Elementary decomposition of soliton automata*

Miklós Bartha † Miklós Krész ‡

Abstract

Soliton automata are the mathematical models of certain possible molecular switching devices. In this paper we work out a decomposition of soliton automata through the structure of their underlying graphs. These results lead to the original aim, to give a characterization of soliton automata in general case.

1 Introduction

One of the most important goals of research in bioelectronics is to develop a molecular computer (see e.g. [3]). The soliton automaton introduced in [4] is the mathematical model of so-called "soliton valves" having the potential to serve as a molecular switching device in such a computer architecture.

The underlying object of a soliton automaton is a soliton graph, which is the topological model of a hydrocarbon molecule-chain in which the appropriate soliton waves travel along. Any soliton graph has a perfect internal matching, i.e. a matching that covers all the vertices with degree at least two. These vertices model the carbon atoms, whereas vertices with degree one (external vertices) represent an interface with the outside world. The states of the corresponding automaton – also called the states of the graph – are the perfect internal matchings of the underlying graph, while the transitions are realized by making soliton walks. Intuitively, a soliton walk is an alternating walk with respect to some state M of the graph G, which starts and ends at an external vertex. However, the status of each edge in the walk regarding its presence in M changes dynamically step by step while making the walk, so that by the time the walk is finished, a new state of G is reached.

The analysis of soliton automata is a very complex task. So far only a few special cases have been described. In [4], [5] and [6], the transition monoids were determined for strongly deterministic soliton automata, deterministic soliton automata with a single external vertex or with one cycle. Following a different approach, in

*This work was partially supported by Soros Foundation. Presented at the Conference of PhD Students on Computer Science, July 20-23 2000, Szeged.

†Department of Informatics, University of Szeged, 6720, Szeged, Árpád tér 2, Hungary, e-mail: bartha@inf.u-szeged.hu

‡Department of Computer Science, University of Szeged, Faculty of Juhász Gyula Teacher Training College, 6725, Szeged, Boldogasszony sgt. 6, Hungary, e-mail: kresz@jgytf.u-szeged.hu

[8], the computational power of strongly deterministic soliton automata have been investigated by automata products. However, the general case is still open.

The main contribution of this paper is to reduce the general problem to a simpler one by working out a decomposition of soliton automata into elementary ones. For this goal we make use of the elementary structure of soliton graphs found in [2]. In Section 3 we describe the automata based on the internal parts of this decomposition, then characterize the relationship of component automata by α_0^ε-products. In Section 4 the self-transitions – transitions from a state to itself – induced by non-trivial walks are investigated. This problem will be analyzed also through the elementary decomposition.

2 Basic concepts and preliminaries

By a graph we mean, unless otherwise specified, a finite undirected graph in the most general sense, i.e. with multiple edges and loops allowed. For a graph G, $V(G)$ and $E(G)$ will denote the set of vertices and the set of edges of G, respectively. An edge $e = (v_1, v_2) \in E(G)$ connects two vertices $v_1, v_2 \in V(G)$, which are called the *endpoints* of e, and e is said to be *incident* with v_1 and v_2. If $v_1 = v_2$, then e is called a *loop* around v_1. Two edges sharing at least one endpoint are said to be *adjacent* in G. A subgraph G' of G is a graph such that $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$. If $X \subseteq V(G)$ then $G[X]$ denotes the subgraph of G induced by X, i.e. $V(G[X]) = X$ and E(G[X]) consists of the edges of G having both endpoints in X. Moreover, we say that the set $E \subseteq E(G)$ *spans* the subgraph G' if $G' = G[X]$, where X is the set of vertices incident with some edge of E.

If the vertex set of a graph G can be partitioned into two disjoint non-empty sets A and B such that all edges of G join a vertex in A to a vertex in B, we call G *bipartite* and refer to (A, B) as the *bipartition* of G.

The *degree* of a vertex v in graph G is the number of occurences of v as an endpoint of some edge in $E(G)$. According to this definition, every loop around v contributes two occurences to the count. The vertex v is called *external* if its degree $d(v)$ is one, *internal* if $d(v) \geq 1$, and *isolated* otherwise. *External edges* are those that are incident with at least one external vertex, whereas an edge is *internal*, if it is not external. The sets of external and internal vertices of G will be denoted by $Ext(G)$ and $Int(G)$, respectively.

A *matching* M of graph G is a subset of $E(G)$ such that no vertex of G occurs more than once as an endpoint of some edge in M. Again, it is understood that loops are not allowed to participate in M. The endpoints of the edges contained in M are said to be covered by M. A *perfect internal matching* is one that covers all the internal vertices of G. An edge $e \in E(G)$ is *allowed (mandatory)* if e is contained in some (respectively, all) perfect internal matching(s) of G. *Forbidden edges* are those that are not allowed. We will also use the name *constant edge* as a common reference to forbidden and mandatory edges. A perfect internal matching in G will be also referred to as a *state* of G, and the set of states of G is denoted by $S(G)$. For a complete account on matching theory the reader is refered to [9].

To follow the matching theoretic terminology, a *soliton graph* G is defined as a graph having at least one external vertex and a perfect internal matching. (See [1]). A connected soliton graph G is said to be *essentially internal* if either G consists of one edge or every external edge of G is forbidden.

An *elementary component* C of soliton graph G is a maximal connected subgraph of G spanned by allowed edges only. Then C is called *external* or *internal* depending on whether it contains an external vertex or not. An elementary component is said to be *trivial*, if it contains at most one edge. *Elementary graphs* are those which consist of one elementary component. Note that the decomposition into elementary components determines a partition on $V(G)$.

Now let G' be a subgraph of soliton graph G. Then for any state M of G, by $M_{G'}$ we mean the restriction of M to G'. If, in addition, G' is also a soliton graph with $M_{G'} \in S(G')$, and either $Int(G') = V(G') \cap Int(G)$ or G' is essentially internal, then G' will be called a *soliton subgraph with respect to M.*

In a graph G, a *walk* of length n is a sequence $\alpha = v_0, e_1, \ldots, e_n, v_n$, $n \geq 0$, of alternating vertices and edges. This sequence indicates the starting point $v_0 \in V(G)$ of α and the vertex v_j, $j \in [n] = \{1, \ldots n\}$, that α reached after traversing the j-th edge e_j. The notation $\alpha[v_i, v_j]$ with $1 \leq i \leq j \leq n$ will be used for the subwalk of α between v_i and v_j, i.e., $\alpha[v_i, v_j] = v_i, e_{i+1}, \ldots, e_j, v_j$. Furthermore α^{-1} will represent the reverse of α. For every $j \in [n]$, $n_\alpha(j)$ will denote the number of occurences of the edge e_j in the prefix $v_0, e_1, \ldots, e_j$. By a *backtrack* in a walk we mean the traversal of the same edge twice in a consecutive way. However, as the only exception, the traversal of a looping edge in the above way is not considered to be a backtrack. If all edges in a walk are distinct, the walk is called a *trail*, and if, in addition, the vertices are also distinct, the trail is a *path*. We define a *cycle* to be a path together with an edge joining the first and the last vertex. Note, that a looping edge is also a (trivial) cycle according to the above definition. An *external trail (path)* is a trail (path) having an external endpoint , while a path between two external vertices is said to be *crossing*. *Internal trails (paths)* are those that are not external.

A trail $\alpha = v_0, e_1, \ldots, e_n, v_n$, $n \geq 0$ is an *alternating trail* with respect to state M (or M-alternating trail, for short) if for every $i \in [n-1]$, $e_i \in M$ iff $e_{i+1} \notin M$. If $v_0, v_n \in Int(G)$, then α is called *internal*, otherwise α is *external*. Moreover, α is said to be *positive* (*negative*) if either α is internal with $e_1, e_n \in M$ ($e_1, e_n \notin M$, respectively) or it is external with $v_n \in Int(G)$ such that $e_n \in M$ ($e_n \notin M$, respectively). Observe that at most the endpoints of α can be traversed twice by an alternating trail α. Based on the above fact, any maximal external M-alternating trail α starting from vertex v, different from a crossing, can be decomposed in the form $\alpha = \alpha_h + \alpha_c$, where α_h, the *handle of* α, is an external M-alternating path, whereas α_c, the *cycle of* α, is an M-alternating cycle. With these parameters, α is called an *alternating v-racket* or an *alternating v-loop* depending on whether α_c is even or odd.

