Acta Cybernetica 15 (2001) 33-44.

On isomorphic representations of generalized definite automata*

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To Professor Magnus Steinby on his 60th birthday

Abstract

In this paper, the generalized definite automata are studied. In particular, systems which are isomorphically complete for this class with respect to the α_i-products are characterized.

1 Introduction

Generalized definite languages and recognizers were introduced by Ginzburg in [5]. Generalized definite automata were also studied in the works [2], [7], [8]. This class is so wide that the classes of definite and reverse definite automata are its proper subclasses (see [7]). Here, we deal with the isomorphic representations of the generalized definite automata with respect to the α_i-products. In particular, necessary and sufficient conditions are given for systems of automata to be isomorphically complete for this class with respect to the α_i-products. The paper is organized as follows. After the preliminaries of Section 2, we recall the characterizations of the subdirectly irreducible definite, reverse definite, and generalized definite automata in Section 3. Then we describe the isomorphically complete systems for the class of all generalized definite automata with respect to the α_i-products.

2 Preliminaries

In what follows, X always denotes a finite alphabet, and as usual X^* denotes the set of all words over X. For every nonnegative integer j, let $X^j = \{w : w \in X^* \text{ and } |w| = j\}$, where $|w|$ denotes the length of the word w.

By an *automaton* we mean a pair $\mathbf{A} = (A, X)$, where A is a finite nonempty set of *states*, X is a finite nonempty set of the *input symbols*, and every $x \in X$ is realized as a unary operation $x^{\mathbf{A}} : A \to A$. For any word $w = x_1 \dots x_s \in X^*$,

*his work has been supported by the the Hungarian National Foundation for Scientific Research, Grant T030143, and the Ministry of Culture and Education of Hungary, Grant FKFP 0704/1997.

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$w^{\mathbf{A}} : A \to A$ is defined as the composition of the mappings $x_1^{\mathbf{A}}, \ldots, x_s^{\mathbf{A}}$. If $\mathbf{A}$ is known from the context, we write simply aw for $aw^{\mathbf{A}}$.

By the definition above, an automaton can be considered as a unoid. Therefore, such notions as subautomata, congruences, homomorphisms, isomorphisms, embeddings, direct products, subdirect products, subdirect irreducibility can be defined in the usual way (see *e.g.* [1] or [6]). We shall use a particular isomorphism defined as follows. Let $\mathbf{A} = (A, X)$ and $\mathbf{B} = (B, Y)$ be two automata, μ a one-to-one mapping of A onto B and τ a one-to-one mapping of X onto Y. Then the pair μ, τ of mappings is called an (A, X)-*isomorphism* of $\mathbf{A}$ onto $\mathbf{B}$ if $ax^{\mathbf{A}}\mu = a\mu(x\tau)^{\mathbf{B}}$ is valid, for all $a \in A$ and $x \in X$. In this case, it is said that $\mathbf{A}$ and $\mathbf{B}$ are (A, X)-*isomorphic*.

We introduce some particular congruence relations which we need in the sequal. For this purpose, let $\mathbf{A} = (A, X)$ be an arbitrary automaton with at least three states. A congruence ρ of $\mathbf{A}$ is called *elementary* if $\rho = \omega_A \cup \{(a, b), (b, a)\}$ for two distinct states $a, b \in A$, where ω_A denotes the *diagonal relation*, *i.e.*, $\omega_A = \{(a, a) : a \in A\}$. Let us denote by $Con_e(\mathbf{A})$ the set of all elementary congruences of $\mathbf{A}$.

Now, let $j \geq 0$ be an arbitrary integer. Let us define the relation ρ_j on A by

$$a \rho_j b \text{ if and only if } ap = bp, \text{ for all } p \in X^j.$$

It is easy to see that for every nonnegative integer j, ρ_j is a congruence relation of $\mathbf{A}$.

For every integer $j \geq 0$, let us define the state set A_j as follows:

$$A_0 = \{a : a \in A \text{ and } ax = a, \text{ for all } x \in X\},$$

$$A_{j+1} = \{a : a \in A \text{ and } ax \in A_j, \text{ for all } x \in X\}.$$

Then $\mathbf{A}_j = (A_j, X)$ is a subautomaton of $\mathbf{A}$ provided that $A_0 \neq \emptyset$, and the Rees congruences defined by

$$a \sigma_j b \text{ if and only if } a, b \in A_j \text{ or } a = b$$

are congruence relations of $\mathbf{A}$.

For any integer $k \geq 0$, an automaton $\mathbf{A} = (A, X)$ is called *weakly k-definite*, if $|Ap| = 1$, for every $p \in X^k$. Moreover, it is said that $\mathbf{A}$ is *definite* if it is weakly k-definite for some integer $k \geq 0$. In particular, if a weakly k-definite automaton $\mathbf{A}$ has such a state a^*, called *dead state*, that $a^* x = a^*$, for all $x \in X$, then $\mathbf{A}$ is called a *nilpotent automaton*. In this case $Ap = \{a^*\}$ holds, for every $p \in X^k$.

