbitrary integer and let us suppose that the statement is valid for every $\mathbf{A} \in \mathcal{K}_{nd}$ with $|A| \leq n$. Let us consider an arbitrary n.d. automaton $\mathbf{A} = (A, X) \in \mathcal{K}_{nd}$ with $|A| = n + 1$. Corollary 2 provides that the reachability relation is a partial ordering on the set A. Since $\mathbf{A}$ is finite, $(A, \leq)$ contains maximal elements. We distinguish two cases depending on the number of the maximal elements.

Case 1. The number of the maximal elements in $(A, \leq)$ is not less than 2. Then there are at least 2 maximal elements, which are denoted by c, d. Now, let us define the α_0-product $\mathbf{D} = \mathbf{A}/\Theta(c,d) \times \mathbf{B}(X, \Phi)$ as follows.

For every $z \in X$ and $a \in A \setminus \{c, d\}$, let

$$\varphi_1(z) = z$$

$$\varphi_2(\{a\}, z) = \begin{cases} y & \text{if } az^{\mathbf{A}} \cap \{c,d\} = \{c\}, \\ u & \text{if } az^{\mathbf{A}} \cap \{c,d\} = \{d\}, \\ x & \text{otherwise,} \end{cases}$$

$$\varphi_2(\{\{c,d\}\}, z) = x.$$

Let us define the mapping $\mu : A \to A/\Theta(c,d) \times \{0,1,2\}$ as follows:

$c\mu = (\{c,d\}, 1),$
$d\mu = (\{c,d\}, 2),$
$a\mu = (\{a\}, 0),$ for all $a \in A \setminus \{c,d\}$.

and let $S = \{(\{a\}, 0) : a \in A \setminus \{c,d\}\} \cup \{(\{c,d\},1), (\{c,d\},2)\}$.

We prove that μ is an isomorphism of $\mathbf{A}$ into the α_0-product $\mathbf{D}$, more precisely, $\mathbf{A}$ is isomorphic to the subautomaton of $\mathbf{D}$ which is determined by the subset S.

First, let $a \in A \setminus \{c,d\}$ and $z \in X$ be arbitrary state and input sign, respectively. If $az^{\mathbf{A}} \cap \{c,d\} = \emptyset$, then $az^{\mathbf{A}}\mu = a\mu z^{\mathbf{D}} \cap S = a\mu z^{\mathbf{S}}$ is obviously valid. If $az^{\mathbf{A}} \cap \{c,d\} \neq \emptyset$, then let us investigate separately the three cases corresponding to the elements of the intersection. For the sake of simplicity, let us denote by Q the set $\{c,d\}$ and for every $R \subseteq A \setminus Q$, let $R' = \{(\{r\}, 0) : r \in R\}$.

(1) $az^{\mathbf{A}} = R \cup \{c\}$, where $R \subseteq A \setminus Q$. Then $az^{\mathbf{A}}\mu = R' \cup \{(Q, 1)\}$. On the other hand,

$$(\{a\}, 0)z^{\mathbf{D}} = \{a\}z^{\mathbf{A}/\Theta(c,d)} \times \{0,1\} = (R' \cup \{Q\}) \times \{0,1\}.$$

But $((R' \cup \{Q\}) \times \{0,1\}) \cap S = R' \cup \{(Q,1)\}$, and hence, $az^{\mathbf{A}}\mu = a\mu z^{\mathbf{S}}$ is valid for the case under consideration.

(2) $az^{\mathbf{A}} = R \cup \{d\}$, where $R \subseteq A \setminus Q$. Then $az^{\mathbf{A}}\mu = R' \cup \{(Q,2)\}$. Furthermore,

$$(\{a\},0)z^{\mathbf{D}} = \{a\}z^{\mathbf{A}/\Theta(c,d)} \times \{0,2\} = (R' \cup \{Q\}) \times \{0,2\}.$$

Now, $((R' \cup \{Q\}) \times \{0,2\}) \cap S = R' \cup \{(Q,2)\}$, and therefore, $az^{\mathbf{A}}\mu = a\mu z^{\mathbf{S}}$ is valid for this case as well.