We say that an internal vertex w is *accessible in state M from external vertex v* (or simply w is *M-accessible from* v) if there exists a positive external M-alternating path with endpoints v and w. We will call an edge e *viable* from external vertex

v in state M if e is traversed by an external M-alternating trail starting from v. *Impervious* edges are those which are not viable in any state from any external vertex. Furthermore, a cycle is said to be *M-accessible* from external vertex v if some of its edges are viable from v in state M.

For a state M of G, an M-alternating trail α is called *complete* if α is either a crossing or it is an even length cycle. An *alternating network* with respect to M (or M-alternating network, for short) is a set of nonempty, pairwise disjoint, complete M-alternating trails. Note that, although an M-alternating network Γ consists of nonempty trails only, the network Γ itself can be empty.

Let M be a state of graph G and α be a complete M-alternating trail. By *making* α in state M we mean exchanging the status of the edges in α regarding their being present or not being present in M, thus creating a new state M'. The state M' created in this way will be denoted by $S_G(M,\alpha)$ or simply $S(M,\alpha)$ if G is understood. Making an M-alternating network Γ in state M means making all the trails of Γ simultaneously in M. Since the trails of Γ do not intersect each other, the resulting state, denoted by $S_G(M,\Gamma)$, is well-defined. Finally, let G' be a subraph of G and $M \in S(G)$. Then by an $M_{G'}$-alternating trail (network) we mean one that is entirely contained in graph G'.

Now we quote two results from [1] related to alternating networks.

Theorem 2.1 *For any two states M_1, M_2 of graph G, there exists a unique mediator alternating network Γ between M_1 and M_2, i.e. $S_G(M_1,\Gamma) = M_2$ and $S_G(M_2,\Gamma) = M_1$*

Corollary 2.2 *An edge e is non-constant iff it is traversed by a complete M-alternating trail in each state M.*

In our decomposition results we will make use of the α_0^ε-products of finite automata, therefore we now recall the necessary definitions from [7]. An *alphabet* is a finite, non-empty set. If X is an alphabet, then X^* denotes the set of words over X, including the empty word ε. A *non-deterministic finite automaton* is a triple $\mathcal{A} = (S,X,\delta)$, where S is a non-empty finite set, the *set of states*, X is an alphabet, the *input alphabet*, and $\delta : A \times X \to 2^A$ is the *transition function.* We can extend δ in such a way that $\delta(s,\varepsilon) = s$ for all $s \in S$.
For $i = 1,2$, let $\mathcal{A}_i = (S_i,X_i,\delta_i)$ be finite automata. An *isomorphism between $\mathcal{A}_1$ and $\mathcal{A}_2$* is a pair $\psi = (\psi_S,\psi_X)$ of bijective mappings $\psi_S : S_1 \to S_2$ and $\psi_X : X_1 \to X_2$ which satisfies the equation

$$\{\psi_S(s') \mid s' \in \delta_1(s,x)\} = \delta_2(\psi_S(s),\psi_X(x)),$$

for every $s \in S_1$ and every $x \in X_1$. The existence of an isomorphism between $\mathcal{A}_1$ and $\mathcal{A}_2$ is denoted by $\mathcal{A}_1 \cong \mathcal{A}_2$.

Definition 2.3 Let $\mathcal{A}_i = (S_i,X_i,\delta_i)$ ($i = 1,\dots,k$; $k > 0$) be a system of automata. Their *α_0^ε-product* with respect to alphabet X and feedback function ϕ — notation $\prod_{i=1}^k \mathcal{A}_i[X,\phi]$ — is the automaton

$$\mathcal{A} = (S,X,\delta), \text{ where}$$

(a) $S = S_1 \times \ldots \times S_k$

(b) $\phi = (\phi_1, \ldots, \phi_k)$ is a mapping, such that
$\phi_i : S_1 \times \ldots \times S_k \times X \to X_i \cup \{\varepsilon\}$, and ϕ_i is independent of its j^{th} component whenever $i \leq j \leq k$, $(i = 1, \ldots, k)$

(c) $\delta((s_1, \ldots, s_k), x) =$
$$\delta_1(s_1, \phi_1(s_1, \ldots, s_k, x)) \times \ldots \times \delta_k(s_k, \phi_k(s_1, \ldots, s_k, x))$$
for every $x \in X$, $s_i \in S_i$ $(i = 1, \ldots, k)$

Moreover, if every ϕ_i $(1 \leq i \leq k)$ depends only on the input signal, then we speak of the *quasi-direct ε-product* of $\mathcal{A}_1 \ldots \mathcal{A}_k$.

The following definitions are the matching theoretic formalizations of soliton walk and soliton automata introduced in [4].

Definition 2.4 A *partial soliton walk* in graph G with respect to state M is a backtrack-free walk $\alpha = v_0, e_1, \ldots, e_n, v_n$ subject to the following conditions:

(a) v_0 is an external vertex

(b) for every $j \in [n-1]$, $n_\alpha(j)$ and $n_\alpha(j+1)$ have the same parity iff e_j and e_{j+1} are M-alternating, i.e., $e_j \in M$ iff $e_{j+1} \notin M$.

Furthermore if v_n is also external then α is called *total soliton walk*, or simply *soliton walk*.

Note that the case of $n = 0$ is also possible; then the soliton walk is called *trivial*.

Making the walk α in state M means creating $M' = S(M, \alpha)$ by setting for every $e \in E(G)$

$e \in M'$ iff $e \in M$ and e occurs an even number of times in α, or $e \notin M$ and e occurs an odd number of times in α.

In the light of [4, Lemma 3.3] it should be clear that $S(M, \alpha)$ is indeed a state.

In the rest of the paper we will use the following notation. If M is a state of graph G and $v_1, v_2 \in Ext(G)$, then

$\mathcal{S}_G(M, v_1, v_2) = \{S(M, \alpha) \mid \alpha$ is a soliton walk with respect to M, which starts at v_1 and ends at $v_2\}$

Definition 2.5 A *soliton automaton* with underlying graph G is a non-deterministic finite automaton

$$\mathcal{A}(G) = ((S(G), (X \times X), \delta)$$

subject to the following conditions:

(a) G is a soliton graph

(b) $S(G)$, the set of states of $\mathcal{A}(G)$, is the set of states of G

(c) $(X \times X)$ is the input alphabet, where $X = Ext(G)$

(d) $\delta : S(G) \times (X \times X) \to 2^{S(G)}$ is the transition function, such that
$\delta(M, (v_1, v_2)) = \mathcal{S}_G(M, v_1, v_2)$, if $\mathcal{S}_G(M, v_1, v_2) \neq \emptyset$
$\delta(M, (v_1, v_2)) = \{M\}$, otherwise
for any $M \in S(G)$ and $v_1, v_2 \in X$.

A soliton automaton is said to be *elementary* if its underlying object is an elementary graph.

Note that, without loss of generality, we can assume that all constant external edges of a soliton graph G are mandatory. Indeed, attaching an extra mandatory edge to each forbidden external edge of G results in a graph G^* for which $\mathcal{A}(G) \cong \mathcal{A}(G^*)$. We shall use this assumption throughout the paper without any further reference.

In [4] an edge is called *impervious* if it is not traversed by any partial soliton walk. The following proposition states that our definition of impervious edge is equivalent to this.

