Generation and Reconstruction of hv-convex 8-connected Discrete Sets

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Abstract

An algorithm is given to generate 2-dimensional *hv*-convex 8-connected discrete sets uniformly. This algorithm is based on an extension of a theory previously used for a more special class of *hv*-convex discrete sets. The second part of the paper deals with the reconstruction of *hv*-convex 8-connected discrete sets. The main idea of this algorithm is to rewrite the whole reconstruction problem as a 2SAT problem. Using some a priori knowledge we reduced the number of iterations and the number of clauses in the 2SAT expression which results in reduction of execution time.

Keywords: Discrete tomography; Reconstruction from projections; Convex discrete set; Generation at random;

1 Introduction

One of the most important problems of *Discrete Tomography* is the reconstruction of 2-dimensional discrete sets from their two orthogonal projections. Often the reconstructed object should fulfil some additional properties including connectivity or convexity. In certain classes, the reconstruction is NP-hard [8, 19], therefore, the most frequently studied classes are those, where the reconstruction can be performed in polynomial time, like the *hv*-convex polyominoes and *hv*-convex 8-connected discrete sets. In this paper we present an algorithm for the reconstruction in the class of *hv*-convex 8-connected discrete sets.

Chrobak and Dürr [5] showed that the reconstruction of an *hv*-convex polyomino is equivalent to the evaluation of a suitable constructed 2SAT expression. This method was extended by Kuba [16] for the class of *hv*-convex 8-connected discrete sets. In this paper we give the description of a modified algorithm following the same idea as Chrobak and Dürr and Kuba for the class of *hv*-convex 8-connected discrete sets. We reduced the number of cases and modified the 2SAT expression introducing some preliminary information concerning the object which are obtained from the two orthogonal projections. The class of 8-connected discrete sets includes

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the class of convex polyominoes. It has applications in image processing, biplane angiography, electron microscopy, and others.

In Section 2 we introduce some basic definitions and notations. To test the reconstruction algorithm we need to generate such sets with uniform distribution. In [14] Hochstättler, Loebl, and Moll presented an algorithm for the class of *hv*-convex polyominoes. In Section 3 we extend the more general class of *hv*-convex 8-connected discrete sets. Finally, in Section 4 we present the reconstruction algorithm.

2 Definitions and Notations

Let $\mathbb{Z}^2$ denote the 2-dimensional integer lattice, its elements can be represented by unit squares called cells. The finite subsets of $\mathbb{Z}^2$ are called discrete sets. Let S be a discrete set. Then there is a discrete rectangle Q of size $m \times n$ where m and n are positive integers, $Q = \{1, \ldots, m\} \times \{1, \ldots, n\}$, such that Q is the smallest discrete rectangle containing S. The number of cells of S is called the *area* of S. The *perimeter* of S is the number of pairs of cells sharing a common side, where one cell is an element of S and the other one is not. Note that if $Q = \{1, \ldots, m\} \times \{1, \ldots, n\}$ is the smallest discrete rectangle containing S then the perimeter of S is $2m + 2n$.

The discrete set can be represented as a binary matrix $(s_{ij})_{m \times n}, s_{ij} \in \{0, 1\}$, such that $s_{ij} = 1$ if $(i, j) \in S$, and $s_{ij} = 0$, otherwise.

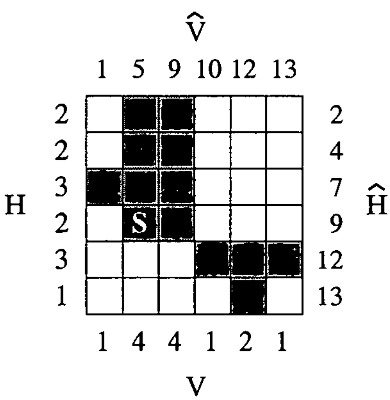

Figure 1: An *hv*-convex 8-connected discrete set S, its elements are marked by dark grey squares. H and V are the row and column sum vectors of S. The cumulated sums of H and V are denoted by $\widehat{H}$ and $\widehat{V}$ (their first elements, $\widehat{h}_0$ and $\widehat{v}_0$ are not indicated in the figure).

Let $\mathbb{N}$ denote the set of positive integers. For any discrete set S we define its *projections* by the operations $\mathcal{H}$ and $\mathcal{V}$ as follows. $\mathcal{H} : S \longrightarrow \mathbb{N}^m$, $\mathcal{H}(S) = H = (h_1, \ldots, h_m)$ where $h_i = \sum_{j=1}^{n} s_{ij}, i = 1, \ldots, m$, and $\mathcal{V} : S \longrightarrow \mathbb{N}^n$, $\mathcal{V}(S) = V = (v_1, \ldots, v_n)$ where $v_j = \sum_{i=1}^{m} s_{ij}, j = 1, \ldots, n$. The vectors H and V are called the *projections* or *row* and *column sum vectors* of S, respectively (see Fig 1). The cumulated vectors of H and V are denoted by $\widehat{H} = (\widehat{h}_0, \widehat{h}_1, \ldots, \widehat{h}_m)$ and $\widehat{V} = (\widehat{v}_0, \widehat{v}_1, \ldots, \widehat{v}_n)$, that is, $\widehat{h}_0 = 0$, $\widehat{h}_i = \widehat{h}_{i-1} + h_i$, $i = 1, \ldots, m$, and $\widehat{v}_0 = 0$, $\widehat{v}_j = \widehat{v}_{j-1} + v_j$, $j = 1, \ldots, n$ (see Fig. 1). Let $T = \sum_{i=1}^{m} h_i = \sum_{j=1}^{n} v_j$. Let S and S' be discrete sets.