(3) $az^{\mathbf{A}} = R \cup Q$, with $R \subseteq A \setminus Q$. In this case, $az^{\mathbf{A}}\mu = R' \cup \{(Q,1),(Q,2)\}$. Furthermore,

$$(\{a\},0)z^{\mathbf{D}} = \{a\}z^{\mathbf{A}/\Theta(c,d)} \times \{0,1,2\} = (R' \cup \{Q\}) \times \{0,1,2\}.$$

Now, $((R' \cup \{Q\}) \times \{0,1,2\}) \cap S = R' \cup \{(Q,1),(Q,2)\}$, and hence, $az^{\mathbf{A}}\mu = a\mu z^{\mathbf{S}}$ is valid for the case considered.

Finally, it is easy to see that $cz^{\mathbf{A}}\mu = c\mu z^{\mathbf{S}}$ and $dz^{\mathbf{A}}\mu = d\mu z^{\mathbf{S}}$. By the cases considered above, we get that μ is an isomorphism of $\mathbf{A}$ into the α_0-product $\mathbf{D}$. On the other hand, it is easy to check that $\mathbf{A}/\Theta(c,d)$ is a homomorphic image of $\mathbf{A}$, and thus, Corollary 1, Lemma 4 and the induction hypothesis result in that $\mathbf{A}$ can be embedded into an α_0-power of $\mathbf{B}$.

Case 2. $(A,\leq)$ has only one maximal element which is denoted by c. Then the partially ordered set $(A \setminus \{c\},\leq)$ contains at least one maximal element. Let us denote it by b. For the sake of simplicity, let Q denote the set $\{b,c\}$. Now, let us define the α_0-product $\mathbf{A}/\Theta(b,c) \times \mathbf{B}(X,\Phi)$ as follows.

For every $z \in X$ and $a \in A \setminus Q$, let

$$\varphi_1(z) = z,$$

$$\varphi_2(\{a\},z) = \begin{cases} u & \text{if } az^{\mathbf{A}} \cap Q = \{c\}, \\ y & \text{if } az^{\mathbf{A}} \cap Q = \{b\}, \\ x & \text{otherwise}, \end{cases}$$

$$\varphi_2(Q,z) = \begin{cases} y & \text{if } bz^{\mathbf{A}} = \{b\}, \\ v & \text{otherwise}. \end{cases}$$

Define the mapping of A into $A/\Theta(b,c) \times \{0,1,2\}$ as follows:

$c\mu = (Q,2)$,

$b\mu = (Q,1)$,

$a\mu = (\{a\},0)$, for all $a \in A \setminus Q$,

and let $S = \{(\{a\},0) : a \in A \setminus Q\} \cup \{(Q,1),(Q,2)\}$.

Then it can be seen that μ is an isomorphism of $\mathbf{A}$ into the α_0-product considered, namely, $\mathbf{A}$ is isomorphic to the subautomaton determined by the set S. On the other hand, $\mathbf{A}/\Theta(b,c)$ is a homomorphic image of $\mathbf{A}$. Then Corollary 1, Lemma

4 and the induction hypothesis yield that **A** can be embedded into an α_0-power of **B** which ends the proof of Proposition 2.

It is interesting to note that we need the monotone n.d. automaton of three states not only for the convenience. This assertion is vitnissed by a commutative asynchronous n.d. automaton which can not be embedded into any general product of two-state monotone n.d. automata.