Proposition 2.6 *Let $\alpha = v_0, e_1, \ldots, e_n, v_n$ be a partial soliton walk with respect to state M with $v_0 \neq v_n$. Then there exists an external M-alternating trail β from v_0 such that β terminates in e_n and $E(\beta) \subseteq E(\alpha)$. Furthermore, if $n_\alpha(n)$ is odd, then the other endpoint of β is v_n.*

Proof. First suppose that $e_n \notin M$. Extend G by new external edges $e = (v_{n-1}, v)$ and $e' = (v_n, v')$, such that $v, v' \notin V(G)$. Furthermore let e_m denote the last edge of α for which $e_m = e_n$ and $n_\alpha(m)$ is odd. Observe that $\alpha[v_0, v_{m-1}] + e$ or $\alpha[v_0, v_{m-1}] + e'$ is a total soliton walk in $G + e + e'$ depending on whether $v_{m-1} = v_{n-1}$ or $v_{m-1} = v_n$. Therefore, based on Theorem 2.1, there exists an M-alternating network Γ such that making Γ and making the appropriate part of the above walks results in the same state of $G + e + e'$. Clearly, Γ will contain an M-alternating crossing β' between v_0 and v (between v_0 and v', respectively). Then replacing e (respectively, e') in β' by e_n, we obtain the required M-alternating trail β.

Now consider the case when $e_n \in M$. Then $e_{n-1} \notin M$, thus we can construct the appropriate external alternating trail β described above, which terminates at e_{n-1}. If $e_n \in E(\beta)$, then we are ready. Otherwise $\beta + e_n$ will provide a suitable alternating path. Finally, based on the first part of the proof, observe that $e_n \notin E(\beta)$, when $n_\alpha(n) = n_\alpha(n-1)$ is odd, which makes the proof complete. □

It is clear that impervious edges have no effect on the operations of soliton automata. Thus, without loss of generality, we can restrict our investigation to soliton graphs without impervious edges. The above fact in more precise form is stated in [4, Proposition 4.5]. Therefore, throughout the paper, unless otherwise specified, G will denote a soliton graph without impervious edges.

In the rest of this section we summerize some results from [2].

Definition 2.7 For any two internal vertices $u, v \in V(G)$, $u \sim v$ if u and v belong to the same elementary component of G and the edge $e = (u, v)$ becomes forbidden in $G + e$.

For an elementary component C of G, $\sim_C$ will denote $\sim$ on C separately. Note that generally $\sim_C$ is not equal to the restriction of $\sim$ to C.

Theorem 2.8 *The relation $\sim$ is an equivalence on $Int(G)$.*

The classes of the partition determined by $\sim$ are called *canonical classes*. In particular a *canonical class of elementary component C* is a canonical class contained in $V(C)$.

Proposition 2.9 *Let u and v be arbitrary vertices of a non-trivial internal elementary component C of G. Then $u \not\sim_C v$ iff for any state M of G there exists a positive internal M_C-alternating path connecting u and v.*

In the following we shall use the phrase "external alternating path γ enters elementary component C" in the strict sense, meaning that γ enters C for the first time. Note that in this case γ must be negative.

Definition 2.10 An internal elementary component C is *one-way* if all external alternating paths enter C in the same canonical class of C. This unique class is called *principal*. Further to this, every external elementary component is a priori one-way by the present definition (with no principal canonical class). An elementary component is *two-way* if it is not one-way.

Proposition 2.11 *There exists no edge connecting two internal vertices contained in principal canonical classes.*

Let C be an elementary component of G, and consider a state M in G. An *M-alternating C-loop* (just C-loop if M is understood) is a negative internal M-alternating path or odd M-alternating cycle in G having both endpoints, but no other vertices, in C. Note that the endpoints of a C-loop α must belong to the same canonical class which is called the *domain* of α. We say that α *covers* the elementary component D if some edge of D is traversed by α.

Definition 2.12 Let M and C be a state and an external elementary component of G, respectively. A *hidden edge* of C is an edge $e = (v_1, v_2)$, not necessarily in $E(G)$, for which v_1, v_2 are the endpoints of an M-alternating C-loop.

An elementary graph C consisting of an external elementary component and its hidden edges will also be considered elementary component throughout the paper. In this case we will call C *augmented external elementary component.* In [2] it was proved that the augmentation of a soliton graph G by its hidden edges preserves the elementary structure of G with the same canonical partition for each elementary component.

The hidden edges have important role in the external alternating paths, which is expressed below.

Proposition 2.13 *Let M be a state of G, $w \in V(G)$ and $v \in Ext(G)$. Furthermore let α be a positive (negative) M-alternating trail between v and w such that $E(\alpha)$ contains hidden edges. Then there exists a positive (respectively, negative) M-alternating trail between v and w which does not traverse any hidden edge.*

Elementary components are structured according to their accessibility by external alternating paths. The rest of this section is an extract of some results obtained in [2] relating to this structure.

Definition 2.14 Let C be an elementary component with a non-principal canonical class P. We say that the couple (C, P) are the *parents* of elementary component D if a C-loop with domain P covers D but there does not exist a C'-loop α for any elementary component C' such that α covers both C and D. In that case C and P are called the *father* and the *mother* of D, respectively.

Theorem 2.15 *Each two-way elementary component has a unique father and a unique mother. One-way components have no parents.*

The following property of fathers will play important role in the paper.

Proposition 2.16 *Let (C, P) be the parents of elementary component D and let α be an alternating trail starting from external vertex v entering D at a vertex w. Then $\alpha[v, w]$ will go through C such that the last common vertex of $\alpha[v, w]$ and of C belongs to P.*

By Theorem 2.15, elementary components can be grouped into disjoint family trees according to the father-son relationship. Then a *family* $\mathcal{F}$ is defined as a block of elementary components belonging to the same family tree. The root, denoted by $r(\mathcal{F})$, is the ultimate forefather of $\mathcal{F}$. Then Theorem 2.15 implies the following result.

Theorem 2.17 *Every family contains a unique one-way elementary component, which is $r(\mathcal{F})$.*

A family $\mathcal{F}$ is *external* if $r(\mathcal{F})$ is such, otherwise it is internal. Moreover, for the family containing some elementary component C, the notation $\mathcal{F}_C$ will be used.

Now we describe the relationship of families with the help of a binary relation. For this we need the following observation.

Proposition 2.18 *Let e be a forbidden edge of G connecting two different families $\mathcal{F}_1$ and $\mathcal{F}_2$. Then exactly one endpoint of e belongs to the principal canonical class of the root of either $\mathcal{F}_1$ or $\mathcal{F}_2$.*

Making use of the above claim, the binary relation $\mapsto$ is defined in the following way.

Definition 2.19 For any two different families $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_1 \mapsto \mathcal{F}_2$ if there exists an edge e connecting $\mathcal{F}_1$ and $\mathcal{F}_2$ such that the principal endpoint of e is in $\mathcal{F}_2$. In this case we say that e points to family $\mathcal{F}_2$.

Let $\overset{*}{\mapsto}$ denote the reflexive and transitive closure of $\mapsto$.

Theorem 2.20 *The relation $\overset{*}{\mapsto}$ is a partial order on the collection of all families of G, by which the external families are maximal elements.*

Finally we give an important consequence of the above results, which will be used throughout the paper.

Corollary 2.21 *An edge e connecting two families is traversed by any external alternating trail α in G by reflecting the relation $\mapsto$, that is, if α enters family $\mathcal{F}_2$ from family $\mathcal{F}_1$, then e points to $\mathcal{F}_2$.*

Proof. Suppose by way of contradiction that $\mathcal{F}_2 \mapsto \mathcal{F}_1$ holds in the situation described in the statement of the corollary for some M-alternating trail α starting from an external vertex u. In that case let (v_1, v_2) denote the edge traversed by the above way with v_1 being contained in the principal canonical class P_1 of $r(\mathcal{F}_1)$. Now let extend G by edge $f = (v_1, v_1)$. Then f will be clearly viable by $\alpha[u, v_1] + (v_1, v_1)$, thus $G + f$ has no impervious edges. Observe that P_1 will be a canonical class in $G + f$, too. Indeed, if we assumed that an extra edge g connecting two vertices $w_1, w_1' \in P_1$ would be allowed in $G + f + g$, then it is easy to see, by making use of Corollary 2.2, that there would exist in $G + f + g$ a complete M-alternating trail β containing both f and g. However, f is a loop, consequently the above situation is not possible. Therefore the elementary component $r(\mathcal{F}_1) + f$ is also one-way in $G + f$ with f being in its principal canonical class P_1; which is a contradiction in Proposition 2.11. □

3 The decomposition of soliton automata

It is a central question to establish the correspondence between alternating networks and soliton walks. The first result gives the characterization of this problem.

