Acta Cybernetica **15** (2001) 45-52.

Languages Recognized by a Class of Finite Automata *

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Abstract

We consider automata defined by left multiplications in graph algebras, and describe all languages recognized by these automata in terms of combinatorial properties of words which belong to these languages, regular expressions and linear grammars defining these languages. This description is applied to investigate closure properties of the obtained family of languages.

A *language* over an alphabet X is a subset of the free monoid X^* generated by X. We use standard concepts of automata and languages theory (see [7], [11]).

Graph algebras make it possible to apply methods of universal algebra to various problems of discrete mathematics and computer science. They have been investigated by several authors (see [2], [4], [5], [6], [8], [9], and [10] for references). Throughout the word *graph* means a finite directed graph without multiple edges but possibly with loops. The *graph algebra* $\mathrm{Alg}(D)$ of a graph $D = (V, E)$ is the set $V \cup \{\infty\}$ equipped with multiplication given by the rule

$$xy = \begin{cases} x & \text{if } (x, y) \in E, \\ \infty & \text{otherwise,} \end{cases}$$

for all $x, y \in V$. In this paper we use graph algebras as language recognizers.

Let $\mathrm{Alg}(D)$ be a graph algebra. Put $\mathrm{Alg}^1(D) = \mathrm{Alg}(D) \cup \{1\}$, and extend the multiplication of $\mathrm{Alg}(D)$ to the whole set $\mathrm{Alg}^1(D)$ by assuming that 1 acts as an identity on all elements of $\mathrm{Alg}^1(D)$. Let T be a subset of $\mathrm{Alg}^1(D)$, and let $f\colon X \to \mathrm{Alg}^1(D)$ be any mapping. We consider the *graph algebra automaton* $\mathrm{Atm}(D, T)$, where

- the set of states is $\mathrm{Alg}^1(D)$;
- 1 is the initial state;

*The author has been supported by the Estonian Science Foundation, Grant 4912.

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- T is the set of terminal states;
- the next-state function is given by $a \cdot x = f(x)a$, for $a \in \mathrm{Alg}^1(D)$, $x \in X$.

This automaton is defined by left multiplications of elements of the graph algebra.

Our main theorem describes all languages recognized by graph algebra automata in terms of combinatorial properties of words which belong to these languages, as well as in terms of regular expressions for these languages or their complements (see Theorem 1). This description allows us to answer several natural questions concerning the class $\mathcal{G}$ of languages recognized by graph algebra automata. First, we show that this class is not empty and, moreover, contains certain fairly large subclasses (see Corollary 4). Second, we verify that $\mathcal{G}$ is a proper subclass of the class of languages recognizable by finite state automata (see Corollary 6). Third, we give examples which demonstrate that the whole $\mathcal{G}$ is not closed for union, intersection, and product (see Example 7). However, we represent $\mathcal{G}$ as a union of two classes $\mathcal{G}_a$ and $\mathcal{G}_b$ such that $\mathcal{G}_a$ is closed under intersection and left derivative, and $\mathcal{G}_b$ is closed under union and right derivative. Finally, we show that the whole class $\mathcal{G}$ is closed under the Kleene $*$-operation and complement (see Corollaries 8 and 9).

Theorem 1 *For any language L over an alphabet X, the following are equivalent:*

(i) *L is recognized by a graph algebra automaton;*

(ii) *at least one of the following two conditions is satisfied for all $x, y \in X$, and $u, v \in X^*$:*

  (a) *$xyu \in L$ implies $yu \in L$, and*

  $xu, yxv \in L$ implies $yxu \in L$;

  (b) *$yu \in L$ implies $xyu \in L$, and*

  $yxu \in L$ implies $xu \in L$ or $yxv \in L$.