We say that S and S' are *tomographically equivalent* (w.r.t. the row and column sum vectors) if $\mathcal{H}(S) = \mathcal{H}(S')$ and $\mathcal{V}(S) = \mathcal{V}(S')$.

The *4-neighbours* of a cell $(i,j) \in \mathbb{Z}^2$ are $(i-1,j)$, $(i,j-1)$, $(i,j+1)$, $(i+1,j)$ and the cell (i,j) itself. The *8-neighbours* of a cell $(i,j) \in \mathbb{Z}^2$ are the 4-neighbours and $(i-1, j-1$, $(i-1, j+1)$, $(i+1, j-1)$, $(i+1, j+1)$. The sequence of distinct cells $(i_0, j_0), \ldots, (i_k, j_k)$ is a *4-path/8-path* from cell (i_0, j_0) to cell (i_k, j_k) in a discrete set S if each cell of the sequence is in S and (i_l, j_l) is 4-adjacent/8-adjacent, respectively, to (i_{l-1}, j_{l-1}) for each $l = 1, \ldots, k$. Two points are *4-connected/8-connected* in the discrete set S if there is a 4-path/8-path, respectively, in S between them. A discrete set S is *4-connected/8-connected* if any two points in S are *4-connected/8-connected*, respectively. The 4-connected set is also called *polyomino*. From the definitions it follows that the class of 4-connected sets is a subset of the class of 8-connected sets (see Fig. 2).

The discrete set S is *horizontally convex* (or shortly, *h-convex*) if its rows are 4-connected. Similarly, a discrete set S is *vertically convex* (or, shortly, *v-convex*) if its columns are 4-connected. If a discrete set is both h-convex and v-convex then it is called hv-convex.

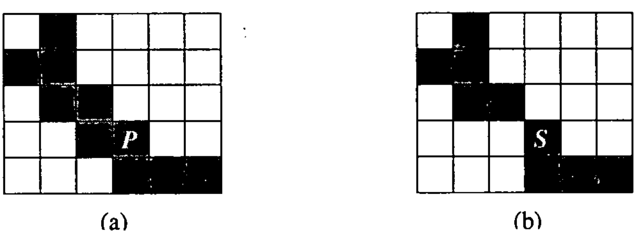

Figure 2: (a) P is a 4-connected hv-convex discrete set. (b) S is an 8-connected but not 4-connected hv-convex discrete set.

3 Generation of hv-convex 8-connected Discrete Sets at Random

Hochstättler, Loebl, and Moll [14] gave an algorithm for the generation of hv-convex polyominoes. In this paper, we extend this algorithm to the more general class of hv-convex 8-connected sets. First we describe the algorithm for calculating the number of hv-convex 8-connected discrete sets with fixed perimeter which is greater than the number of hv-convex 4-connected discrete sets with the same perimeter. Then using the same method as presented in [14] we construct a bijection between a set of hv-convex 8-connected discrete sets of a given perimeter and a finite interval of natural numbers will be given. This bijection allows us to generate such sets at random with uniform distribution in polynomial time. We use almost the same notation as in [14].

3.1 Definitions

A *strip* is a discrete rectangle of height 1. Let $L(s)$ and $R(s)$ denote the leftmost and rightmost cell of a strip s. The *length* of the strip s is $R(s) - L(s) + 1$. Each hv-convex 8-connected set can be considered as a sequence of strips $(s_1, \ldots, s_k)$, $k \geq 1$, where s_1 is the topmost strip and s_k is the downmost strip of the set.

Definition 1 *Let $S = (s_1, \ldots, s_k)$ be an hv-convex 8-connected set, s_l the downmost strip with $L(s_l)$ minimal and s_r the downmost strip with $R(s_r)$ maximal. We partition the set $(s_1, \ldots, s_k)$ as follows (see Fig. 3):*

- *the top $(s_1, \ldots, s_{min(l,r)})$,*
- *the interior $(s_{min(l,r)} + 1, \ldots, s_{max(l,r)})$,*
- *the bottom $(s_{max(l,r)} + 1, \ldots, s_k)$.*

We say the interior is of eastern type if $l < r$ and of western type if $l > r$.

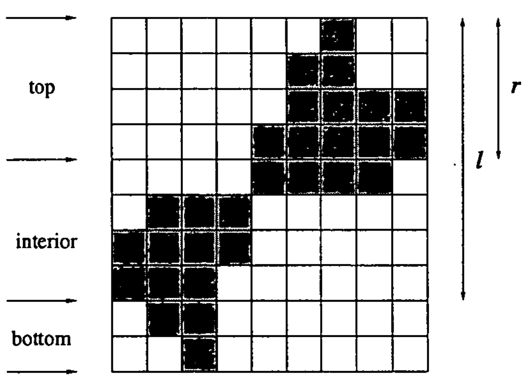

Figure 3: An hv-convex 8-connected discrete set with non-empty bottom. In this case $l = 8$ and $r = 4$.