Definition 3.1 Let v, w be external vertices and M be a state of G. An *M-transition network* Γ *from* v *to* w is an M-alternating network with the following conditions:

(a) If $\Gamma = \emptyset$, then $v = w$.

(b) All elements of Γ, except one crossing from v to w if $v \neq w$, are alternating cycles accessible from v in M.

Let $\mathcal{T}_G(M, v, w)$ denote the set of M-transition networks from v to w in graph G.

Theorem 3.2 *Let M be a state of G, $v, w \in Ext(G)$, and Γ be an M-alternating network. Then $S_G(M, \Gamma) \in \mathcal{S}_G(M, v, w)$ iff $\Gamma \in \mathcal{T}_G(M, v, w)$.*

Proof. The "only if" part is straightforward from Theorem 2.1 and from Proposition 2.6. To prove the "if" part, let us construct for each cycle β of Γ an appropriate v-racket β' with respect to M, such that the length of the handle of β' is minimal. Let Γ' denote the set of the above v-rackets and , in the case of $v \neq w$, of the crossing of Γ.

We will show by an inductive argument on $| \Gamma' |$ that there exists a soliton walk α for which $E(\alpha) \subseteq E(\cup\Gamma')$ and $S(M, \alpha) = S(M, \Gamma)$. The basis step with Γ' being empty or a singleton is trivial.

Now let $| \Gamma' | > 1$ and assume that the assertion holds for each soliton walk set Γ_1' constructed in the described way from an appropriate alternating network Γ_1 with $| \Gamma_1' | < | \Gamma' |$. Let γ denote the v-racket with longest handle in Γ'. It is evident that γ_c is disjoint from $\cup(\Gamma' \setminus \{\gamma\})$. Now using the induction hypothesis consider a soliton walk $\alpha = v, e_1, v_1, \ldots, v_{n-1}, e_n, w$ traversing $\Gamma' \setminus \{\gamma\}$ by the required way. If $\gamma_h = v, f_1, w_1, \ldots, f_m, w_m$, then let w_i be the first vertex of γ_h such that $f_{k+1} \neq e_{k+1}$. Then it is easy to see that $\alpha[v, v_i] + \gamma_h[w_i, w_m] + \gamma_c + \gamma_h^{-1}[w_m, w_i] + \alpha[v_i, w]$ will be a soliton walk with the required properties, which makes the proof complete. □

Note that based on Theorem 3.2, the transitions of a soliton automaton can be effectively computed from its underlying soliton graph. Indeed, for any two states

M_1, M_2, the mediator alternating network between M_1 and M_2 is given by $\Gamma = M_1 \oplus M_2$, where $\oplus$ denotes the symmetric difference of M_1 and M_2, while the shortest handle for each alternating cycle in Γ can be found in a straightforward way. Moreover, the proof of Theorem 3.2 is constructive, it yields a soliton walk between two given states. The above facts are important from simulation point of view.

We will work out products of automata based on elementary components, thus first we characterize the automata constructed from these components.

Definition 3.3 For an elementary component C of graph G, the *component automaton* determined by C is the soliton automaton based on the graph C^*, where
$C^* = C$, if C is external
$C^* = C + (v, w), v \in C, w \notin V(G)$, if C is internal

Definition 3.3 might give the impression that an internal component automaton depends on the choice of vertex v. However, Theorem 3.5 will show that all component automata determined by the same internal elementary component are isomorphic, thus $\mathcal{A}(C^*)$ is unambiguous.

Definition 3.4 An $\mathcal{A} = (S, X, \delta)$ automaton is called *full*, if

(i) $X = \{x\}$
(ii) $\delta(s, x) = S$, for each $s \in S$

Theorem 3.5 *Every internal component automaton is a full automaton. Conversely, for any full automaton there exists an isomorphic internal component automaton.*

Proof. We start with proving the first statement. To this end let C be an internal elementary component, $v \in V(C)$ and (v, w) an extra external edge attached to C in order to form C^*. As any state of C^* has a transition to itself by a trivial soliton walk, we have to prove the "full-property" only for any two different states M_1,M_2 of C. If Γ is the mediator alternating network between M_1 and M_2, then clearly Γ consits of $M_1(M_2)$-alternating cycles. Any cycle β of Γ contains a vertex u for which $u \not\sim_C v$, thus there exists an internal positive $M_1(M_2)$-alternating path α between u and v in the graph C. Therefore β is accessible from w in $M_1(M_2)$ by $(w, v) + \alpha$. As v and β were arbitrary, we obtain the first claim with the help of Theorem 3.2.

To prove the second statement, we only have to show that there exists an internal elementary component with n states for every $n \in N$. The case $n = 1$ is satisfied by an elementary component consisting of one internal mandatory edge. If $n \geq 2$, then consider an even cycle β, two adjacent vertices $v, w \in V(\beta)$ and construct a graph G such that it has a representation in the form $G = \beta + \alpha_1 + \ldots + \alpha_{n-2}$, where

(i) α_i, $i \in [n-2]$ is an odd path with endpoints v and w
(ii) $V(\alpha_i) \cap V(\beta) = \{v, w\}$, $i \in [n-2]$
(ii) $V(\alpha_i) \cap V(\alpha_j) = \{v, w\}$, $i, j \in [n-2]$

Observe that for any edge e being incident with v, there is a unique state M of G such that $e \in M$. Thus, it is easy to see, that each edge of G is allowed and G has n states, as expected. $\square$

For the description of the product automaton we need the following concepts.

Definition 3.6 Let P be a canonical class of some external component. Then the set ρ_P is the smallest set of elementary components such that:

(i) if C' is an internal elementary component and (v, w) is an edge for which $v \in P$ and $w \in V(C')$, then $C' \in \rho_P$.

(ii) if C_1, C_2 are internal elementary components such that $\mathcal{F}_{C_1} \overset{*}{\mapsto} \mathcal{F}_{C_2}$, $C_1 \in \rho_P$ and there is an edge between C_1 and C_2, then $C_2 \in \rho_P$.

Note that (ii) may also hold if C_1 and C_2 are in the same family, as $\overset{*}{\mapsto}$ is reflexive. Moreover, based on the structure of the families it can be easily showed, that if $C \in \rho_P$ and $\mathcal{F}_C$ is internal, then $C' \in \rho_P$ for any elementary component C' of $\mathcal{F}_C$.

For the main result of the section we introduce some technical notations and prove a lemma. For these we need the following simple observation.

Claim 3.7 *Let P be a canonical class of some elementary component C. An internal vertex of P is accessible from external vertex v in state M iff all vertices of P are M-accessible from v .*

Proof. Let us assume that α is a positive external M-alternating path from v to w and let u be an arbitrary vertex of P different from w. We claim that there exists an internal M-alternating path β between u and some vertex of α such that β is positive on the end of vertex u. If C is external, then according to [2, Proposition 2.3] there exists a positive external M_C-alternating path γ with endpoint u. Observe that $E(\alpha) \cap E(\gamma) \neq \emptyset$, because otherwise $\alpha' = \alpha + (w, u) + \gamma$ would form an alternating crossing indicating that $u \not\sim w$ by $S(M, \alpha')$. Therefore an appropriate subpath of γ is suitable for β. Now assume that C is internal. Then let w' denote the vertex incident with w by the edge covered by M. Clearly, $u \not\sim_C w'$, thus, based on Proposition 2.9, there exists a positive internal M_C-alternating path between u and w', from which the existence of β is straightforward again.

