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Affine matching of two sets of points in arbitrary dimensions

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Abstract

In many applications of computer vision, image processing, and remotely sensed data processing, an appropriate matching of two sets of points is required. Our approach assumes one-to-one correspondence between these sets and finds the optimal global affine transformation that matches them. The suggested method can be used in arbitrary dimensions. A sufficient existence condition for a unique transformation is given and proven.

1 Introduction

Many applications lead to the following mathematical problem: Two corresponding sets of points $\{p_i\}$ and $\{q_i\}$ $(i = 1, 2, \ldots, n)$ are given in the k-dimensional Euclidean space $\mathbb{R}^k$, and the transformation $T : \mathbb{R}^k \to \mathbb{R}^k$ is to be found that gives the minimal mean squared error

$$\sum_{i=1}^{n} \|T(q_i) - p_i\|^2.$$

The dimension k is usually 2 or 3. Some solutions have been proposed for this problem assuming rigid-body transformation (i.e., where only rotations and translations are allowed) [1, 3, 6, 7, 13], affine transformation (i.e., which maps straight lines to straight lines, parallelism is preserved, but angles can be altered) [8], and non-linear transformation (i.e., which can map straight lines to curves) [2, 5, 8]. In [10], a solution is proposed when the correspondence between the

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point sets is unknown, assuming affine transformation. It is mentioned, that if the correspondence was known, a simpler solution is possible e.g., using least squares method, but neither such a method nor a sufficient existence condition for unique solution is given or referenced.

In this paper, we present a method for solving the problem assuming affine transformation, which can be used in arbitrary dimensions. The method is described in Section 2. We state and prove a sufficient existence condition for a unique solution in Section 3. A related open problem concerning degeneracy is presented in Section 4.

2 Method for affine matching of two sets of points

Given a matrix

$$\mathcal{T} = \begin{pmatrix} t_{11} & t_{12} & \cdots & t_{1k} & t_{1,k+1} \\ t_{21} & t_{22} & \cdots & t_{2k} & t_{2,k+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ t_{k1} & t_{k2} & \cdots & t_{kk} & t_{k,k+1} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix},$$

it determines an affine transformation $T : \mathbb{R}^k \to \mathbb{R}^k$ as follows: For $x = (x_1, \ldots, x_k)$ and $y = (y_1, \ldots, y_k)$ in $\mathbb{R}^k$ we have $y = T(x)$ if and only if

$$\begin{pmatrix} y_{i1} \\ y_{i2} \\ \vdots \\ y_{ik} \\ 1 \end{pmatrix} = \begin{pmatrix} t_{11} & t_{12} & \cdots & t_{1k} & t_{1,k+1} \\ t_{21} & t_{22} & \cdots & t_{2k} & t_{2,k+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ t_{k1} & t_{k2} & \cdots & t_{kk} & t_{k,k+1} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x_{i1} \\ x_{i2} \\ \vdots \\ x_{ik} \\ 1 \end{pmatrix}.$$

Note that homogeneous coordinates are used. Each affine transformation T can uniquely be represented in this form [4]. The transformation has $k \cdot (k+1)$ degrees of freedom according to the non-constant matrix elements.

Let us fix an affine transformation $T : \mathbb{R}^k \to \mathbb{R}^k$ and the corresponding $\mathcal{T}$ as above. Let $\{p_i\}$ and $\{q_i\}$ be two sets of n points, where

$$\begin{aligned} p_i &= (p_{i1}, p_{i2}, \ldots, p_{ik}) \in \mathbb{R}^k \quad \text{and} \\ q_i &= (q_{i1}, q_{i2}, \ldots, q_{ik}) \in \mathbb{R}^k \quad (i = 1, 2, \ldots, n). \end{aligned}$$

Let $\{p'_i\}$ be a set of n points in $\mathbb{R}^k$, where $p'_i = T(q_i)$ $(i = 1, 2, \ldots, n)$. Define the merit function $\mathcal{S}$ of $k \cdot (k+1)$ variables as follows:

$$\mathcal{S}(t_{11}, \ldots, t_{k,k+1}) = \sum_{i=1}^{n} \|p'_i - p_i\|^2 = \sum_{i=1}^{n} \sum_{j=1}^{k} (t_{j1} \cdot q_{i1} + \ldots + t_{jk} \cdot q_{ik} + t_{j,k+1} - p_{ij})^2.$$

which is generally regarded as the *matching error*.