For any integer $k \geq 0$, an automaton $\mathbf{A} = (A, X)$ is called *weakly reverse k-definite* if $apx = ap$ is valid, for all $a \in A$, $p \in X^k$, and $x \in X$. $\mathbf{A}$ is *reverse definite* if it is weakly reverse k-definite for some $k \geq 0$.

Following [8], for any pair of integers $h, k \geq 0$, an automaton $\mathbf{A} = (A, X)$ is called *weakly (h, k)-definite* if $aupv = auv$ is valid, for all $a \in A$, $u \in X^h$, $v \in X^k$, and $p \in X^*$. It is worth noting that for every pair of integers $h' \geq h$, $k' \geq k$, an automaton $\mathbf{A}$ is weakly (h', k')-definite if it is weakly (h, k)-definite. An automaton is called *generalized definite* if it is weakly (h, k)-definite for some integers $h, k \geq 0$. Let us denote by $\mathcal{G}$ the class of all generalized definite automata. By the definitions, one can obtain the following observation.

Lemma 1. *If* $\mathbf{A} \in \mathcal{G}$ *and* $\mathbf{B}$ *is a homomorphic image of* $\mathbf{A}$, *then* $\mathbf{B} \in \mathcal{G}$ *as well.*

We recall here the notion of α_i-products (see *e.g.* [3], [4]). This product family is a natural generalization of the serial connection or cascade product of automata.

Let i be an arbitrary nonnegative integer. Let us consider the automata $\mathbf{A} = (X, A)$, $\mathbf{A}_j = (X_j, A_j)$, $j = 1, \ldots, m$, and let Φ be a family of feedback functions below

$$\varphi_j : A_1 \times \ldots \times A_{j+i-1} \times X \to X_j, \quad j = 1, \ldots, m.$$

It is said that $\mathbf{A}$ is the *α_i-product* of $\mathbf{A}_j$, $j = 1, \ldots, m$, if the following conditions are satisfied:

(1) $A = \prod_{j=1}^{m} A_j$,

(2) for all $(a_1, \ldots, a_m) \in A$ and $x \in X$,

$$(a_1, \ldots, a_m)x^{\mathbf{A}} = (a_1 x_1^{\mathbf{A}_1}, \ldots, a_m x_m^{\mathbf{A}_m})$$

is valid where $x_j = \varphi_j(a_1, \ldots, a_{j+i-1}, x)$, for all $j \in \{1, \ldots, m\}$.

For the α_i-product introduced above, we use the notation

$$\mathbf{A} = \prod_{j=1}^{m} \mathbf{A}_j(X, \Phi) .$$

When the component automata $\mathbf{A}_j$ are equal, say $\mathbf{A}_j = \mathbf{B}$, $j = 1, \ldots, m$, then it is said that the α_i-product $\mathbf{A}$ is an *α_i-power* of $\mathbf{B}$ and it is denoted by $\mathbf{B}^m(X, \Phi)$.

In particular, if each of the feedback functions is independent of the states, *i.e.*, if the feedback functions have the forms $\varphi_j : X \to X_j$, $j = 1, \ldots, m$, then the α_i-product is called *quasi-direct product*. It corresponds to the parallel connection of automata where the sign transformation is allowed.

Lemma 2. *If an automaton* $\mathbf{A}$ *can be embedded into an* α_0*-product of automata* $\mathbf{A}_j$, $j = 1, \ldots, k$, *moreover, each automaton* $\mathbf{A}_j$ *can be embedded into an* α_0*-product of automata* $\mathbf{A}_{jt}$, $t = 1, \ldots, m_j$, *then* $\mathbf{A}$ *can be embedded into an* α_0*-product of automata* $\mathbf{A}_{jt}$, $t = 1, \ldots, m_j$; $j = 1, \ldots, k$.

Let $\mathcal{N}$ be an arbitrary class and $\mathcal{M}$ a system of automata. It is said that $\mathcal{M}$ is *isomorphically complete* for $\mathcal{N}$ with respect to the α_i-product if for any automaton $\mathbf{B} \in \mathcal{N}$, there exist automata $\mathbf{A}_j \in \mathcal{M}$, $j = 1, \ldots, m$, such that $\mathbf{B}$ can be embedded into an α_i-product of the automata $\mathbf{A}_j$, $j = 1, \ldots, m$.

3 Isomorphic representations

We shall use the subdirectly irreducible definite, reverse definite, and generalized definite automata, whose charcterizations can be found in [2].

Proposition 1 ([2]). *A definite automaton* $\mathbf{A}$ *with* $|A| \geq 3$ *is subdirectly irreducible if and only if* $Con_e(\mathbf{A}) = \{\rho_1\}$.