Now starting from u let u_α denote the first vertex along β for which $u_\alpha \in V(\alpha)$. $\alpha_1 = \alpha[w, u_\alpha] + \beta[u_\alpha, u]$ cannot form a positive internal alternating path, as it would contradict $u \sim w$. Therefore $\alpha[v, u_\alpha] + \beta[u_\alpha, u]$ gives a positive external M-alternating path, as desired. $\square$

By Claim 3.7 it is justified to say that a *canonical class is accessible* from an external vertex in a given state.

For any internal elementary component C' of graph G:

$$\mathcal{R}_G(C') = \{P \mid P \text{ is a canonical class of some external elementary component and } C' \in \rho_P\}$$

For any external vertex v of a (possibly augmented) external elementary component C and state M of C in graph G:

$$\mathcal{P}_C(M,v) = \{P \mid P \text{ is a canonical class of } C\text{, which is } M\text{-accessible from } v \text{ in the graph } C\ \}.$$

and

$$\mathcal{C}_G(M,v) = \{C' \mid C' \text{ is an internal elementary component such that } \mathcal{R}_G(C') \cap \mathcal{P}_C(M,v) \neq \emptyset.\}$$

Note that if G is understood then the subscript G is omitted from the above notations. Furthermore, if C is an augmented external elementary component, then it is indicated with a superscript $'h'$, i.e. using $\mathcal{C}^h(M,v)$.

Lemma 3.8 *Let P' be a non-principal canonical class of some internal elementary component C' and v be an external vertex of an elementary component C. Then an edge e incident with a vertex of P' is viable from v in state M iff $C' \in \mathcal{C}^h(M_C,v)$.*

Proof. During the proof the notation C^h will be used for the augmented external elementary component constructed from C. Furthermore, for any external alternating trail α starting from C, w_α will denote the last vertex of α for which $w_\alpha \in V(C)$.

'Only if' Let α be a positive external M-alternating trail starting from v and terminating at vertex w, where w is an endpoint of e. Moreover, let P_α denote the canonical class containing w_α. Then substituting the C-loops for hidden edges in α, we obtain that $P_\alpha \in \mathcal{P}_{C^h}(M_C,v)$. Now using Corollary 2.21, it is easy to see that $C_s \in \rho_{P_\alpha}$ for each internal elementary component C_s reached by $\alpha[w_\alpha,w]$. Hence $P_\alpha \in \mathcal{R}(C') \cap \mathcal{P}_{C^h}(M_C,v)$, which gives the result.

'If' Suppose that $C' \in \rho_P$ for some canonical class $P \in \mathcal{P}_{C^h}(M_C,v)$. Then based on the definition of ρ_P there exist families $\mathcal{F}_1,\dots,\mathcal{F}_m$ containing members of ρ_P such that $\mathcal{F}_1 = \mathcal{F}_C$, $\mathcal{F}_m = \mathcal{F}_{C'}$ and for each $1 \leq s \leq m-1$ $\mathcal{F}_s \mapsto \mathcal{F}_{s+1}$ with some edges connecting elements of $\rho_P \cap \mathcal{F}_s$ and $\rho_P \cap \mathcal{F}_{s+1}$. Let α be an external M-alternating trail terminating at w, where w is an endpoint of e. Note that such an α exists, because [1, Corollary 3.3] states that an edge is impervious in one state iff it is impervious in all states. The proof will apply an induction on m.

Basis step. Applying Theorem 2.15 iteratively, we obtain that each two-way elementary component C_1 of $\mathcal{F}_1$ has a unique ultimate foremother – in notation $m(C_1)$– as a class of C. Then, making use of Proposition 2.16, it is clear that for any external M-alternating trail β reaching C_1, w_β is contained in $m(C_1)$.

It is clear, by Proposition 2.18, that $\rho' = \rho_P \cap \mathcal{F}_1$ can be built up iteratively according to Definition 3.6 $(i) - (ii)$. We will show by a structural induction based on the building procedure of ρ', that for any elementary component $C_1 \in \rho'$, $m(C_1) = P$ holds. First suppose that C_1 is added to ρ' in a step of type (i). As $P \in \mathcal{P}_{C^h}(M_C,v)$, we obtain with the help of Proposition 2.13, that in this case $w_\gamma \in P$ holds, which implies $m(C_1) = P$ by the previous paragraph. Continuing the procedure with (ii) such that edge e connects C_1 with an elementary component C_2 already in ρ', let us consider an external alternating trail γ terminating at e. According to the hypothesis for C_2, w_γ must belong to P. Thus applying the observation of the previous paragraph again, we obtain that $m(C_1) = P$, as desired.

Summarizing the foregoings we conclude that $w_\alpha \in P$. Now choosing a positive M_C-alternating path α_1 between v and w_α and applying Proposition 2.13 for $\alpha_1 + \alpha[w_\alpha, w]$ we obtain a suitable alternating trail.

Induction step. Let u denote the first vertex of α which is also in $C_m = r(\mathcal{F}_m)$. Moreover, let u' denote a vertex of C_m which is connected by an edge to a vertex w' of some elementary component of $\mathcal{F}_{m-1} \cap \rho_P$. According to the induction hypothesis there exists an appropriate M-alternating trail β running from v and terminating at (u', w'). Based on Corollary 2.21 the following facts hold: the internal endpoint of β is u' such that β is a negative path, $\alpha[u, w]$ avoids $\mathcal{F}_1, \ldots, \mathcal{F}_{m-1}$ and $\beta - (w', u')$ does not "touch" $\mathcal{F}_m$. Now consider a positive internal M_C alternating path γ, which starts from u' and terminates in some vertex u_1 of $V(\alpha) \cap V(C_m)$ such that $u' \not\sim_C u_1$. By Corollary 2.21, $u \sim u'$, therefore if u_1' denotes the vertex where γ hits α first time, then we can conclude that $\beta + \gamma[u', u_1'] + \alpha[u_1', w]$ provides the desired alternating trail. □

Theorem 3.9 *Let $C_1, \ldots, C_l$ be the augmented external elementary components, $C_{l+1}, \ldots, C_k$ be the internal elementary components of G, and $\mathcal{A}(C_i^*) = (S(C_i), (X_i \times X_i), \delta_i)$ $(i = 1, \ldots, k)$, with $X_i = \{x_i\}$, if $i > l$. Then:*

$$\mathcal{A}(G) \cong \mathcal{A}^*(G), \text{ where}$$

$$\mathcal{A}^*(G) = \prod_{i=1}^{k} \mathcal{A}(C_i^*)[Y, \phi] \text{ is an } \alpha_0^\varepsilon\text{-product such that}$$

(a) $Y = (Ext(G) \times Ext(G))$

(b) $\phi = (\phi_1, \ldots, \phi_k)$ *is defined in the following way:*

For each $1 \le i \le k$, $M_1 \in S(C_1), \ldots, M_k \in S(C_k)$ *and* $(y_1, y_2) \in Y$

(b/1) if $1 \le i \le l$ *and* $(y_1, y_2) \in X_i \times X_i$, *then*

$$\phi_i(M_1, \ldots, M_k, (y_1, y_2)) = (y_1, y_2)$$

(b/2) if $l + 1 \le i \le k$, $(y_1, y_2) \in X_j \times X_j$ *for some* $1 \le j \le l$, $C_i \in \mathcal{C}^h(M_j, y_1)$, *and either* $y_1 = y_2$, *or* $y_1 \ne y_2$ *with* $\delta_j(M_j, (y_1, y_2)) \ne \{M_j\}$, *then*

$$\phi_i(M_1, \ldots, M_k, (y_1, y_2)) = (x_i, x_i)$$

(b/3) Otherwise:

$$\phi_i(M_1, \ldots, M_k, (y_1, y_2)) = \varepsilon.$$

Proof. Let δ and δ^* denote the transition function of $\mathcal{A}(G)$ and that of $\mathcal{A}^*(G)$, respectively. Moreover, let $(y_1, y_2) \in Y$ and $M \in S(G)$ be arbitrary, such that $y_1 \in V(C_r)$, $y_2 \in V(C_s)$ for some $r, s \le l$. Since the mapping

$$\psi(M) = (M_{C_1}, \ldots, M_{C_k}) \tag{1}$$

is clearly a bijection between $S(G)$ and $S(C_1) \times \ldots S(C_k)$, we only have to prove that

$$\{\psi(M') \mid M' \in \delta(M, (y_1, y_2))\} = \delta^*(\psi(M), (y_1, y_2)) \tag{2}$$

For each $1 \le i \le k$ let z_i denote $\phi_i(M_{C_1}, \ldots, M_{C_k}, (y_1, y_2))$. Consider first the right side of (2). Then based on (1) and Definition 2.3, we have

First we provide a characterization of non-trivial self-transitions by alternating trails. For this result we need the following definition.