The least square solution of matrix $\mathcal{T}$ is determined by minimizing the function $\mathcal{S}$. Function $\mathcal{S}$ may be minimal if all of the partial derivatives $\frac{\partial \mathcal{S}}{\partial t_{11}}, \ldots, \frac{\partial \mathcal{S}}{\partial t_{k,k+1}}$ are equal to zero. The required $k \cdot (k+1)$ equations:

$$\begin{aligned}
\frac{\partial \mathcal{S}}{\partial t_{uv}} &= 2 \cdot \sum_{i=1}^{n} q_{iv} \cdot (t_{u,k+1} - p_{iu} + \sum_{l=1}^{k} t_{ul} \cdot q_{il}) = 0 \\
&\quad (u = 1, 2, \ldots, k\,,\ v = 1, 2, \ldots, k), \\
\frac{\partial \mathcal{S}}{\partial t_{u,k+1}} &= 2 \cdot \sum_{i=1}^{n} (t_{u,k+1} - p_{iu} + \sum_{l=1}^{k} t_{ul} \cdot q_{il}) = 0 \\
&\quad (u = 1, 2, \ldots, k).
\end{aligned}$$

We get the following system of linear equations:

$$\begin{pmatrix}
\begin{matrix} a_{11} & \cdots & a_{1k} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{k1} & \cdots & a_{kk} & b_k \\ b_1 & \cdots & b_k & n \end{matrix} & & & \mathbf{0} \\
& \begin{matrix} a_{11} & \cdots & a_{1k} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{k1} & \cdots & a_{kk} & b_k \\ b_1 & \cdots & b_k & n \end{matrix} & & \\
& & \ddots & \\
\mathbf{0} & & & \begin{matrix} a_{11} & \cdots & a_{1k} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{k1} & \cdots & a_{kk} & b_k \\ b_1 & \cdots & b_k & n \end{matrix}
\end{pmatrix} \cdot \begin{pmatrix} t_{11} \\ \vdots \\ t_{1k} \\ t_{1,k+1} \\ t_{21} \\ \vdots \\ t_{2k} \\ t_{2,k+1} \\ \vdots \\ t_{k1} \\ \vdots \\ t_{kk} \\ t_{k,k+1} \end{pmatrix} = \begin{pmatrix} c_{11} \\ \vdots \\ c_{1k} \\ d_1 \\ c_{21} \\ \vdots \\ c_{2k} \\ d_2 \\ \vdots \\ c_{k1} \\ \vdots \\ c_{kk} \\ d_k \end{pmatrix},$$

where

$$\begin{aligned}
a_{uv} = a_{vu} &= \sum_{i=1}^{n} q_{iu} \cdot q_{iv}\,, \\
b_u &= \sum_{i=1}^{n} q_{iu}\,, \\
c_{uv} &= \sum_{i=1}^{n} p_{iu} \cdot q_{iv}\,,
\end{aligned}$$

$$d_u = \sum_{i=1}^{n} p_{iu}$$
$$(u = 1, 2, \ldots, k, \; v = 1, 2, \ldots, k).$$

The above system of linear equations can be solved by using an appropriate numerical method [9]. There exists a unique solution if and only if $det(M) \neq 0$, where

$$M = \begin{pmatrix} a_{11} & \ldots & a_{1k} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{k1} & \ldots & a_{kk} & b_k \\ b_1 & \ldots & b_k & n \end{pmatrix}.$$

Note that if a problem is close to singular (i.e., $det(M)$ is close to 0), the method can become unstable.