To characterize the subdirectly irreducible reverse definite automata we need some preparations.

For any $m \geq 2$, let $X_m = \{x_1, \ldots, x_m\}$ and define the sets $A(m,k)$, $k = 0, 1, \ldots$, inductively so that

$A(m,0) = \{0,1\}$,

$A(m,1) = \{0,1\} \cup \{(i_1, \ldots, i_m) \in \{0,1\}^m : \ i_r \neq i_s \text{ for some } 1 \leq r < s \leq m\}$,

and for any $k \geq 1$,

$A(m,k+1) =$

$A(m,k) \cup \{(i_1, \ldots, i_m) \in A(m,k)^m : \{i_1, \ldots, i_m\} \cap (A(m,k) \setminus A(m,k-1)) \neq \emptyset\}$.

For any $m \geq 2$ and $k \geq 1$, we define an automaton $\mathbf{A}(m,k) = (A(m,k), X_m)$ as follows:

(1) for both $i \in A(m,0)$ and $x \in X_m$, let $ix = i$;

(2) for all $(i_1, \ldots, i_m) \in A(m,k) \setminus A(m,0)$ and $x_s \in X_m$, let $(i_1, \ldots, i_m)x_s = i_s$.

It is clear that for any $m \geq 2$ and $k \geq 1$, $\mathbf{A}(m,k)$ is a subautomaton of $\mathbf{A}(m,k+1)$.

We recall that an automaton $\mathbf{A} = (A, X)$ is *input reduced* if $x^{\mathbf{A}} \neq y^{\mathbf{A}}$ for all pairs of distinct input symbols $x, y \in X$.

Proposition 2 ([2]). *Let* $\mathbf{A} = (A, X)$ *be an input reduced automaton such that* $|A| \geq 3$ *and* $|X| = m$. *If* $\mathbf{A}$ *is subdirectly irreducible and reverse* k*-definite, but not nilpotent, then* $m \geq 2$, $k \geq 1$ *and* $\mathbf{A}$ *is* (A,X)*-isomorphic to a subautomaton of* $\mathbf{A}(m,k)$.

From this statement the next observation follows immediately.

Corollary 1. *If* $\mathbf{A}$ ($|A| \geq 3$) *is subdirectly irreducible and reverse* k*-definite, but not nilpotent automaton, then there is an* $m \geq 2$ *such that* $\mathbf{A}$ *can be embedded into a quasi-direct product of* $\mathbf{A}(m,k)$ *with a single factor.*

Proposition 3 ([2]). *A generalized definite automaton* $\mathbf{A}$ *with at least three states is subdirectly irreducible if and only if* $Con_e(\mathbf{A}) = \{\rho_1\}$ *or* $Con_e(\mathbf{A}) = \{\sigma_0\}$.

We also need some particular automata. Let $\mathbf{R} = (\{0,1\}, \{x,y\})$ denote the two-state reset automaton defined by $0x^{\mathbf{R}} = 1x^{\mathbf{R}} = 1$ and $0y^{\mathbf{R}} = 1y^{\mathbf{R}} = 0$. Finally, for every positive integer $s \geq 2$, let $\mathbf{I}_s = (\{0, \ldots, s\}, \{x_1, \ldots, x_s\})$ denote the automaton defined as follows: for all $i \in \{0, 1, \ldots, s\}$, $i \neq s-1$ and $x_j \in \{x_1, \ldots, x_s\}$, let

$$ix_j^{\mathbf{I}_s} = \begin{cases} i+j & \text{if } i+j \leq s, \\ s & \text{otherwise,} \end{cases}$$

$$(s-1)x_j^{\mathbf{I}_s} = s-1.$$

It is easy to see that each of the automata defined above is generalized definite.

Now, we are ready to characterize the isomorphically complete systems for $\mathcal{G}$.

Theorem 1. *A system $\mathcal{M}$ of generalized definite automata is isomorphically complete for $\mathcal{G}$ with respect to the α_0-product if and only if there exists an automaton $\mathbf{R}' \in \mathcal{M}$ such that $\mathbf{R}$ can be embedded into a quasi-direct product of $\mathbf{R}'$ with a single factor, moreover, for every positive integer $s \geq 2$, there exists an automaton $\mathbf{I}'_s \in \mathcal{M}$ having at least $s+1$ distinct states denoted by $0, 1, \ldots, s$, such that for every $i < j \in \{0, 1, \ldots, s\}$, $\{i, j\} \neq \{s-1, s\}$, there exists an input symbol x_{ij} of $\mathbf{I}'_s$ with $ix_{ij} = j$, and there is an input symbol x of $\mathbf{I}'_s$ with $sx = s$ and $(s-1)x = s-1$.*