Definition 4.2 Let M be a state of G and $v \in Ext(G)$. An M- *alternating double v-racket* α is a pair of M-alternating v-rackets (α^1, α^2) with branching handles, i.e. with neither of α_1^h and α_2^h being a prefix of the other. The maximal common external subpath – denoted by α_h – of α_h^1 and of α_h^2 is called the *handle* of α, whereas the last vertex of α_h is referred to be as the *branching vertex* of α.

Note that the handle of a double v-racket is a positive external alternating path.

Theorem 4.3 *There exists a non-trivial self-transition of external vertex v with respect to state M of G, iff G contains either an M-alternating v-loop or an M-alternating double v-racket.*

Proof. During the proof if we refer to an alternating cycle α as a part of a decomposed form of a soliton walk β, then we mean that α as a subwalk of β is traversed in an appropriate way.

For an M-alternating v-loop α it is easy to check that $\alpha_h + \alpha_c + \alpha_c + \alpha_h^{-1}$ is a non-trivial self-transition of v. Therefore, we can suppose for the rest of the proof that G does not contain an M-alternating v-loop .

'Only if' Let $\alpha = v, e_1, v_1, \dots, e_n, v_n$ be a non-trivial self-transition of v with respect to M, and let i be the smallest index for which there exists an index $j > i$ such that $v_j = v_i$, $n_\alpha(j) = 1$ and each edge of $\alpha[v, v_i]$ is traversed exactly once by $\alpha[v, v_j]$. In other words, v_i is the closest vertex to v where α returns to itself. Now, based on Proposition 2.6, there exists an M-alternating trail β such that β terminates at e_j and $E(\beta) \subseteq E(\alpha[v, v_j])$. By assumption, β is an alternating v-racket with $\beta_h = \alpha[v, v_i]$. Observe that $\alpha[v, v_j] + \alpha[v, v_i]^{-1}$ is a soliton walk from v to itself. Therefore, it is obvious that the edges traversed by $\alpha[v_i, v_j]$ an odd number of times will constitute an M-alternating network Γ consisting of alternating cycles. By the above facts we obtain that $e_{j+1} = e_i$. The edge e_j must be traversed by $\alpha[v_j, v_n]$, consequently there is a first edge e_m with $m > j$ which is not on $\alpha[v, v_i]$. Then, let e_r denote the edge for which $e_r = e_{m-1}$ with $r \leq i$. It is easy to see, that because of the choice of v_i, any vertex v_l with $l < i$ is incident with exactly two edges of $\alpha[v, v_j]$. Therefore $n_\alpha(m) = 1$ and we can select the first edge e_k of $\alpha[v_{m-1}, v_n]$ for which $n_\alpha(k)$ is even. Again, by the choice of v_i, we conclude that $\alpha[v, v_{r-1}]$ and $\alpha[v_{m-1}, v_k]$ are edge-disjoint. Furthermore, observe that $e_r \notin M$, therefore $\alpha' = \alpha[v, v_{r-1}] + \alpha[v_{m-1}, v_{k-1}]$ is a partial soliton walk with respect to M. As we have seen, there exists an M-alternating cycle γ' of Γ containing e_k. Making use of the former observations for Γ and for α' we obtain that γ' and α' are edge-disjoint. Now applying Proposition 2.6 for α', an M-alternating v-racket γ can be constructed such that $\delta = (\beta, \gamma)$ is a double v-racket with $\gamma_c = \gamma'$ and $\delta_h = \alpha[v, v_{r-1}]$.

'If' Let $\alpha = (\alpha^1, \alpha^2)$ be a double v-racket, and let w denote the branching vertex of α. Moreover, let us introduce the notation $\alpha_s^i = \alpha_h^i - \alpha_h$ and $\alpha_w^i = \alpha_s^i + \alpha_c^i + (\alpha_s^i)^{-1}$ for $i = 1, 2$. If $\alpha^2 - \alpha_h$ is edge-disjoint from α^1 we obtain that $\alpha_h + \alpha_w^1 + \alpha_w^2 + \alpha_w^1 + \alpha_w^2 + \alpha_h^{-1}$ is a soliton walk with the desired properties.

Otherwise let u be the first vertex of α_w^2 which is also on α^1 and extend $\alpha^2[w,u]$ to an M-alternating path α_u by continuing its way appropriately on α_h^2 until it reaches α_c^2. Note that $\alpha_u = \alpha^2[w,u]$ holds if $u \in V(\alpha_c^2)$. Also note that the construction described above is feasible, as G has no M-alternating v-loops. Then $\alpha_h + \alpha_w^1 + \alpha_u + \alpha_c^1 + \alpha_u^{-1} + \alpha_h^{-1}$ will result in the requested soliton walk. □

We now turn to the characterization of v-loops.

Proposition 4.4 *Let v be an external vertex of graph G and $M \in S(G)$. Then G contains an M-alternating v-loop iff there exists an internal edge (u, w) such that both u and w are accessible from v in M.*

Proof. It is sufficient to prove the 'If' part. Let α and β be positive external M-alternating paths from v to internal vertices u and w, respectively, such that $(u,w) \in E(G)$ and $| E(\alpha) \cup E(\beta) |$ is minimal. Then let w_β denote the last vertex of β with $w_\beta = u_\alpha$ for some $u_\alpha \in V(\alpha)$. We claim that $\alpha[v, u_\alpha]$ is positive. Indeed, otherwise both endpoints of the last edge of $\alpha[v, u_\alpha]$ would be accessible from v in M by the appropriate subpaths of α and β, which would be a contradiction in the choice of u and w. Therefore an M-alternating v-loop can be formed from the edges of the set $E(\alpha) \cup E(\beta[w_\beta, w]) \cup \{(u,w)\}$, as desired. □

To state the following important consequence of Proposition 4.4, let us call two states M_1, M_2 *compatible* if M_1 and M_2 cover the same external edges.

Corollary 4.5 *Let M_1 and M_2 be compatible states of G and $v \in Ext(G)$. Then G contains an M_1-alternating v-loop iff G contains an M_2-alternating v-loop.*

Proof. The role of M_1 and M_2 is symmetric, so we need to prove one direction only. To this end let (v_1, v_2) be an edge of the cycle of an M_1-alternating v-loop α and let α_1 and α_2 denote the appropriate positive external M_1-alternating subpaths of α running to vertices v_1 and v_2, respectively. According to Theorem 2.1, there exists a mediator alternating network Γ between M_1 and M_2 containing only alternating cycles. Clearly, we can suppose without loss of generality that Γ consists of one M_1-alternating cycle β. We claim that for α_i, $i = 1, 2$, either α_i is accessible from v in M_2 or an M_2-alternating v-loop can be formed from the edges of $E(\alpha_i) \cup E(\beta)$. If the latter case holds for at least one of α_1 and α_2, then we are ready. Otherwise the Corollary can be obtained by Proposition 4.4.