3 Discussion

In this section we state and prove a sufficient existence condition for a unique solution for the system of linear equations.

By a hyperplane of the Euclidean space $\mathbb{R}^k$ we mean a subset of the form $\{a + x : x \in S\}$ where S is a $(k-1)$–dimensional linear subspace. Given some points $q_1, \ldots, q_n$ in $\mathbb{R}^k$, we say that these points *span* $\mathbb{R}^k$ if no hyperplane of $\mathbb{R}^k$ contains them. If any $k+1$ points from $q_1, \ldots, q_n$ span $\mathbb{R}^k$ then we say that $q_1, \ldots, q_n$ are in *general position.*

Theorem 1. If $q_1, \ldots, q_n$ span $\mathbb{R}^k$ then $\det(M) \neq 0$.

Proof. Suppose $\det(M) = 0$. Consider the vectors $v_j = (q_{1j}, q_{2j}, \ldots, q_{nj})$ $(1 \leq j \leq k)$ in $\mathbb{R}^n$, and let $v_{k+1} = (1, 1, \ldots, 1) \in \mathbb{R}^n$. With the notation $m = k+1$ observe that $M = \Big(\langle v_i, v_j \rangle\Big)_{m \times m}$ where $\langle\ ,\ \rangle$ stands for the scalar multiplication. Since the columns of M are linearly dependent, we can fix a $(\beta_1, \ldots, \beta_m) \in \mathbb{R}^m \setminus \{(0, \ldots, 0)\}$ such that $\sum_{j=1}^{m} \beta_j \langle v_i, v_j \rangle = 0$ holds for $i = 1, \ldots, m$. Then

$$0 = \sum_{i=1}^{m} \beta_i \cdot 0 = \sum_{i=1}^{m} \beta_i \sum_{j=1}^{m} \beta_j \langle v_i, v_j \rangle = \sum_{i=1}^{m} \beta_i \Big\langle v_i, \sum_{j=1}^{m} \beta_j v_j \Big\rangle =$$
$$\Big\langle \sum_{i=1}^{m} \beta_i v_i, \sum_{j=1}^{m} \beta_j v_j \Big\rangle = \Big\langle \sum_{i=1}^{m} \beta_i v_i, \sum_{i=1}^{m} \beta_i v_i \Big\rangle,$$

whence $\sum_{i=1}^{m} \beta_i v_i = 0$. Therefore all the q_j, $1 \leq j \leq n$, are solutions of the following (one element) system of linear equations:

$$\beta_1 x_1 + \cdots + \beta_k x_k = -\beta_m. \tag{1}$$

Since the system has solutions and $(\beta_1, \ldots, \beta_m) \neq (0, \ldots, 0)$, there is an $i \in \{1, \ldots, k\}$ with $\beta_i \neq 0$. Hence the solutions of (1) form a hyperplane of $\mathbb{R}^k$. This hyperplane contains $q_1, \ldots, q_n$. Now it follows that if $q_1, \ldots, q_n$ span $\mathbb{R}^k$ then $\det(M) \neq 0$. Q.e.d.

4 Conclusions

In real applications, it is assumed that both $p_1, \ldots, p_n$ and $q_1, \ldots, q_n$ span $\mathbb{R}^k$. Then, if the matching error is zero (i.e., $p'_i = T(q_i) = p_i$ for $i = 1, 2, \ldots n$), the transformation is necessarily non-degenerate, i.e., $\det(\mathcal{T}) \neq 0$. Moreover, in this case the following property is fulfilled:

Observation 2. For all $I \subseteq \{1, \ldots, n\}$ with $k+1$ elements, the p_i, $i \in I$, span $\mathbb{R}^k$ if and only if the q_i, $i \in I$, span $\mathbb{R}^k$.