The following four different types of hv-convex sets can arise (see Fig. 4):

- *type t* : hv-convex 8-connected sets with empty interior and empty bottom,
- *type i^w*: hv-convex 8-connected sets with empty bottom and western interior,
- *type i^e*: hv-convex 8-connected sets with empty bottom and eastern interior,
- *type b* : hv-convex 8-connected sets with non-empty bottom.

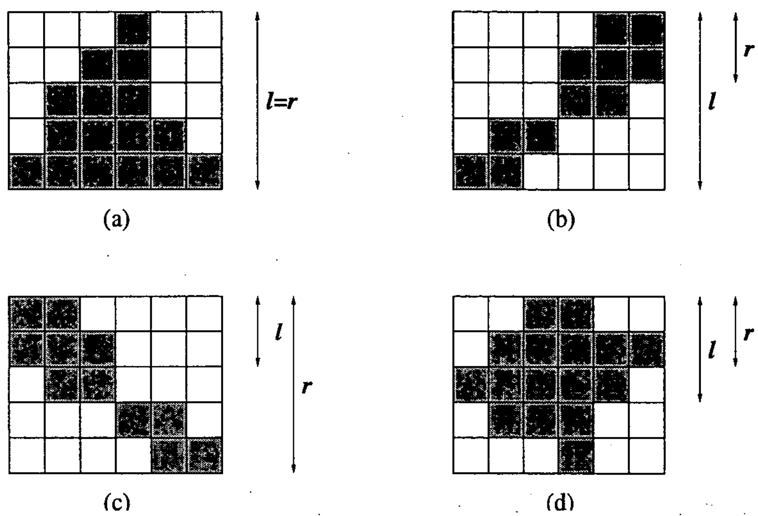

Figure 4: Four different types of hv-convex 8-connected sets: (a) type t; (b) type i^w; (c) type i^e ; (d) type b .

3.2 The Number of hv-convex 8-connected Discrete Sets

In order to count the hv-convex 8-connected discrete sets with fixed perimeter we construct them strip by strip starting from the top. Given a partially constructed set, we calculate its all possible extensions of this set to an hv-convex 8-connected discrete set.

Lemma 1 *Let $\tilde{S}(s_1, \ldots, s_j)$ be an hv-convex 8-connected set. Then the set of all extensions $(s_{j+1}, \ldots, s_k)$ of $\tilde{S}$ to an hv-convex set $S = (s_1, \ldots, s_k)$ is determined by the type of $\tilde{S}$ and the last strip s_j.*

Proof. It is similar to the proof of Lemma 1 in [14].

The number of all possible extensions of an hv-convex 8-connected set $\tilde{S}$ of perimeter $\tilde{n}$ to an hv-convex 8-connected set S of perimeter n depends on

- the type of $\tilde{S}$,
- $\tilde{m}$, the length of the downmost strip, and
- the remaining length $\tilde{l} = n - \tilde{n} + \tilde{m}$.

The cases i^e and i^w are symmetric so let denote by $N_i(\tilde{m}, \tilde{l})$ the number of possible extensions of hv-convex 8-connected discrete sets of interior type. Let

$N_b(\tilde{m},\tilde{l})$ and $N_t(\tilde{m},\tilde{l})$ be the number of possible extensions in the remaining two cases.

If $\tilde{l} = \tilde{m}$ then the only extension is the empty set. If $\tilde{l} < \tilde{m}$ then we cannot extend $\tilde{S}$ to an hv-convex 8-connected discrete sets S of perimeter n. This means, that

$$N_b(\tilde{m},\tilde{l}) = N_i(\tilde{m},\tilde{l}) = N_t(\tilde{m},\tilde{l}) = \begin{cases} 1, & \text{if } \tilde{l} = \tilde{m}, \\ 0, & \text{if } \tilde{l} < \tilde{m}. \end{cases}$$

Now we count the number of extensions of $\tilde{S}$ of different types with bottom length $\tilde{m}$ and perimeter $\tilde{n}$ to hv-convex 8-connected sets with perimeter n ($n = \tilde{n} - \tilde{m} + \tilde{l}$ for a given remaining length $\tilde{l}$). Note that $\tilde{l} + \tilde{m}$ is always even.

- hv-convex 8-connected sets of type b (see Fig 5):

$$N_b(\tilde{m},\tilde{l}) = \sum_{m=1}^{\tilde{m}} (\tilde{m} - m + 1) N_b(m, \tilde{l} - 2 - \tilde{m} + m) \quad \text{for } \tilde{l} > \tilde{m},\ \tilde{l} + \tilde{m} \text{ even.}$$

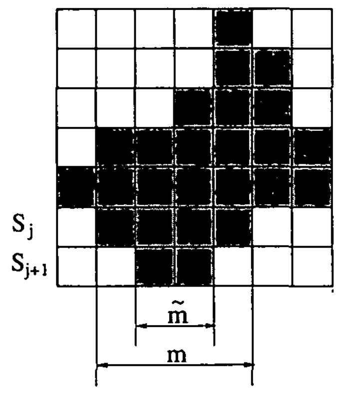

Figure 5: A possible extension of an hv-convex 8-connected set of type b to an hv-convex 8-connected set of type b.