Our claim is obviously enough to be proved for α_1 with the assumption that $E(\alpha_1) \cap E(\beta) \neq \emptyset$. Let w and w' denote the first and the last vertex of α_1 which are also in $V(\beta)$. If w and w' are in odd distance on β, then the requested M_2-alternating path is obtained by combining $\alpha[v,w]$, the positive M_2-alternating subpath of β between w and w', and $\alpha[w', v_1]$. Otherwise it is easy to see that there must exist a subpath α' having its endpoints x and y, but no other vertices, in $V(\beta)$ such that both x and y are in an odd distance from w on β . This allows an M_2-alternating v-loop to be constructed from $\alpha[v,w]$, α' and an appropriate M_2-alternating subpath of β. Hence the proof is complete. □

For a further analysis of v-loops we introduce the graph C_v^M, where C is an external elementary component of G containing external vertex v and $M \in S(G)$. The graph C_v^M is the subgraph of C spanned by the edges that are M_C-viable from v in the subgraph C. Moreover, let $\mathcal{C}_v^M$ denote the set $\mathcal{C}(M_C, v) \cup \{C_v^M\}$. Finally, G_v^M will denote the graph which consists of $\cup\mathcal{C}_v^M$ plus the edges connecting different elements of $\mathcal{C}_v^M$.

Note that generally G_v^M is not equal to the graph $G[V(\cup\mathcal{C}_v^M)]$. Moreover, we might have the impression that G_v^M contains all the edges M-viable from v. However, by Lemma 3.8 and by Proposition 2.13, it is easy to see that the above fact is true iff $\mathcal{C}(M_C, v) = \mathcal{C}^h(M_C, v)$ and any edge of C M_C-viable from v in the augmentation of C is also M_C-viable from v in C.

Proposition 4.6 *G_v^M is a soliton subgraph with respect to M. Furthermore, the set of elementary components of G_v^M is $\mathcal{C}_v^M$.*

Proof. The first sentence is evidently tru if C is a trivial component. Furthermore, if C is non-trivial, then any maximal M-alternating trail starting from v is entirely contained in C_v^M, which implies that G_v^M is indeed a soliton subgraph with respect to M. To verify the second sentence, observe that if an edge of a soliton subgraph G' of G is forbidden in G, then it is also forbidden in G'. For this reason, all we have to prove is that C_v^M is elementary. To this end we will make use of the following two claims.

Claim A *If an edge of C_v^M is part of an even M-alternating cycle α of G, then α is enterily contained in C_v^M.*

Proof. Straightforward.

Claim B *Any edge of C_v^M traversed by an M-alternating crossing in G is in the unique external elementary component of C_v^M.*

Proof. It is clear that C_v^M has a unique external elementary component. Let $\alpha = v_0, e_1, \ldots, e_n, v_n$ be a fixed M-alternating crossing and e_i be an arbitrary edge of α which is also in C_v^M. Furthermore let β be an external M-alternating trail starting from v and terminating at e_i such that $k = \mid E(\beta)\backslash E(\alpha) \mid$ is minimal. We will prove the claim by induction on k. The basis step $k = 0$ is trivial. For the induction step, consider the last edge e_β of β not on α and let w denote the endpoint of e_β contained in $V(\alpha)$. We can assume without loss of generality that $w = v_j$ with $j < i$. Then clearly $e_{j+1} \in M$. If $\beta[v, w]$ does not overlap with $\alpha[v_i, v_n]$, then the crossing $\beta[v, w] + \alpha[v_j, v_n]$ does the job. Otherwise, let v_k be the first vertex of $\alpha[v_i, v_n]$ incident with an edge e of $E(\beta)\backslash E(\alpha)$ and let u denote the vertex of β with $u = v_k$. Observe that, starting from v, β must go through e before reaching u. Indeed, if not, then – since $e_{k+1} \in M$ – $\beta[v, u] + \alpha[v_k, v_{i-1}]^{-1}$ would contradict the choice of β. Therefore $\alpha' = \beta[u, w] + \alpha[v_j, v_k]$ will form an even M-alternating cycle, which shows by Claim A that e_i and e_{k+1} are in the same elementary component of C_v^M. Finally, by applying the induction hypothesis for e_{k+1} we obtain Claim B. □

Continuing the proof of Proposition 4.6, let us suppose by way of contradiction that C_v^M has an internal elementary component C'. Then, there must exist an allowed edge e of C having exactly one endpoint in C'. Let f denote the edge of C' incident with e such that $f \in M$. Clearly $e \notin M$, consequently, by Corollary 2.2, a complete M-alternating trail α must go through e. Applying Claim A and Claim B for f and α, we obtain a contradiction, which makes the proof complete. □

Corollary 4.7 *Each edge of G_v^M is viable from v in $M_{G_v^M}$.*

Proof. Based on Proposition 4.6, we have $\mathcal{C}_{G_v^M}(M_{C_v^M}, v) = \mathcal{C}_G(M_C, v)$, i.e. $\mathcal{C}_{G_v^M}(M_{C_v^M}, v)$ contains all elementary components of G_v^M which are different from C_v^M. Then the claim is obtained with the help of Lemma 3.8. □

Proposition 4.8 *For any external vertex v of G, there exists an alternating v-loop with respect to state M iff G_v^M is non-bipartite.*

Proof.

'Only if' Let C be the elementary component containig v, and α be an M-alternating v-loop. If each edge of α is also contained in C_v^M, then we are ready. Otherwise, starting from v, let w be the first vertex of α such that an appropriate subpath α' of α forms a C-loop with one of its endpoints being w. Then, it is easy to see with the help of Corollary 2.21, that each edge of $\alpha[v, w]$ is contained in G_v^M. Therefore α' is also a C-loop in G_v^M, consequently, because of Claim 3.7, both endpoints of α' are $M_{G_v^M}$-accessible from v. Finally, applying Proposition 4.4 for the endpoints of the last edge of $\alpha[v, w]$, we obtain that G_v^M has a v-loop, indicating that it is non-bipartite.

'If' Let us suppose by way of contradiction that G_v^M does not contain M-alternating v-loops. Then let G' denote a maximal bipartite soliton subgraph of G_v^M with respect to $M_{G_v^M}$ such that $v \in V(G')$ and each edge of G' is viable from v in $M_{G'}$. Note that such a subgraph G' exists under our assumption, because any maximal external alternating trail starting from v as a v-racket or a crossing from v has the required properties. Based on Corollary 4.7, there exists a maximal external $M_{G_v^M}$-alternating trail β from v to some vertex v' traversing an edge not in G'. Let e denote the first edge of β not in $E(G')$. Moreover, let w be the endpoint of e belonging to $V(G')$ with A being the bipartition class of G' containing w. Observe that $E(\beta[w, v']) \cap E(G') \neq \emptyset$ and starting from w, the first overlap will occur at a vertex u in A. Indeed, checking any other possible cases, because of $G' + \beta[w, v']$, we would obtain a contradiction with the choice of G'. Furthermore, every edge is viable from v in $M_{G'}$, consequently there exists an $M_{G'}$-alternating trail γ from v to u. Observe that γ is also positive, as the parity of the length of γ and that of $\beta[v, w]$ must be equal because of the bipartition of G'. Finally, applying Proposition 4.4 for any edge of $\beta[w, u]$, we obtain a contradiction. Hence the proof is complete. □

Considering double v-rackets too, we can describe non-trivial self-transitions via the elementary structure of soliton automata. We also obtain that, similarly to Theorem 3.9, the problem can be reduced to elementary automata. For this final result we introduce the following concept.

Definition 4.9 Let $\{C_1, \ldots, C_n\}$ be the set of the elementary components of G with C_1 being external. G is a *component-chain graph* if it can be decomposed in the *chain-form* $G = C_1 + (w_1, v_2) + C_2 + (w_2, v_3) + \ldots + (w_{n-1}, v_n) + C_n$ such that for each $2 \leq i \leq n-1$, $(w_{i-1}, v_i), (w_i, v_{i+1}) \in E(G)$ with the vertices v_i and w_i belonging to different canonical classes of C_i.