This raises the question whether the transformation is necessarily non–degenarete in general or when Observation 2 holds or at least when Observation 2 "strongly" holds in the following computational sense: each simplex with vertices in $\{p_1, \ldots, p_n\}$ or with vertices in $\{q_1, \ldots, q_n\}$ has a large volume (k–dimensional measure) compared with its edges.

Surprisingly, all these questions have a negative answer, for we have the following three dimensional example.

Example 3. With $n = 5$ and $k = 3$ let $q_1 = (0, 0, 24)$, $q_2 = (24, 0, 0)$, $q_3 = (0, 24, 0)$, $q_4 = (0, 0, 0)$, and $q_5 = (-24, -48, 16)$. These five points determine five tetrahedra with reasonably large volumes, the smallest of them being 1536, the volume of the tetrahedron (q_2, q_3, q_4, q_5). Let $p_1 = (0, 0, 0)$, $p_2 = (3, 0, 0)$, $p_3 = (0, 3, 0)$, $p_4 = (0, 0, 3)$, $p_5 = (3, 3, 3)$, these are some vertices of a cube, so the tetrahedra they determine are at least of volume $9/2$. Yet,

$$\mathcal{T} = \begin{pmatrix} 2 & -6 & -6 & 12 \\ -9 & -1 & -9 & 18 \\ 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

which is degenerate.

Experience shows that in real applications the choice of points always guarantees that the transformation is non–degenerate [11, 12]. However, from theoretical point of view the following open problem is worth raising: Find a meaningful sufficient condition to ensure non–degeneracy.

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References

[1] Arun, K.S., T.S. Huang, S.D. Blostein, Least squares fitting of two 3-D point sets, *IEEE Trans. Pattern Analysis and Machine Intelligence* **9** (1987), 698–703.

[2] Bookstein, F.L., Principal warps: thin–plate splines and the decomposition of deformations, *IEEE Trans. Pattern Analysis and Machine Intelligence* **11** (1989), 567–585.

[3] Faugeras, O.D., M. Hebert, A 3-D recognition and positioning algorithm using geometrical matching between primitive surfaces, Proc. Int. Joint Conf. Artificial Intelligence, Karlsruhe, 1983, 996–1002.

[4] Foley, J.D., A. van Dam, S.K. Feiner, J.F. Hughes, *Computer Graphics — Principles and practice*, Addison–Wesley Publishing Company, Reading, Massachusetts 1991.

[5] Fornefett, M., K. Rohr, H.S. Stiehl, Radial basis functions with compact support for elastic registration of medical images, Proc. Int. Workshop on Biomedical Image Registration, Bled, 1999, 173–185.

[6] Horn, B.K.P., Closed-form solution of absolute orientation using unit quaternions, *J. Opt. Soc. Amer. A* **4** (1987), 629–642.

[7] Horn, B.K.P., Closed-form solution of absolute orientation using orthonormal matrices, *J. Opt. Soc. Amer. A* **5** (1987), 1127–1135.

[8] Maguire, D., M.F. Goodchild, D.W. Rhind (eds.), *Geographical information systems — Principles and Applications*, Longman Scientific and Technical, 1991.

[9] Press, W.H., B.P. Flannery, S.A. Teukolsky, W.T. Vetterling, *Numerical Recipes in C: The Art of Scientific Computing*, Cambridge University Press, 1992.

[10] Sprinzak, J., M. Werman, Affine Point Matching, *Pattern Recognition Letters* **4** (1994), 337–339.

[11] Tanács, A., K. Palágyi, A. Kuba, Medical image registration based on interactively defined anatomical landmark points, *Int. J. Machine Graphics & Vision* **7** (1998), 151–158.

[12] Tanács, A., K. Palágyi, A. Kuba, Target Registration Error of Point–Based Methods Assuming Rigid–Body and Linear Motions, Proc. Int. Workshop on Biomedical Image Registration, Bled, 1999, 223–233.

[13] Umeyama, S., Least-squares estimation of transformation parameters between two point patterns, *IEEE Trans. Pattern Analysis and Machine Intelligence* **13** (1991), 376–380.

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