- hv-convex 8-connected sets of type i (see Fig 6):

$$N_i(\tilde{m},\tilde{l}) = N_i^b(\tilde{m},\tilde{l}) + N_i^i(\tilde{m},\tilde{l})$$

where

$$N_i^b(\tilde{m},\tilde{l}) = \sum_{m=1}^{\tilde{m}-1}(\tilde{m}-m)N_b(m,\tilde{l}-2-\tilde{m}+m),$$

$$N_i^i(\tilde{m},\tilde{l}) = \sum_{a=0}^{\tilde{m}} \sum_{m=\tilde{m}-a}^{\frac{(\tilde{l}+\tilde{m})}{2}-a-1} N_i(m,\tilde{l}+\tilde{m}-2a-2-m).$$

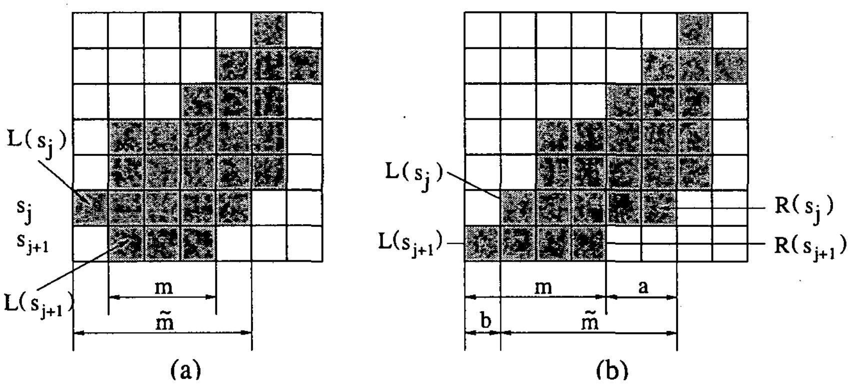

Figure 6: A possible extension of an *hv*-convex 8-connected set of type i^w to type b (a) and to type i^w (b). Notation: $a = R(s_j) - R(s_{j+1})$, $b = L(s_j) - L(s_{j+1})$.

- *hv*-convex 8-connected sets of type t (see Fig 7):

$$N_t(\tilde{m},\tilde{l}) = N_t^b(\tilde{m},\tilde{l}) + N_t^i(\tilde{m},\tilde{l}) + N_t^t(\tilde{m},\tilde{l})$$

where

$$N_t^b(\tilde{m},\tilde{l}) = \sum_{m=1}^{\tilde{m}-2}(\tilde{m}-m-1)N_b(m,\tilde{l}-2-\tilde{m}+m),$$

$$N_t^i(\tilde{m},\tilde{l}) = 2\sum_{a=1}^{\tilde{m}} \sum_{m=\tilde{m}-a}^{\frac{\tilde{l}+\tilde{m}}{2}-a-1} N_i(m,\tilde{l}+\tilde{m}-2a-2-m),$$

$$N_t^t(\tilde{m},\tilde{l}) = \sum_{m=\tilde{m}}^{\frac{\tilde{l}+\tilde{m}}{2}-1} N_t(m,\tilde{l}-2-\tilde{m}+m).$$

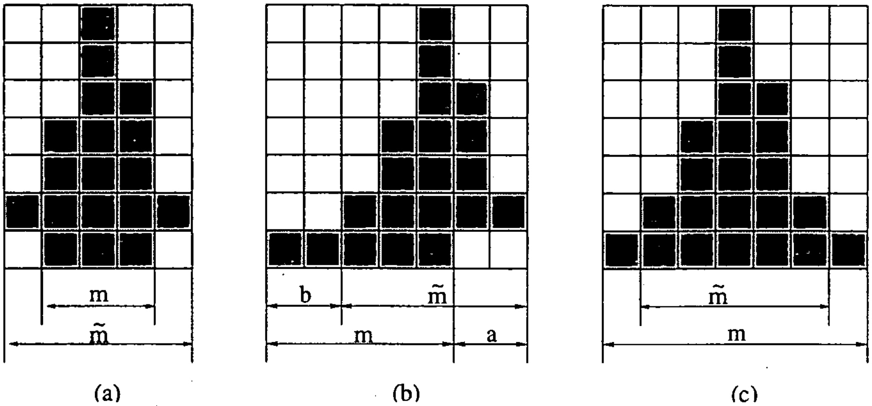

Figure 7: A possible extension of an hv-convex 8-connected set of type t to type b (a), to type i^w (b), and to type t (c).