(iii) *there exist disjoint subsets X_1, X_2 of X and a relation $G \subseteq X \times X$ such that the language $L \setminus \{1\}$ or $X^+ \setminus L$ has the following regular expression:*

$$X^* X_1 X^* + X^* X_2 + \sum_{(x_j, x_i) \in G} X^* x_i x_j X^*; \tag{1}$$

(iv) *there exist subsets $Q \subseteq X$ and $P \subseteq X \times X$ such that the language $L \setminus \{1\}$ or $X^+ \setminus L$ is generated by the right linear grammar with the alphabet X, the set $W = \{x' \mid x \in X\} \cup \{s_0\}$ of nonterminal symbols, the start symbol s_0, and productions*

$$\begin{array}{rcll} s_0 & \to & xx' & \text{for all } x \in X, \\ x' & \to & yy' & \text{for all } (y, x) \in P, \\ x' & \to & 1 & \text{for all } x \in Q. \end{array} \tag{2}$$

Proof.

(ii)⇒(i): First, suppose that the language L satisfies (a). Introduce a graph $D = (V, E)$ with the set $V = X$ of vertices, and the set E of edges consisting of all pairs (x, y) such that $yxu \in L$ for some $u \in X^*$. Let f be the mapping from X to $\mathrm{Alg}^1(D)$ defined by $f(x) = x$. Put $T = (\{1\} \cup X) \cap L$.

Take an arbitrary word $u = x_1 \cdots x_n$ in L. If $u = 1$, i.e. $n = 0$, then $1 \in T$, and the automaton $\mathrm{Atm}(D, T)$ accepts u. Further, assume that $n > 0$. The first implication of condition (a) shows that $x_k \cdots x_n \in L$, for all $k = 1, \ldots, n$. It follows that $(x_2, x_1), (x_3, x_2), \ldots, (x_n, x_{n-1}) \in E$ by the definition of E, and $x_n \in T$ by the definition of T. We get $1 \cdot u = f(x_n)(\cdots f(x_1)) = f(x_n) = x_n \in T$, and so the automaton $\mathrm{Atm}(D, T)$ recognizes u.

Consider any word $u = x_1 \cdots x_n$ accepted by $\mathrm{Atm}(D, T)$. If $u = 1$, then $1 \in T$, and so $u = 1 \in L$. Further, assume that $n > 0$. As above, $1 \cdot u \in T$ means that $x_n, \ldots, x_1$ is a directed path in D and $x_n \in T$. By reversed induction on k we show that $x_k \cdots x_n \in L$, for all $k = 1, \ldots, n$. If $k = n$, then $x_n \in T \subseteq L$ by definition. Assume that $x_k x_{k+1} \cdots x_n \in L$, for some $1 < k \leq n$. Since $(x_k, x_{k-1}) \in E$, we have $x_{k-1} x_k v \in L$, for some $v \in X^*$. Then the second implication of (a) yields $x_{k-1} x_k \cdots x_n \in L$, as required. Therefore $u = x_1 \cdots x_n \in L$. Thus L is the language recognized by $\mathrm{Atm}(D, T)$.

Second, suppose that L satisfies condition (b). Observe that the complement $\overline{L} = X^* \setminus L$ of L satisfies condition (a) if and only if L satisfies condition (b). Indeed, denoting the logical negation of a proposition P by $\overline{P}$, we get

$$
\begin{aligned}
&(xyu \notin L \Rightarrow yu \notin L) \equiv \\
&\qquad \equiv \overline{xyu \in L} \Rightarrow \overline{yu \in L}) \\
&\qquad \equiv (\overline{\overline{xyu \in L}} \vee \overline{yu \in L}) \\
&\qquad \equiv (xyu \in L \vee \overline{yu \in L}) \\
&\qquad \equiv (yu \in L \Rightarrow xyu \in L) \quad \text{and} \\
&(xu, yxv \notin L \Rightarrow yxu \notin L) \equiv \\
&\qquad \equiv (\overline{xu \in L} \wedge \overline{yxv \in L} \Rightarrow \overline{yxu \in L}) \\
&\qquad \equiv (\overline{\overline{xu \in L} \wedge \overline{yxv \in L}} \vee \overline{yxu \in L} \\
&\qquad \equiv (xu \in L \vee yxu \in L \vee \overline{yxv \in L}) \\
&\qquad \equiv (yxu \in L \Rightarrow xu \in L \vee yxv \in L).
\end{aligned}
$$

Therefore $\overline{L} = X^* \setminus L$ satisfies (a), and so it is recognized by some automaton $\mathrm{Atm}(D, T)$. Hence L is recognized by the automaton $\mathrm{Atm}(D, \overline{T})$, where $\overline{T} = \mathrm{Alg}^1(D) \setminus T$.