Proof. To prove the necessity of the conditions, let us suppose that $\mathcal{M}$ is an isomorphically complete system of generalized definite automata for $\mathcal{G}$ with respect to the α_0-product. Since $\mathbf{R} \in \mathcal{G}$, there exist automata $\mathbf{A}_j = (A_j, X_j) \in \mathcal{M}$, $j = 1, \ldots, m$, such that $\mathbf{R}$ can be embedded into an α_0-product $\prod_{j=1}^m \mathbf{A}_j(X, \Phi)$. Then it is easy to see that $\mathbf{R}$ can be embedded into a quasi-direct product of some $\mathbf{A}_j$ with a single factor. Similarly, by our assumption, $\mathbf{I}_s$ can be embedded into an α_0-product $\prod_{j=1}^m \mathbf{A}_j(X, \Phi)$ of automata in $\mathcal{M}$, since $\mathbf{I}_s \in \mathcal{G}$. Let μ denote a suitable embedding and let $t\mu = (a_{t1}, \ldots, a_{tm})$, $t = 0, \ldots, s$. Moreover, let us denote by r the least integer for which $a_{s-1,r} \neq a_{sr}$.

First we show that $a_{ir} \notin \{a_{s-1,r}, a_{sr}\}$, for any $0 \leq i < s-1$. Contrary, let us suppose that $a_{ir} = a_{sr}$ for some $0 \leq i < s-1$. Then there exists an input symbol $y = \varphi_r(a_{i1}, \ldots, a_{i,r-1}, x_{s-1-i}) \in X_r$ such that $a_{sr}y = a_{ir}y = a_{s-1,r}$. On the other hand, by our assumption, $a_{s-1,r}z = a_{s-1,r}$ and $a_{sr}z = a_{sr}$, where $z = \varphi_r(a_{s1}, \ldots, a_{s,r-1}, x_1)$. Therefore, $a_{sr}z^h y z^k = a_{s-1,r}$ and $a_{sr}z^h z^k = a_{sr}$ is valid for every pair of integers $h, k \geq 0$. This contradicts the fact that $\mathbf{A}_r$ is a generalized definite automaton. Consequently, $a_{ir} \neq a_{sr}$, for any $0 \leq i < s-1$. One can prove in similar way that $a_{ir} \neq a_{s-1,r}$, for all $i \in \{0, 1, \ldots, s-2\}$.

Next we show that the elements a_{tr}, $t = 0, 1, \ldots, s-2$, are pairwise different. Contrary, let us suppose that $a_{ir} = a_{jr}$ for some integers $0 \leq i < j \leq s-2$. Since $i < j$ and μ is an embedding, there exists an input symbol $x \in X_r$ such that $a_{ir}x^h = a_{ir}$ holds, for every nonnegative integer h. Moreover, there are such input symbols $y, \bar{x}_1, \bar{x}_2 \in X_r$ for which $a_{ir}y = a_{s-2,r}$, $a_{s-2,r}\bar{x}_1 = a_{s-1,r}$, and $a_{s-2,r}\bar{x}_2 = a_{sr}$. Finally, $a_{sr}z = a_{sr}$ and $a_{s-1,r}z = a_{s-1,r}$, where $z = \varphi_r(a_{s1}, \ldots, a_{s,r-1}, x_1)$ again. Then $a_{ir}x^h y \bar{x}_1 z^k = a_{s-1,r}$ and $a_{ir}x^h y \bar{x}_2 z^k = a_{s,r}$ hold, for every pair of integers $h, k \geq 0$, which contradicts the fact that $\mathbf{A}_r$ is generalized definite. Therefore, $a_{ir} \neq a_{jr}$, for any integers $0 \leq i < j \leq s-2$. By the two observations given above, we obtain that the elements $a_{0,r}, a_{1r}, \ldots, a_{sr}$ are pairwise different. This results in that $\mathbf{A}_r$ can be considered as the required automaton $\mathbf{I}'_s$.

In order to prove the sufficiency of the conditions, let us suppose that $\mathcal{M}$ has the required properties. We prove that $\mathcal{M}$ is an isomorphically complete system for $\mathcal{G}$ with respect to the α_0-product. For this reason, let us consider an arbitrary generalized definite automaton $\mathbf{A} = (A, X)$ of n states. We prove by induction on n that $\mathbf{A}$ can be embedded into an α_0-product of automata in $\mathcal{M}$. If $n = 1$ or $n = 2$, then the statement is obviously valid. Now, let $n > 2$ and suppose that the statement is valid for every $m < n$. If $\mathbf{A}$ is subdirectly reducible, then it can be embedded into a direct product of generalized definite automata having fewer states than n. By our induction hypothesis, each component automaton of this direct product can be embedded into an α_0-product of automata in $\mathcal{M}$, and thus, by Lemma 2, $\mathbf{A}$ can be also embedded into an α_0-product of automata in $\mathcal{M}$.