Theorem 2 *The total number of hv-convex 8-connected sets of perimeter n is*

$$S_n = \sum_{\tilde{m}+2+\tilde{l}=n} N_t(\tilde{m}, \tilde{l}).$$

Remark 1 *A discrete set of type t is always 4-connected (see Fig 4.a). Extending an hv-convex 4-connected discrete set of type i into type i the result is 4-connected, if* $a = R(s_j) - R(s_{j+1}) < \tilde{m}$ *and it is 8-connected, if* $a = \tilde{m}$ *(see Fig 6.b). Extending an hv-convex 4-connected discrete set of type t into type i the result is 4-connected, if* $a < \tilde{m}$ *and it is 8-connected, if* $a = \tilde{m}$ *(see Fig 7.b). In any other cases the connectedness of the set determines the connectedness of the extended set.*

3.3 An Algorithm for Generating *hv*-convex 8-connected Discrete Sets

According to the presented way of constructing hv-convex 8-connected discrete sets we can construct a tree. The nodes are hv-convex 8-connected sets, an edge between nodes A and B means that the set represented by node A can be extended to the set represented by node B by the inclusion of a new last strip, the leaves are the hv-convex 8-connected sets of a given perimeter, interior nodes are partially constructed hv-convex 8-connected sets and the root is the empty set, which can also be considered as a trivial hv-convex 8-connected discrete set. In order to construct an embedding of such a tree into a finite interval of natural numbers we can use the same technique described in [14] and so we obtain the bijection between the set of hv-convex 8-connected sets of perimeter n and the interval $[1, S_n]$.

We fix an ordering of the summands in the presented equations which gives a numbering of the sets. This means that for any node the set of numbers of the leaves in its subtree is an interval.

To construct a bijection between the set of *hv*-convex 8-connected sets of perimeter n and the interval $[1, P_n]$ we have to do the following:

1. Using the formulas in Subsection 3.2 compute S_n, the number of *hv*-convex 8-connected sets of perimeter n, $N_b(\tilde{m}, \tilde{l})$, $N_i(\tilde{m}, \tilde{l})$, and $N_t(\tilde{m}, \tilde{l})$ for $\tilde{m} + \tilde{l}$ even and $\tilde{m} + \tilde{l} \leq n - 2$, which give for each interior node the number of leaves of the corresponding subtrees.

2. Given an *hv*-convex 8-connected set $s = (s_1, s_2, ..., s_k)$, compute I_1, the partition of this discrete sets of perimeter n induced by s_1. Taking into consideration the type of the partially constructed part of the set, apply this procedure recursively until reaching the leaf of the tree with interval $[j, j]$.

3. Given a number j in $[1, P_n]$, compute the partition of the interval $[1, P_n]$, fix the corresponding strip s_1 and proceed recursively.

Using this bijection we can generate *hv*-convex 8-connected sets with fixed perimeter with uniform probability:

1. Compute S_n, $N_b(\tilde{m}, \tilde{l})$, $N_i(\tilde{m}, \tilde{l})$, and $N_t(\tilde{m}, \tilde{l})$ for $\tilde{m}+\tilde{l}$ even and $\tilde{m}+\tilde{l} \leq n-2$.

2. Compute a random number j in $[1, S_n]$.

3. Apply the above presented procedure.

Remark 2 *Instead of working with perimeters we can extend the sets taking account of their area. This yields to an algorithm for calculating the number of hv-convex 8-connected discrete sets with fixed area. Working with perimeters and areas we can calculate the number of hv-convex 8-connected discrete sets with fixed perimeter and fixed area. Then with the same method as presented in this section we construct a bijection between this set and the corresponding finite interval of natural numbers. This means that we can generate hv-convex 8-connected discrete sets with fixed area or even with fixed area and fixed perimeter at random with uniform distribution.*

4 Reconstruction of *hv*-convex 8-connected Discrete Sets from Two Orthogonal Projections

In the reconstruction problem we are given two vectors $H \in \mathbb{N}^m$ and $V \in \mathbb{N}^n$. The task is to construct an *hv*-convex 8-connected discrete set S such that $\mathcal{H}(S) = H$ and $\mathcal{V}(S) = V$. The same notation is used as in [16].

4.1 Definitions

Definition 2 *An (H, V) pair of vectors is said to be compatible if there exist positive integers m, n, and A such that*

(i) $H \in \mathbb{N}^m$ *and* $V \in \mathbb{N}^n$*;*

(ii) $h_i \leq n$*, for* $1 \leq i \leq m$*, and* $v_j \leq m$*, for* $1 \leq j \leq n$*;*

(iii) $\sum_{i=1}^{m} h_i = \sum_{j=1}^{n} v_j = A$*, i.e., the two vectors have the same total sum* A*.*

Definition 3 *The north foot of an hv-convex 8-connected set S denoted by P_N is the set of columns of S that have elements in the first row of the rectangle Q. The column indices of P_N determine a set of consecutive integers: $\{n_f, n_f + 1, \ldots, n_l\}$.*

Similar definition can be given for the *south foot*, P_S, taking the columns of S, $\{s_f, s_f+1, \ldots, s_l\}$, which have an element in the last row of Q (see Fig. 8). The *east* and *west feet*, P_E and P_W, respectively, and their column indices, $\{e_f, e_f+1, \ldots, e_l\}$ and $\{w_f, w_f + 1, \ldots, w_l\}$, can be defined analogously.