(i)⇒(iii): Suppose that L is recognized by a graph algebra automaton $\mathrm{Atm}(D, T)$ of a graph $D = (V, E)$. First, assume that $\infty \in T$. Let us define the sets:

$$X_1 = \{x \in X \mid f(x) = \infty\} \qquad X_2 = \{x \in X \mid f(x) \in T \setminus \{\infty\}\}$$

and the relation

$$G = \{(x_i, x_j) \in X \times X \mid (f(x_i), f(x_j)) \notin E\}.$$

Take an element $u = x_1 \cdots x_n$ in $L \setminus \{1\}$. Since $u \neq 1$, we get $n > 0$. By the definition, $1 \cdot u = f(x_n)(\cdots f(x_1)) \in T$. The following three cases are possible:

Case 1: $1 \cdot u \in T \setminus \{\infty\}$. Then $f(x_n) = 1 \cdot u \in T \setminus \{\infty\}$, and so $u \in X^*X_2$.

Case 2: $1 \cdot u = \infty$ and $f(x_i) = \infty$, for some $i = 1, \ldots, n$. Then $u \in X^*X_1X^*$.

Case 3: $1 \cdot u = \infty$ and all $f(x_i) \neq \infty$. Then there exists $i = 1, \ldots, n-1$ such that $(f(x_{i+1}), f(x_i)) \notin E$. It follows that $u \in X^*x_ix_{i+1}X^*$.

On the other hand, consider an arbitrary element $u = x_1 \cdots x_n$ of the language defined by (1). If $u \in X^*X_1X^*$ or $u \in X^*x_ix_{i+1}X^*$, then $1 \cdot u = \infty$, and so $u \in L$. If $u \in X^*X_2$, then either $1 \cdot u = \infty \in T$, and so $u \in L$, or $1 \cdot u = f(x_n) \in T \setminus \{\infty\}$, and $u \in L$, again.

Thus $L \setminus \{1\}$ is given by the regular expression (1).

Second, assume that $\infty \notin T$. Then the complement of L is defined by the regular expression (1).

(iii)$\Rightarrow$(ii): Let L be a language defined by the regular expression (1). We are going to verify condition (b) of Theorem 1. If $yu \in L$, then obviously $xyu \in L$. Now, suppose that $yxu \in L$. First, assume that $yxu \in X^*X_1X^*$. If $y \in X_1$, then $yxv \in X_1X^* \subseteq L$. Otherwise, $xu \in X^*X_1X^* \subseteq L$. Second, assume that $yxu \in X^*X_2$. Then $xu \in X^*X_2 \subseteq L$. Third, assume that $yxu \in X^*x_ix_jX^*$, for some $(x_j, x_i) \in G$. If $y = x_i$, then $yxv \in x_ix_jX^* \subseteq L$. Otherwise, $xu \in X^*x_ix_jX^* \subseteq L$. Thus condition (b) of Theorem 1 holds, and hence L is recognized by a graph algebra automaton.

(i)$\Rightarrow$(iv): Suppose that the graph algebra automaton $\mathrm{Atm}(D, T)$ of a graph $D = (V, E)$ recognizes L. First, assume that $\infty \notin T$. The standard method gives us a right linear grammar which generates L (see, for example, the proof of Proposition 6.2.3 in [3]). Removing redundant productions from this grammar, and simplifying notation of states, we get the right linear grammar mentioned in condition (iv) with $Q = T$ and $P = E$. Note that s_0 is a nontermainal state. Thus this grammar generates the same language.

Second, assume that $\infty \in T$. It can be easily seen that the grammar specified in (iv) generates $X^+ \setminus L$.