Let us suppose now that $\mathbf{A}$ is subdirectly irreducible. Then, by Proposition 3, $Con_e(\mathbf{A}) = \{\rho_1\}$ or $Con_e(\mathbf{A}) = \{\sigma_0\}$. We distinguish two cases depending on $Con_e(\mathbf{A})$.

Case 1. $Con_e(\mathbf{A}) = \{\rho_1\}$. Let $c \neq d \in A$ with $c\rho_1 d$ for some states $c, d \in A$. Then, by the definition of ρ_1, $cx = dx$, for all $x \in X$. Let $X_1 = \{x : x \in X \text{ and } cx = c\}$. Moreover, let 0, 1 and u, v denote the states and the input symbols of $\mathbf{R}'$ for which $0u = 1u = 0$ and $0v = 1v = 1$ hold. Let us consider the α_0-product $\mathbf{A}/\rho_1 \times \mathbf{R}'(X, \Phi)$ defined as follows. For all $a \in A \setminus \{c, d\}$ and $x \in X$, let

$$\varphi_1(x) = x$$

and

$$\varphi_2(\{a\}, x) = \begin{cases} u & \text{if } ax^{\mathbf{A}} \notin \{c, d\} \text{ or } ax^{\mathbf{A}} = c, \\ v & \text{otherwise,} \end{cases}$$

$$\varphi_2(\rho_1(c), x) = \begin{cases} u & \text{if } cx^{\mathbf{A}} \notin \{c, d\} \text{ or } x \in X_1, \\ v & \text{otherwise.} \end{cases}$$

Let us define the mapping $\mu : A \to A/\rho_1 \times \{0, 1\}$ by

$\mu : a \to (\{a\}, 0)$, for all $a \in A \setminus \{c, d\}$,
$\mu : c \to (\rho_1(c), 0)$,
$\mu : d \to (\rho_1(c), 1)$,

where $\rho_1(c)$ denotes the equivalence class containing c. Then it is easy to see that μ is an embedding of $\mathbf{A}$ into the α_0-product $\mathbf{A}/\rho_1 \times \mathbf{R}'(X, \Phi)$, and thus, our induction hypothesis and Lemmas 1 and 2 imply that $\mathbf{A}$ can be embedded into an α_0-product of automata in $\mathcal{M}$.

Case 2. $Con_e(\mathbf{A}) = \{\sigma_0\}$. Let $c \neq d \in A$ with $c\sigma_0 d$. Then $cx = c$ and $dx = d$, for all $x \in X$. Let us suppose that $\mathbf{A}$ is weakly (h, k)-definite for some $h \geq 0$,

$k \geq 0$. For all $a \in A$ and $u \in X^h$, let us define the subautomaton $\mathbf{A}_{au} = (A_{au}, X)$, where $A_{au} = \{aup : p \in X^*\}$. Then $\mathbf{A}_{au}$ is a weakly k-definite automaton. Indeed, let $v \in X^k$ be an arbitrary word and $a' \in A_{au}$. Then there is a word $p \in X^*$ such that $a' = aup$. Since $\mathbf{A}$ is weakly (h, k)-definite, $a'v = aupv = auv$ is valid for all $a' \in A_{au}$, and hence, $A_{au}v = \{auv\}$.

Now, we distinguish two subcases depending on the subautomata $\mathbf{A}_{au}$.

Subcase 1. There exist a state $a \in A$ and a word $u \in X^h$ such that $\mathbf{A}_{au}$ is not singleton. Then $\mathbf{A}_{au}$ is a subdirectly irreducible definite subautomaton of $\mathbf{A}$, since for any congruence γ of $\mathbf{A}_{au}$, the relation $\gamma \cup \omega_A$ is a congruence relation of $\mathbf{A}$. If $|A_{au}| \geq 3$, then by Proposition 1, $Con_e(\mathbf{A}_{au}) = \{\rho'_1\}$, where ρ'_1 denotes the corresponding relation belonging to $\mathbf{A}_{au}$. Then there are states $e, f \in A_{au}$ such that $ex = fx$, for all $x \in X$. Then it is easy to see that the relation θ defined on A by

$$a' \Theta a'' \text{ if and only if } \{a', a''\} \subseteq \{e, f\} \text{ or } a' = a''$$

is an elementary congruence of $\mathbf{A}$ distinct from σ_0 which contradicts our assumption that $Con_e(\mathbf{A}) = \{\sigma_0\}$. If $|A_{au}| = 2$, then $A_{au} = \{e, f\}$ for some states $e, f \in A$ and $ex = fx$ is valid, for all $x \in X$. Then one can define Θ in the same way as above, and Θ is an elementary congruence of $\mathbf{A}$ which results in a contradiction again. Consequently, this subcase is impossible.