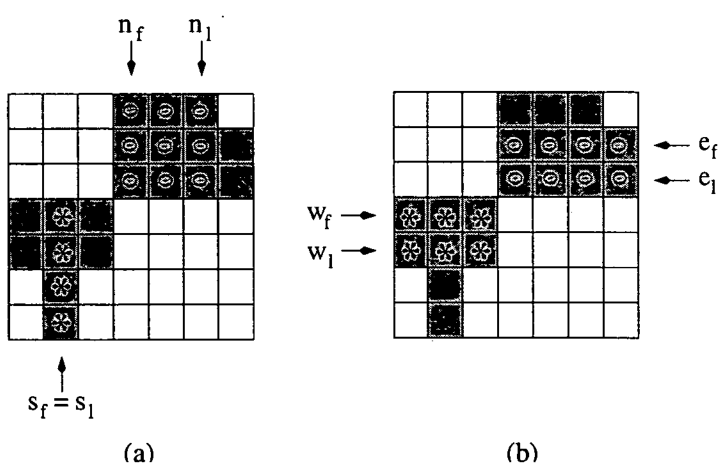

Figure 8: (a) The north and south feet of an hv-convex discrete set marked by o and *, respectively. (b) The east and west feet of the same discrete set marked by o and *, respectively.

Let S be an hv-convex 8-connected set with projections (H, V). Let $l = \max\{1 \leq j \leq n \mid v_p \leq v_q$ for all $1 \leq p < q\}$, the index of the last element of the first nondecreasing subsequence of V and $r = \min\{1 \leq j \leq n \mid v_p \geq v_q$ for all $1 \leq p < q \leq j\}$, the index of the first element of the last non-increasing subsequence of V. Let, furthermore, $l_1 = \min\{1 \leq j \leq l \mid v_j = v_l\}$ and,

$r_1 = \max\{r \le j \le n \mid v_j = v_r\}$, $l_N = \min\{l_1 + h_1 - 1, l\}$, $l_S = \min\{l_1 + h_m - 1, l\}$, $r_N = \max\{r_1 - h_1 + 1, r\}$, and $r_S = \max\{r_1 - h_m + 1, r\}$.

Then the following two propositions are true.

Proposition 1 [3] *If there is an hv-convex 8-connected set S that satisfies (H, V) with $v_j < m$ for all $j = 1, \dots, n$, then*

1. *if P_N is to the left of P_S then $n_l \le l_N$ and $s_f \ge r_S$,*
2. *if P_N is to the right of P_S then $s_l \le l_S$ and $n_f \ge r_N$.*

Proof. See [3].

Proposition 2 [3] *If there is an hv-convex 8-connected set S that satisfies (H, V) with $v_j = m$ for some $j = 1, \dots, n$ then the following four cases are possible.*

1. *If $h_1 > l - r + 1$ and $h_m > l - r + 1$ then (i) $n_f = l - h_1 + 1$, $n_l = l$, $s_f = r$, and $s_l = r + h_m - 1$, or (ii) $n_f = r$, $n_l = r + h_1 - 1$, $s_f = l - h_m + 1$, and $s_l = l$.*
2. *If $h_1 = l - r + 1$ and $h_m > l - r + 1$ then $n_f = r$, $n_l = l$, $s_f \ge \max\{1, l - h_m + 1\}$, and $s_l \le \min\{r + h_m - 1, n\}$.*
3. *If $h_1 > l - r + 1$ and $h_m = l - r + 1$ then $n_f \ge \max\{1, l - h_1 + 1\}$, $n_l \le \min\{r + h_1 - 1, n\}$, $s_f = r$, and $s_l = l$.*
4. *If $h_1 = h_m = l - r + 1$ then $n_f = s_f = r$, $n_l = s_l = l$.*

Proof. See [3].

Certain elements in the middle of an *hv*-convex 8-connected set can be recognized easily from the row and column sums by the following

Definition 4 *The spine of an hv-convex 8-connected set with row and column sums (H, V) is the set of positions (i, j) in T where one of the following conditions are satisfied:*

- $(n_f \le j \le s_l$ or $w_f \le i \le e_l)$ and $\widehat{v}_j \ge \widehat{h}_{i-1}$, $\widehat{h}_i \ge \widehat{v}_{j-1}$,
- $(s_f \le j \le n_l$ or $e_f \le i \le w_l)$ and $\widehat{h}_i \ge T - \widehat{v}_j$, $T - \widehat{v}_{j-1} \ge \widehat{h}_{i-1}$.

The spine of S will be denoted by S_p.

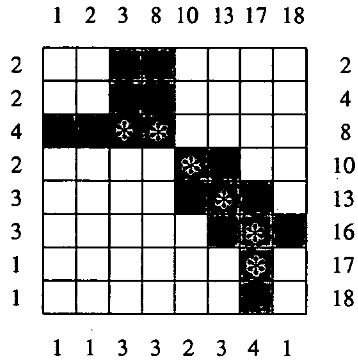

Figure 9: The spine of an hv-convex 8-connected discrete set marked by stars.