(iv)$\Rightarrow$(i): Suppose that $L \setminus \{1\}$ is generated by the right linear grammar given in (iv), and so we are given Q, P and T. Consider the graph $D = (V, E)$ with the set $V = X$ of vertices and the set $E = P$ of edges. Let f be the mapping from X to $\mathrm{Alg}^1(D)$ defined by $f(x) = x$. Put $T = Q$. It is routine to verify that L is recognized by the graph algebra automaton $\mathrm{Atm}(D, T)$ of D.

Suppose that $\overline{L} \setminus \{1\} = X^* \setminus (L \cup \{1\})$ is generated by the right linear grammar specified in (iv). Then $\overline{L}$ is recognized by the automaton $\mathrm{Atm}(D, T)$, and hence L is recognized by the automaton $\mathrm{Atm}(D, \overline{T})$, where $\overline{T} = \mathrm{Alg}^1(D) \setminus T$. □

Corollary 2 *It is decidable whether a regular language belongs to the class $\mathcal{G}$.*

Proof. follows from condition (iii) of our main theorem. Indeed, given a regular language, for all disjoint subsets X_1, X_2 of X and all relations $G \subseteq X \times X$, we can use well known algorithms to verify whether the regular language (1) is equal to $L \setminus \{1\}$ or $X^+ \setminus L$. □

Denote by $\mathcal{G}_a$ the subclass of $\mathcal{G}$ containing the languages over X satisfying condition (a) of Theorem 1 and by $\mathcal{G}_b$ the subclass of $\mathcal{G}$ containing the languages over X satisfying condition (b). Let $\tilde{\mathcal{G}}_i = \{L \subseteq X^* \mid \overline{L} = X^* \setminus L \in \mathcal{G}_i\}$, for $i = a, b$. It follows from the proof of Theorem 1 that

$$\tilde{\mathcal{G}}_a = \mathcal{G}_b \quad \text{and} \quad \tilde{\mathcal{G}}_b = \mathcal{G}_a, \tag{3}$$

the regular expression (1) describes languages of the class $\mathcal{G}_b$, and the right linear grammar specified in condition (iv) of Theorem 1 describes languages of the class $\mathcal{G}_a$.

Corollary 3 *A language L belongs to the class $\mathcal{G}_a \cap \mathcal{G}_b$ if and only if there exists a subset X_2 of X such that $L \setminus \{1\}$ is given by the regular expression:*

$$X^* X_2 \tag{4}$$

Proof. It is easily seen that the regular expression (4) is a particular case of the regular expression (1), and so every language defined by it belongs to $\mathcal{G}_b$.

Further, it is easy to verify that every language defined by the regular expression (4) satisfies condition (a) of Theorem 1, and therefore belongs to $\mathcal{G}_a$. Thus every language described by the regular expression (4) belongs to the class $\mathcal{G}_a \cap \mathcal{G}_b$

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Conversely, consider an arbitrary language L of the class $\mathcal{G}_a \cap \mathcal{G}_b$. By Theorem 1, there exist disjoint subsets X_1, X_2 of X and a relation $G \subseteq X \times X$ such that $L \setminus \{1\}$ has the regular expression (1). Moreover, we can choose $X_1 = \{x \in X \mid f(x) = \infty\}$, $X_2 = \{x \in X \mid f(x) \in T \setminus \{\infty\}\}$ and $G = \{(x_i, x_j) \in X \times X \mid (f(x_i), f(x_j)) \notin E\}$, where L is recognized by a graph algebra automaton $\mathrm{Atm}(D, T)$ of a graph $D = (V, E)$ such that $\infty \in T$.

Since the language X^+ obviously has a regular expression of the form (4), we may assume that L is not equal to any of the languages X^+ and X^*.

First, suppose that $X_1 \neq \emptyset$. Take any $x \in X_1$ and $u \in X^+$. Condition (a) of Theorem 1 implies $u \in L$. This contradiction shows that $X_1 = \emptyset$.

Next, suppose that $G \neq \emptyset$. Choose a pair $(x_j, x_i) \in G$ and $u \in X^+$. Then $x_i x_j u \in L$, and condition (a) of Theorem 1 implies $u \in L$, a contradiction. Hence $G = \emptyset$.