Subcase 2. For all $a \in A$ and $u \in X^h$, $\mathbf{A}_{au}$ is singleton. Then, $aux = au$, for all $u \in X^h$ and $x \in X$, and hence, $\mathbf{A}$ is weakly reverse h-definite, moreover, by $Con_e(\mathbf{A}) = \{\sigma_0\}$, $\mathbf{A}$ is not nilpotent. Since $\mathbf{A}$ is subdirectly irreducible, by Corollary 1, $\mathbf{A}$ can be embedded into a quasi-direct product of $\mathbf{A}(m, h)$ with a single factor for an integer $m \geq 2$. Without loss of generality, we may assume that h is the least integer with this property. Let τ denote a suitable embedding of $\mathbf{A}$ into the quasi-direct product of $\mathbf{A}(m, h)$ with a single factor. Let $B_i = (A\tau \cap A(m, i))\tau^{-1}$, $i = 0, 1, \ldots, h$. Since τ is an embedding, it is easy to show that

(3) $\{c, d\} = B_0 \subset B_1 \subset \ldots \subset B_h = A$

(4) $B_i x^{\mathbf{A}} \subseteq B_{i-1}$, for all $1 \leq i \leq h$ and $x \in X$.

Let us consider now the α_0-product $\mathbf{A}/\sigma_0 \times \mathbf{I}'_{h+1}(X, \Phi)$ defined as follows. For all $a \in A \setminus \{c, d\}$ and $x \in X$, let

$\varphi_1(x) = x,$

$$\varphi_2(a, x) = \begin{cases} x_{h-j,h-i} & \text{if } a \in B_j \setminus B_{j-1} \text{ and } ax^{\mathbf{A}} \in B_i \setminus B_{i-1} \\ & \text{for some } 1 \leq i < j \leq h, \\ x_{h-j,h} & \text{if } a \in B_j \setminus B_{j-1} \text{ and } ax^{\mathbf{A}} = c \text{ for some } 1 \leq j \leq h, \\ x_{h-j,h+1} & \text{if } a \in B_i \setminus B_{i-1} \text{ and } ax^{\mathbf{A}} = d \text{ for some } 1 \leq j \leq h, \end{cases}$$

$\varphi_2(\{c, d\}, x) = x',$

where x' denotes now the input symbol of $\mathbf{I}_{h+1}$ for which $hx' = h$ and $(h+1)x' = h+1$ hold. By (3) and (4), the feedback functions are well-defined.

Let us define the mapping μ of $A \to A/\sigma_0 \times \{0, 1, \dots h+1\}$ as follows. For every $a \in A \setminus \{c, d\}$, let

$\mu : a \to (\{a\}, h - j)$ if $a \in B_j \setminus B_{j-1}$ for some $1 \le j \le h$,
$\mu : c \to (\{c, d\}, h)$,
$\mu : d \to (\{ c, d\}, h + 1)$.

Now, it is easy to check that μ is an embedding of $\mathbf{A}$ into the α_0-product under consideration. Then, the induction assumption and Lemmas 1 and 2 yield that $\mathbf{A}$ can be embedded into an α_0-product of automata in $\mathcal{M}$. This ends the proof of the statement.

From Theorem 1, the next observation follows.

Corollary 2. *There is no finite system $\mathcal{M}$ of generalized definite automata which is isomorphically complete for $\mathcal{G}$ with respect to the α_0-product.*

The following statement shows that we can obtain finite isomorphically complete systems by allowing automata as components which are not necessarily generalized definite.

Theorem 2. *A system $\mathcal{M}$ of automata is isomorphically complete for $\mathcal{G}$ with respect to the α_0-product if and only if $\mathcal{M}$ satisfies the conditions below:*

(1) *there exists an automaton $\mathbf{R}' \in \mathcal{M}$ such that $\mathbf{R}$ can be embedded into a quasi-direct product of $\mathbf{R}'$ with a single factor,*

(2) (a) *for every positive integer $s \ge 2$, there exists an automaton $\mathbf{I}'_s \in \mathcal{M}$ which has $s+1$ distinct states, denoted by $0, 1, \dots, s$, such that for all $i < j \in \{0, 1, \dots, s\}$, $\{i, j\} \ne \{s-1, s\}$, there exists an input symbol x_{ij} of $\mathbf{I}'_s$ with $ix_{ij} = j$, and there is a further input symbol x of $\mathbf{I}'_s$ such that $sx = s$ and $s - 1x = s - 1$,*

or

(b) *there exists an $\mathbf{A} \in \mathcal{M}$ which has a state a and not necessarily distinct input symbols u, v, w, z such that $au = a$, $av \ne aw$, $avz = av$, and $awz = aw$.*

Proof. To prove the necessity of the conditions, let us suppose that $\mathcal{M}$ is isomorphically complete for $\mathcal{G}$ with respect to the α_0-product. The necessity of (1) follows from the proof of Theorem 1.