Definition 5 *Let S be an hv-convex 8-connected set. We say that the discrete set A is an upper-left corner region in the discrete rectangle Q containing S if $(i+1,j) \in A$ or $(i,j+1) \in A$ implies $(i,j) \in A$.*

The *upper-right*, *lower-left* and *lower-right regions*, B, C, and D, respectively, can be defined analogously (see Fig 10). Let $\overline{S}$ denote the complement of S (in Q).

Lemma 3 [5] *S is an hv-convex 8-connected discrete set if and only if $\overline{S} = A \cup B \cup C \cup D$, where A, B, C, and D are disjoint corner regions* (upper-left, upper-right, lower-left and lower-right, respectively).

Proof. See [5].

Remark 3 *$a_{i,j} = 1$ if $(i,j) \in A$, and $a_{i,j} = 0$ otherwise, for every $i = 1, ..., m$ and $j = 1, ..., n$. The elements of corner regions B, C, and D have a similar meaning.*

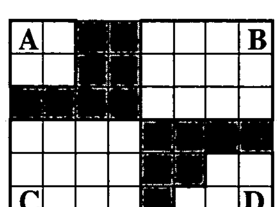

Figure 10: An hv-convex 8-connected discrete set with the four corner regions.

4.2 The Reconstruction Algorithm

The main idea of this algorithm is to rewrite the whole reconstruction problem as a 2-Satisfiability (2SAT) problem which can be solved in polynomial time. A 2SAT expression is a boolean expression in conjunctive normal form with at most two literals in each clause. The 2SAT problem is the following: given a 2SAT expression is there a value (true or false) for each literal that makes the expression true?

First we determine the limitations of the possible positions of the feet. Propositions 1 and 2 can be used to determine two intervals containing the possible column indices of the north and the south feet. Analogous propositions are used to determine the two intervals for the possible row indices of the east and west feet [3]. Choosing two opposite feet we can determine the spine. Without loss of generality let us suppose that we choose the west and east feet. According to the definition of the feet and the spine in this case we have found already at least one element in each column. Let $P_W = \{w_f, w_{f+1}, ..., w_l\}$ and $P_E = \{e_f, e_{f+1}, ..., e_l\}$.

Then we construct a 2SAT expression $F_{w_f,e_f}(H,V)$ such that $F_{w_f,e_f}(H,V)$ is satisfiable if and only if there is an hv-convex 8-connected discrete set S whose west and east feet are P_W and P_E. Let construct $F_{w_f,e_f}(H,V)$ in the following way:

$$F_{k,l}(H,V) = Cor \wedge Dis \wedge FSp \wedge LBC' \wedge UBR',$$

where Cor, Dis, FSp, LBC', and UBR' are sets of clauses describing the properties of "Corners", "Disjointness", "Feet and Spine", "Lower Bound on Column sums", and "Upper Bound on Row sums", respectively, in the following way.

$$\begin{aligned}
Cor &= \bigwedge_{i,j}(a_{ij} \Rightarrow a_{i-1,j} \;\wedge\; a_{ij} \Rightarrow a_{i,j-1})\wedge \\
&\quad \bigwedge_{i,j}(b_{ij} \Rightarrow b_{i-1,j} \;\wedge\; b_{ij} \Rightarrow b_{i,j+1}) \wedge \\
&\quad \bigwedge_{i,j}(c_{ij} \Rightarrow c_{i+1,j} \;\wedge\; c_{ij} \Rightarrow c_{i,j-1})\wedge \\
&\quad \bigwedge_{i,j}(d_{ij} \Rightarrow d_{i+1,j} \;\wedge\; d_{ij} \Rightarrow d_{i,j+1}), \\
Dis &= \bigwedge_{i,j}\{x_{i,j} \Rightarrow \overline{y_{i,j}} \mid \text{for symbols } X, Y \in \{A,B,C,D\},\; X \neq Y\}, \\
FSp &= \bigwedge_{S(i,j)=1} (\overline{a_{i,j}} \wedge \overline{b_{i,j}} \wedge \overline{c_{i,j}} \wedge \overline{d_{i,j}}),
\end{aligned}$$

$$
\begin{aligned}
LBC' &= \bigwedge_{i,j}(a_{ij} \Rightarrow \overline{c_{i+v_j,j}} \ \wedge \ a_{ij} \Rightarrow \overline{d_{i+v_j,j}}) \wedge \\
&\quad \bigwedge_{i,j}(b_{ij} \Rightarrow \overline{c_{i+v_j,j}} \ \wedge \ b_{ij} \Rightarrow \overline{d_{i+v_j,j}}), \\
UBR' &= \bigwedge_{j}\left(\bigwedge_{1\le i\le \min\{w_f,e_f\}} (\overline{a_{i,j}} \Rightarrow b_{i,j+h_i}) \ \wedge \ \bigwedge_{e_f\le i\le w_l} (\overline{a_{i,j}} \Rightarrow d_{i,j+h_i}) \right) \wedge \\
&\quad \bigwedge_{j}\left(\bigwedge_{w_f\le i\le e_l} (\overline{c_{i,j}} \Rightarrow b_{i,j+h_i}) \ \wedge \ \bigwedge_{\max\{w_l,e_l\}\le i\le m} (\overline{c_{i,j}} \Rightarrow d_{i,j+h_i}) \right).
\end{aligned}
$$

Then the reconstruction algorithm can be given as

Algorithm 1 *Reconstructing hv-convex 8-connected sets*

Input: Two compatible vectors $H \in \mathbb{N}^n$ and $V \in \mathbb{N}^m$.