Let us consider the n.d. automaton $\mathbf{A} = (\{0,1,2,3\}, \{x,y,z\})$ which is defined in the following way:

$$0x^{\mathbf{A}} = \{1,3\},\ ix^{\mathbf{A}} = \{i\},\ i = 1,2,3,$$
$$0y^{\mathbf{A}} = \{2,3\},\ 1y^{\mathbf{A}} = \{2\},\ iy^{\mathbf{A}} = \{i\},\ i = 2,3,$$
$$iz^{\mathbf{A}} = \{3\},\ i = 0,1,2,3.$$

It is easy to check that $\mathbf{A} \in \mathcal{K}_{nd}$. Now, we prove that **A** can not be embedded into any general product of two-state monotone n.d. automata. Contrary, let us suppose that **A** can be embedded into a general product $\mathbf{D} = \prod_{j=1}^{k} \mathbf{A}_j(\{x,y,z\}, \Phi)$ of two-state monotone n.d. automata. Without loss of generality, we may assume that the states of the n.d. automaton $\mathbf{A}_j$ are 0 and 1, moreover, there is no edge from 1 into 0 in the corresponding transition graph, for all j, $j = 1,\ldots,k$. Let μ denote a suitable isomorphism and let $i\mu = (e_{i1},\ldots,e_{ik})$, $i = 0,1,2,3$. Obviously, the vectors $(e_{i1},\ldots,e_{ik})$, $i = 0,1,2,3$ are binary vectors. The isomorphism and the monotone property of the components imply that $0\mu \le 1\mu \le 2\mu \le 3\mu$. Let us investigate the equality $0x^{\mathbf{A}}\mu = 0\mu x^{\mathbf{D}} \cap \{(e_{i1},\ldots,e_{ik}) : 0 \le i \le 3\}$. The left side is obviously $\{(e_{11},\ldots,e_{1k}),(e_{31},\ldots,e_{3k})\}$. By the definition of the general product, the right side is equal to the following set:

$$W = (\{e_{11},e_{31}\} \times \{e_{12},e_{32}\} \times \cdots \times \{e_{1k},e_{3k}\}) \cap \{(e_{i1},\ldots,e_{ik}) : 0 \le i \le 3\}.$$

Since $e_{1j} \le e_{2j} \le e_{3j}$, $j = 1,\ldots,k$ and $e_{ij} \in \{0,1\}$, for all $i = 1,2,3$; $j = 1,\ldots,k$, $(e_{21},\ldots,e_{2k}) \in W$ which is a contradiction.

By the observation above, we obtain the following statement.

Corollary 4. *There is no isomorphically complete system for $\mathcal{K}_{nd}$ with respect to the general product which consists of two-state monotone n.d. automata.*

Summarizing, the results presented here illustrate that although $\mathcal{K}_{nd}$ is a small and very particular class, the characterization of the isomorphically complete systems for $\mathcal{K}_{nd}$ with respect to the α_0-product can be very difficult. Proposition 1 shows that some isomorphically complete systems for $\mathcal{K}_{nd}$ must be infinite, while Proposition 2 implies that there are some finite isomorphically complete systems for $\mathcal{K}_{nd}$.

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References

[1] Gécseg, F., Composition of automata, Proceedings of the 2nd Colloquium on Automata, Languages and Programming, Saarbrücken, LNCS **14** (1974), 351-363.

[2] Gécseg, F., *Products of Automata*, Springer-Verlag, Berlin - Heidelberg-New York - Tokyo (1986).

[3] Gécseg, F., B. Imreh, On completeness of nondeterministicautomata, *Acta Math. Hungar.* **68** (1995), 151-159.

[4] Gécseg F., B. Imreh, On the cube-product of nondeterministic automata, *Acta Sci. Math. (Szeged)* **60** (1995), 321-327.

[5] Imreh, B., On α_i-products of automata, *Acta Cybernetica* **3** (1978), 301-307.

[6] Glushkov, V. M., Abstract theory of automata, *Uspekhi Mat. Nauk*, **16:5 101** (1961), 3-62 (in Russian).

[7] Imreh, B., M., Ito, On α_i-products of nondeterministic automata, *Algebra Colloquium*, **4** (1997), 195-202.

[8] Imreh, B., M. Ito, A. Pukler, On commutative asynchronous automata, Proceedings of The Third International Colloquium on Words, Languages, and Combinatorics, Kyoto, 2000, to appear.

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