Now, we prove that (2)(a) or (2)(b) is valid for $\mathcal{M}$. For this purpose, let us suppose that (2)(a) is not valid. Then there is a positive integer $s_0 \ge 2$ such that $\mathbf{I}'_{s_0} \notin \mathcal{M}$. By our assumption, $\mathbf{I}_{s_0}$ can be embedded into an α_0-product $\prod_{j=1}^{m} \mathbf{A}_j(X, \Phi)$ of automata in $\mathcal{M}$ since $\mathbf{I}_{s_0} \in \mathcal{G}$. Let μ denote a suitable isomorphism and let $t\mu = (a_{t1}, \dots, a_{tm})$, $t = 1, \dots, s_0$. Let r be the least integer for which

$a_{s_0-1,r} \neq a_{s_0,r}$. Let us observe that if the elements $a_{0r}, a_{1r}, \ldots, a_{s_0,r}$ are pairwise different, then $\mathbf{A}_r$ can be considered as $\mathbf{I}'_{s_0}$ which is a contradiction. Consequently, there are $i < j \in \{0, 1, \ldots, s_0\}$, $\{i, j\} \neq \{s_0 - 1, s_0\}$ such that $a_{ir} = a_{jr}$. Now, it is easy to show that $\mathbf{A}_r$ satisfies condition (b), and therefore, (2)(b) is valid for $\mathcal{M}$.

In order to prove the sufficiency, let us suppose that $\mathcal{M}$ satisfies the conditions. If (1) and (2)(a) are valid, then Theorem 1 implies that $\mathcal{M}$ is isomorphically complete for $\mathcal{G}$ with respect to the α_0-product. Let us assume now that (1) and (2)(b) are valid for $\mathcal{M}$ with $\mathbf{R}'$ and $\mathbf{A}$, respectively. Then, by Lemma 2 and the sufficiency of conditions (1) and (2)(a), it is suffficient to show that $\mathbf{I}_s$ can be embedded into an α_0-product of automata in $\{\mathbf{R}, \mathbf{A}\}$, for every positive integer $s \geq 2$. For this purpose, let $s \geq 2$ be an arbitrary positive integer. Let us define the α_0-product

$$\mathbf{R} \times \cdots \times \mathbf{R} \times \mathbf{A}(\{x_1, \ldots, x_s\}, \Phi),$$

where the number of the occurences of $\mathbf{R}$ is equal to $s - 1$, in the following way. For every $(r_1, \ldots, r_{s-1}) \in \{0,1\}^{s-1}$, $x_k \in \{x_1, \ldots, x_s\}$, and $j \in \{2, \ldots, s-1\}$, let

$$\varphi_1(x_k) = x,$$

$$\varphi_j(r_1, \ldots, r_{j-1}, x_k) = \begin{cases} x & \text{if } \sum_{t=1}^{j-1} r_t + k \geq j, \\ y & \text{otherwise,} \end{cases}$$

$$\varphi_s(r_1, \ldots, r_{s-1}, x_k) = \begin{cases} u & \text{if } \sum_{t=1}^{s-1} r_t + k < s - 1, \\ v & \text{if } \sum_{t=1}^{s-1} r_t + k = s - 1, \\ w & \text{if } \sum_{t=1}^{s-1} r_t + k \geq s \text{ and } \sum_{t=1}^{s-1} r_t \neq s - 1, \\ z & \text{otherwise,} \end{cases}$$

where a and u, v, w, z denote the suitable state and input symbols of $\mathbf{A}$, respectively. Now, let us consider the mapping μ defined by

$$\begin{aligned}
&\mu : 0 \to (0, 0, 0, \ldots, 0, a), \\
&\mu : 1 \to (1, 0, 0, \ldots, 0, a), \\
&\mu : 2 \to (1, 1, 0, \ldots, 0, a), \\
&\quad \vdots \\
&\mu : s - 2 \to (1, 1, \ldots 1, 0, a), \\
&\mu : s - 1 \to (1, 1, \ldots, 1, 1, av), \\
&\mu : s \to (1, 1, \ldots, 1, 1, aw).
\end{aligned}$$

Then it is easy to see that μ is an isomorphism of $\mathbf{I}_s$ into the α_0-product $\mathbf{R} \times \cdots \times \mathbf{R} \times \mathbf{A}(\{x_1, \ldots, x_s\}, \Phi)$ under consideration which ends the proof of Theorem 2.

The next statement directly follows from the definition of the α_i-products.

Lemma 3. *If the automaton $\mathbf{A}$ can be embedded into an α_0-product of automata $\mathbf{A}_j$, $j = 1, \ldots, n$, and each $\mathbf{A}_j$ can be embedded into an α_1-product of automata $\mathbf{A}_{jt}$, $t = 1, \ldots, m_j$, then $\mathbf{A}$ can be embedded into an α_1-product of automata $\mathbf{A}_{jt}$, $t = 1, \ldots, m_j$; $j = 1, \ldots, n$.*

Now, let $i \geq 1$ be an arbitrary integer. Then the isomorphically complete systems of generalized definite automata with respect to the α_i-product can be characterized as follows.