1. Compute the cumulated sums of $\widehat{h}_i$ and $\widehat{v}_j$ for $i = 1, ..., m$ and $j = 1, ..., n$.
2. Compute the feet limitations.
3. For all possible configuration of P_W and P_E
   3.1. Compute the spine.
   3.2. If $F_{w_f,e_f}(H,V)$ is satisfiable then **Output** $S = \overline{A \cup B \cup C \cup D}$ and **halt**.

Output: Print "No solution".

Theorem 4 *$F_{w_f,e_f}(H,V)$ is satisfiable if and only if there is an hv-convex 8-connected discrete set F having projections H and V and west and east feet $P_W = \{w_f, w_{f+1}, ..., w_l\}$ and $P_E = \{e_f, e_{f+1}, ..., e_l\}$, respectively.*

Proof. Using the concepts of feet and spine, the proof is similar to the proof of Theorem 2 in [5].

In this algorithm we reduced the number of clauses in the 2SAT expression by introducing some apriori knowledge. The spine of an *hv*-convex 8-connected set guarantee the connectedness of the reconstructed object. We can define the possible feet positions from the row and column projections which reduce the number of iterations in the algorithm. The spine and two opposite feet are described by the clauses notated by FSp. The west and east feet together with the spine determine at least one cell in each column belonging to the object. This allows to reduce the number of clauses in LBC'. Using the concept of the feet we redefined the clauses which describe the upper bound on row sums (UBR').

The number of possible feet positions is smaller than m^2n^2. The complexity of the algorithm for constructing the spine for a given feet position is $O(mn)$, the 2SAT expression can be solved in $O(mn)$ time. Therefore the complexity of the algorithm for reconstruction of *hv*-convex 8-connected sets in the worst case is $O(mn \cdot \min\{m^2, n^2\})$.

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References

[1] B. Aspvall, M. F. Plass, and R. E. Tarjan, A linear-time algorithm for testing the truth of certain quantified boolean formulas. *Information Processing Letters* **8(3)** (1979) 121–123.

[2] E. Barcucci, A. Del Lungo, M. Nivat, R. Pinzani, Reconstructing convex polyominoes from horizontal and vertical projections, *Theor. Comput. Sci.* **155** (1996) 321–347.

[3] E. Barcucci, A. Del Lungo, M. Nivat, R. Pinzani, Medians of polyominoes: A property for the reconstruction, *Int. J. Imaging Systems and Techn.* **9** (1998) 69–77.

[4] S. Brunetti, A. Del Lungo, F. Del Ristoro, A. Kuba, M. Nivat, Reconstruction of 8- and 4-connected convex discrete sets from row and column projections, submitted for publication.

[5] M. Chrobak, C. Dürr, Reconstructing hv-convex polyominoes from orthogonal projections, *Information Processing Letters* **69** (1999) 283–289.

[6] M. P. Delest, G. Viennot, Algebraic languages and polyominoes enumeration, *Theor. Comput. Sci.* **34** (1984) 169–206.

[7] A. Del Lungo, Polyominoes defined by two vectors, *Theor. Comput. Sci.* **127** (1994) 187–198.

[8] A. Del Lungo, M. Nivat, Reconstruction of Connected Sets from Two Projections, In [13] (1999) 3-34.

[9] A. Del Lungo, M. Nivat, R. Pinzani, The number of convex polyominoes reconstructible from their orthogonal projections, *Discrete Math.* **157** (1996) 65–78.

[10] R.J. Gardner, P. Gritzmann, Uniqueness and complexity in discrete tomography, In [13](1999) 85–113.

[11] S.W. Golomb. *Polyominoes* (Schribner, New York, 1965).

[12] G.T. Herman, A. Kuba (Eds.), Discrete Tomography, Special Issue. *Int. J. Imaging Systems and Techn.* **9** (1998) No. 2/3.

[13] G.T. Herman, A. Kuba (Eds.), *Discrete Tomography: Foundations, Algorithms and Applications* (Birkhäuser, Boston, 1999).

[14] W. Hochstättler, M. Loebl, C. Moll, Generating Convex Polyominoes at Random, *Discrete Mathematics* **153** (1996) 165–176.

[15] A. Kuba, The reconstruction of two-directionally connected binary patterns from their two orthogonal projections, *Comp. Vision, Graphics, and Image Proc.* **27** (1984) 249–265.

[16] A. Kuba, Reconstruction in different classes of 2d discrete sets, In *Lecture Notes on Computer Sciences* **1586** (1999) 153-163.

[17] A. Kuba, E. Balogh, Reconstruction of convex 2D discrete sets in polynomial time, *Theor. Comput. Sci.*, to appear.

[18] H.J. Ryser, Combinatorial properties of matrices of zeros and ones, *Canad. J. Math.* **9** (1957) 371–377.

[19] G.W. Woeginger, The reconstruction of polyominoes from their orthogonal projections, *Techn. Report, Technische Universität Graz* **65** (1996).