We shall be interested in situations when G_v^M is a component-chain graph for some graph G with external vertex v and $M \in S(G)$. In that case we augment Definition 4.9 by taking $v_1 = v$. We will call a (external or internal) positive M_{C_i}-alternating path *M^v-transit* if it connects v_i and w_i. Component C_i is said to be *M^v-transit* if $i \neq n$ and either C_i has two different M^v-transit paths or there exists an even M_{C_i}-alternating cycle disjoint from the unique M^v-transit path of C_i. Finally, C_i is called an *M^v-terminal* if $i = n$ and C_i' has an M_{C_i}-alternating double w-racket, where either $C_i' = C_i$ with $w = v$ or $C_i' = C_i + (w_i, v_{i-1})$ with $w = v_{i-1}$ depending on whether $n = 1$ or not.

Theorem 4.10 *Let $\mathcal{A}(G)$ be a soliton automaton, M be a state of $\mathcal{A}(G)$ and C be an elementary component of G containing external vertex v. Then, there exists a non-trivial self-transition of v with respect to M, iff one of the following conditions holds.*

(i) G_v^M is not a bipartite component-chain graph.

(ii) G_v^M is a bipartite component-chain graph having an M^v-transit or an M^v-terminal elementary component.

Proof.

'Only if' Based on Theorem 4.3 and Proposition 4.6, it is enough to prove that if G contains an M-alternating double v-racket $\alpha = (\alpha^1, \alpha^2)$ such that $G_v^M$is a bipartite component-chain graph, then *(ii)* holds. To this end, first we claim that in this case α is entirely contained in G_v^M. Indeed, if, on the contrary, e denotes the first edge of α^1 (or α^2) which is not in $E(G_v^M)$, then, based on Corollary 2.21 and the definition of G_v^M e must connect two elementary components belonging to $\{C\} \cup \mathcal{C}(M_C, v)$. Then, clearly, one of the endpoints of e, denoted by w, will be contained in $V(C)$. However, it is a contradiction in the choice of e, because an appropriate subpath of $\alpha[v, w]$ will be a C-loop between w_1 and w, which implies that w is also in the unique canonical class of $\mathcal{P}_C(M_C, v)$, consequently e should be contained in G_v^M.

Therefore let us consider the chain form $G_v^M = C_1 + (w_1, v_2) + \ldots + (w_{n-1}, v_n) + C_n$ with $C_1 = C_v^M$ and $v_1 = v$. Let C_i and C_j, $1 \leq i, j \leq n$, denote the elementary components containing α_c^1 and α_c^2, respectively. Furthermore, let α_l^k, with $k = 1, 2$ and $l = i, j$, denote the subtrail of α^k running entirely in C_l, whereas the notation (A_i, B_i) will be used for the bipartition of C_i with $w_i \in B_i$. We may suppose without loss of generality that $i \leq j$. Now consider the elementary component C_k containing the branching vertex of α. If $k < i$, then it is easy to see that C_k is M^v-transit. Therefore we may suppose for the rest of the proof that $i = k$. Then we distinguish two cases.

Case (a) $i < j$. Then α_i^2 is an M^v-transit path. Therefore, we are ready, if α_i^2 is disjoint from α_c^1. Otherwise, let u' denote the first vertex of α_i^1 incident with an edge of $E(\alpha_i^1) \backslash E(\alpha_i^2)$. Then $u' \neq w_i$, because it is easy to check that α_i^2 is not a subpath of α_i^1. Thus continuing α_i^1 from u', there will be a first vertex u'' of the appropriate subtrail of α_i^1 which is also in $V(\alpha_i^2)$. Now it can be easily observed that $u' \in B_i$ and $u'' \in A_i$, therefore the edges of $E(\alpha_i^2) \cup E(\alpha_i^1[u', u''])$ form two M^v-transit paths, as desired.

Case (*b*) $i = j$. If $i = n$, then C_i is clearly M^v-terminal, thus we are ready. For other alternatives we will prove that C_i is M^v-transit. To this end let β_i be an M^v-transit path of C_i. Furthermore, starting from v_i let u_i denote the last vertex of β_i which is also in $V(\alpha_i^1) \cup V(\alpha_i^2)$. The role of α_i^1 and α_i^2 is symmetric, thus we can assume that $u_i = u$ for some vertex u of α_i^1. Obviously, $\alpha' = \alpha_i^1[v_i, u] + \beta_i[u_i, w_i]$ is an M^v-transit path, because u_i must belong to B_i. Now following the same argument for α' and α_i^2 which was applied in the proof of Case (*a*), we obtain the claim.

'If' By Theorem 4.3 and Proposition 4.8, it is sufficient to prove that a bipartite G_v^M contains an $M_{G_v^M}$-alternating double v-racket, if (*i*) or (*ii*) holds. In this case observe that each family of G_v^M is singleton. Indeed, if family $\mathcal{F}$ is not a singleton, then there must exist an M_1-alternating C'-loop β connecting vertices $v_1, v_2 \in V(C')$, where $M_1 \in S(G_v^M)$ and $C' = r(\mathcal{F})$. It was proved in [1] that any two vertices of an elementary graph is contained in a common complete alternating trail. Consequently, there exists for some $M' \in S(C')$ a complete M'-alternating trail γ traversing both v_1 and v_2. The length of $\gamma[v_1, v_2]$ is clearly even, thus $\beta + \gamma[v_1, v_2]$ indicates that G_v^M is non-bipartite, which contradicts our assumption.

Therefore, if (*i*) holds, then there must be elementary components C_1, C_2, C_3 of G_v^M such that $\mathcal{F}_{C_1} \mapsto \mathcal{F}_{C_i}$ for $i = 2, 3$ by two different edges $e_2 \neq e_3$. Then, as we have seen in the proof of Theorem 3.5, for $i = 2, 3$, the endpoint of e_i in C_i is connected to some vertex of any even M_{C_i}-alternating cycle by an internal positive M_{C_i}-alternating path. Based on Corollary 4.8, both e_2 and e_3 are viable by alternating paths entering C_2 and C_3 through e_2 and e_3, respectively. Summerizing the above facts we can easily obtain the claim, if (*i*) holds.

Finally, making use of Corollary 4.7, we can build an M-alternating double v-racket by an obvious way in a graph with the conditions of (*ii*). Therefore the proof is complete. □

Finally, observe that C_v^M is trivially determined for constant automata, thus Definition 4.9 has a simplified form. Therefore the use of Theorem 4.10 is much easier in this special case.

5 Conclusion

We have worked out a decomposition of soliton automata into elementary automata. As the internal component automata are full and the appropriate α_0^ξ-product is effectively computable, future research will concentrate on elementary automata only. Moreover, with the help of our results, the class of constant soliton automata is fully characterized. Considering practical issues, non-trivial self-transitions have an important role. We have also reduced this problem to elementary components, namely we have proved that to find self-transitions we only need to search for a double v-racket or a pair of disjoint alternating paths in a bipartite elementary graph.

References

[1] M. Bartha, E. Gombás, On graphs with perfect internal matchings,*Acta Cybernetica* **12** (1995), 111-124.

[2] M. Bartha, M. Krész, Structuring the elementary components of graphs having a perfect internal matching, *submitted.*

[3] F. L. Carter (ed.), *Molecular Electronic Devices*, Marcel Dekker, Inc., New York, 1982.

[4] J. Dassow, H. Jürgensen, Soliton automata, *J. Comput. System Sci.* **40** (1990), 154-181.

[5] J. Dassow, H. Jürgensen, Soliton automata with a single exterior node, *Theoretical Computer Science* **84** (1991), 281-292.

[6] J. Dassow, H. Jürgensen, Soliton automata with at most one cycle, *J. Comput. System Sci.* **46** (1993), 155-197.

[7] F. Gécseg, *Products of Automata*, Akademie-Verlag, Berlin, 1986.

[8] F. Gécseg, H. Jürgensen, Automata represented by products of soliton automata, *Theoretical Computer Science* **74** (1990), 163-181.

[9] L. Lovász, M. D. Plummer, *Matching Theory*, North-Holland, Amsterdam, 1986.