Theorem 3. *A system $\mathcal{M}$ of generalized definite automata is isomorphically complete for $\mathcal{G}$ with respect to the α_i-product ($i \geq 1$) if and only if there exists an automaton $\mathbf{R}'' \in \mathcal{M}$ such that $\mathbf{R}''$ has two distinct states denoted by $0, 1$ and four not necessarily distinct input symbols v, x, y, z such that $1v = 0$, $0x = 0$; $0y = 1$, $1z = 1$.*

Proof. The necessity of the conditions can be proved in a similar way as in the case of Theorem 1.

Regarding the sufficience, let us observe, that $\mathbf{R}$ can be embedded into an α_i-product of $\mathbf{R}''$ with a single factor, and this product is an α_1-product of $\mathbf{R}''$ with single factor. Now we show that for every integer $s \geq 2$, the automaton $\mathbf{I}_s$ can be embedded into an α_1-power of $\mathbf{R}''$.

Let $s \geq 2$ be an arbitrary integer. Let us consider the α_1-power $(\mathbf{R}'')^s(\{x_1, \ldots, x_s\}, \Phi)$ defined as follows. For all $1 \leq k < s$, $(v_1, \ldots, v_k) \in \{0, 1\}^k$, and $x_j \in \{x_1, \ldots, x_s\}$, let

$$\varphi_k(v_1, \ldots, v_k, x_j) = \begin{cases} x & \text{if } j + \sum_{t=1}^{k} v_t < k, \\ y & \text{if } j + \sum_{t=1}^{k} v_t \geq k \text{ and } v_k = 0, \\ z & \text{otherwise,} \end{cases}$$

furthermore, for all $(v_1, \ldots, v_s) \in \{0, 1\}^s$ and $x_j \in \{x_1, \ldots, x_s\}$, let

$$\varphi_s((v_1, \ldots, v_s, x_j) = \begin{cases} x & \text{if } \sum_{t=1}^{s} v_t = s - 1, \\ x & \text{if } \sum_{t=1}^{s} v_t < s - 1 \text{ and } j + \sum_{t=1}^{s} v_t \leq s - 1, \\ y & \text{if } \sum_{t=1}^{s} v_t < s - 1 \text{ and } j + \sum_{t=1}^{s} v_t \geq s, \\ z & \text{if } \sum_{t=1}^{s} v_t = s. \end{cases}$$

Then it is easy to see, that the mapping μ given by

$$\begin{aligned} &\mu : 0 \to (0, 0, 0, \ldots, 0), \\ &\mu : 1 \to (1, 0, 0, \ldots, 0), \\ &\mu : 2 \to (1, 1, 0, \ldots, 0), \\ &\quad \vdots \qquad\qquad \vdots \\ &\mu : (s-1) \to (1, 1, \ldots, 1, 0), \\ &\mu : s \to (1, 1, \ldots, 1, 1), \end{aligned}$$

is an embedding of $\mathbf{I}_s$ into the α_1-power under consideration.

By Theorem 1, the system $\mathcal{K} = \{\mathbf{R}\} \cup \{\mathbf{I}_s : s = 2, 3, \ldots\}$ is an isomorphically complete system for $\mathcal{G}$ with respect to the α_0-poduct. Therefore, every generalized definite automaton can be embedded into an α_0-product of automata in $\mathcal{K}$. On the other hand, we have proved that every automaton in $\mathcal{K}$ can be embedded into an α_1-power of $\mathbf{R}''$. Then, by Lemma 3, we obtain that every generalized definite automaton can be embedded into an α_1-power of $\mathbf{R}''$, and consequently, $\{\mathbf{R}''\}$, and also $\mathcal{M}$, are isomorphically complete systems for $\mathcal{G}$ with respect to the α_i-product.

The following assertion shows that we obtain the same characterization of the isomorphically complete systems consisting of not necessarily generalized definite automata with respect to the α_i-product ($i \geq 1$).

Theorem 4. *A system $\mathcal{M}$ of automata is isomorphically complete for $\mathcal{G}$ with respect to the α_i-product ($i \geq 1$) if and only if it contains an automaton $\mathbf{R}''$ such that $\mathbf{R}''$ has two distinct states, denoted by* 0, 1, *and four not necessarily distinct input symbols x, y, z, v with* $0x^{\mathbf{R}''} = 0$, $0y^{\mathbf{R}''} = 1$, $1z^{\mathbf{R}''} = 1$, $1v^{\mathbf{R}''} = 0$.

Proof. The validity of Theorem 4 follows immediately from Theorem 3.

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Received October, 2000