Therefore $L \setminus \{1\}$ is given by the regular expression (4). □

In particular, the class $\mathcal{G}_a \cap \mathcal{G}_b$ contains the languages $\emptyset$, $\{1\}$, X^+, and X^*.

Corollary 4 *The class $\mathcal{G}_a$ contains all regular languages given by the regular expressions of the form*

$$X_1 X_2^* X_3 X_4^* \cdots X_{2n}^* X_{2n+1}, \tag{5}$$

where $X_1, X_2, \ldots, X_{2n+1} \subseteq X \cup \{1\}$, *and* $X_i \cap X_j = \{1\}$, *for all* $1 \le i < j \le 2n+1$.

Proof. Let L be a language defined by the regular expression (5). First, suppose that $xyu \in L$. Since the empty word 1 belongs to all sets X_i, it follows that $yu \in L$. Second, suppose that $xu, yxv \in L$. Since 1 is the only common element of X_i and X_j for $i \neq j$, we see that both occurrences of the letter x in xu and in yxv come from the same set X_i, where $1 \le i \le 2n+1$. It follows that $yxu \in L$. Thus condition (a) of Theorem 1 holds, and therefore $L \in \mathcal{G}_a$. □

Next, we give an example of a language, which belongs to $\mathcal{G}_a$ but cannot be defined by a regular expression of the form (5).

Example 5 Let $X = \{x_1, x_2, \ldots, x_n\}$, and let L be the set of all words x_i^k such that k is a positive integer, and $1 \le i \le n$. It is easily seen that L satisfies condition (a) of Theorem 1, and so $L \in \mathcal{G}_a$. However, for $n > 1$, it is clear that L cannot be described by an expression of the form (5), because all languages described by these expressions have a word containing all the letters $x_1, \ldots, x_n$.

It is easily seen that all languages in the class $\mathcal{G}_b$, except $\emptyset$ and $\{1\}$, are infinite. On the other hand, the class $\mathcal{G}_a$ contains some finite languages, but not all, as the following corollary shows.

Corollary 6 *Let* L *be a finite language over an alphabet* X, *where* $|X| = n$. *If* $L \in \mathcal{G}_a$, *then* $|L| \le 2^n$.

Proof. First, we show that L has no words with two occurrences of the same letter. Suppose to the contrary that L contains a word $w = axbxc$, where $x \in X$, $a, b, c \in X^*$. Since L satisfies condition (a) of Theorem 1, it follows that L contains all words $axbxbxc, axbxbxbxc, \ldots$. This contradicts the finiteness of L.

Second, we show that if two letters x_i, x_j occur together in several words of L, then they occur in the same order in all of these words. Suppose to the contrary that L contains words $w = ax_1bx_2c$ and $w' = a'x_2b'x_1c'$, where $x_1, x_2 \in X$, $a, b, c, a', b', c' \in X^*$. It follows from the second implication of condition (a) of Theorem 1, that L contains the word $w = ax_1bx_2b'x_1c'$, a contradiction.

Therefore every word of L is defined by the set of its letters. Thus

$$|L| \le \sum_{i=0}^{n} \binom{n}{i} = 2^n.$$

□

It is well-known that all finite languages are regular. Hence we see that many regular languages are not recognized by graph algebra automata. The following example shows that the class of languages recognized by graph algebra automata is not closed under union, intersection, or product.

Example 7 The languages $L_1 = \{x_1x_2, x_2\}$ and $L_2 = \{x_2x_3, x_3\}$ are recognized by graph algebra automata, because they satisfy condition (a) of Theorem 1. However, their union $L_1 \cup L_2$ contains the words x_1x_2, x_2x_3, but does not contain $x_1x_2x_3$. Hence $L_1 \cup L_2$ does not satisfy condition (a). Obviously, it does not satisfy (b) either, because all languages with this condition are infinite. Therefore $L_1 \cup L_2$ is not recognized by a graph algebra automaton.

The languages $\overline{L_1} = X^* \setminus L_1$ and $\overline{L_2} = X^* \setminus L_2$ are recognized by graph algebra automata, as well. But their intersection, which is equal to $\overline{L_1 \cup L_2}$, is not recognized by graph algebra automata, because $L_1 \cup L_2 \notin \mathcal{G}$.

The languages $L_3 = \{x_1\}$ and $L_2 = \{x_2x_3, x_3\}$ satisfy condition (a) of Theorem 1, and so they are recognized by graph algebra automata. However, neither (a) nor (b) holds for their product $L_3 \cdot L_2 = \{x_1x_2x_3, x_1x_3\}$.

For any language L and word $u \in X^*$, let $Lu^{-1} = \{w \in X^* \mid wu \in L\}$ and $u^{-1}L = \{w \in X^* \mid uw \in L\}$. A class $\mathcal{L}$ of languages is said to be *closed under left (right) derivative* if $L \in \mathcal{L}$ implies $u^{-1}L \in \mathcal{L}$ (respectively, $Lu^{-1} \in \mathcal{L}$).

Corollary 8 *The class $\mathcal{G}$ is closed under complement, $\mathcal{G}_a$ is closed under intersections and left derivative, and $\mathcal{G}_b$ is closed under union and right derivative.*

Proof. Equations (3) immideately show that $\mathcal{G}$ is closed under complement. It is routine to verify that $\mathcal{G}_a$ is closed for intersection and left derivative and that $\mathcal{G}_b$ is closed under right derivative.

Now, assume that $L_1, L_2 \in \mathcal{G}_b$. Then $\overline{L_1}, \overline{L_2} \in \mathcal{G}_a$, and therefore $\overline{L_1 \cup L_2} = \overline{L_1} \cap \overline{L_2} \in \mathcal{G}_a$, because $\mathcal{G}_a$ is closed under intersection. Hence $L_1 \cup L_2 \in \mathcal{G}_b$, as required. □

Corollary 9 *The classes $\mathcal{G}_a$ and $\mathcal{G}_b$ are closed under the Kleene $*$-operation.*

Proof. Suppose that $L \in \mathcal{G}_a$. First, take any word xyu in L^*, where $x, y \in X, u \in X^*$. We have $xyu \in L^n$, for some $n \geq 1$. Consider the leftmost prefix of this word which is in L. If $x \in L$, then $yu \in L^{n-1} \subseteq L^*$. Further, assume that $xyu_1 \in L$ and $u_2 \in L^{n-1}$, for some factorization $u = u_1u_2$, where $u_1, u_2 \in X^*$. Then $yu_1 \in L$, because $L \in \mathcal{G}_a$. Therefore $yu = yu_1u_2 \in L^*$, again.

Second, take $xu, yxv \in L^*$. If $y \in L$, then clearly $yxu \in L^*$. Hence we may assume that $yxv_1 \in L$ and $v_2 \in L^*$, where $v = v_1v_2$. We get $xu_1 \in L$ and $u_2 \in L^*$, where $u = u_1u_2$. Since $L \in \mathcal{G}_a$, we get $yxu_1 \in L$, and therefore $yxu = yxu_1u_2 \in L^*$, again.

Thus the whole condition (a) of Theorem 1 is satisfied for L^*, and therefore $L^* \in \mathcal{G}_a$.

Now, suppose that $L \in \mathcal{G}_b$. Pick up any word $yu \in L^*$, where $y \in X, u \in X^*$ and $x \in X$. We have $yu_1 \in L$ and $u_2 \in L^*$, for some factorization $u = u_1u_2$. It follows that $xyu_1 \in L$, and so $xyu \in L^*$.

Finally, take $yxu \in L^*$ and $v \in X^*$. If $y \in L$, then obviously $xu \in L^*$. Therefore we may assume that $yxu_1 \in L$ and $u_2 \in L^*$, for some factorization

$u = u_1u_2$, where $u_1, u_2 \in X^*$. Since $L \in \mathcal{G}_b$, it follows that $xu_1 \in L$ or $yxv \in L$. In the former case $xu = xu_1u_2 \in L^*$, and in the latter case $yxv \in L^*$. Thus the whole condition (b) of Theorem 1 is satisfied, and so $L^* \in \mathcal{G}_b$. $\square$

The authors are grateful to two referees for valuable advice and suggestions that helped to improve the first version of this paper.

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Received November, 2000