Acta Cybernetica 15 (2001) 201–211.

A 3D Parallel Shrinking Algorithm

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Abstract

Shrinking is a frequently used preprocessing step in image processing. This paper presents an efficient 3D parallel shrinking algorithm for transforming a binary object into its topological kernel. The applied strategy is called directional: each iteration step is composed of six subiterations each of which can be executed in parallel. The algorithm makes easy implementation possible, since deletable points are given by $3 \times 3 \times 3$ matching templates. The topological correctness of the algorithm is proved for $(26, 6)$ binary pictures.

1 Introduction

A 3D binary picture [6] is a mapping that assigns the value of 0 or 1 to each point with integer coordinates in the 3D digital space denoted by $\mathbb{Z}^3$. Points having the value of 1 are called black points and form the objects in the picture, while 0's are called white points and form the background, the cavities, and the holes in the picture.

Shrinking of binary pictures into similarly connected representations that have smaller foregrounds (i.e., fewer 1's) has found application as a fundamental preprocessing step in image processing [3]. Two forms of such shrinking have been emerged:

1. The picture is transformed into its topological kernel, where the shrunk picture is topologically equivalent to the original one.
2. Each object (connected components of 1's) in the picture is shrunk into an isolated point (i.e., single–point residue which may then be deleted). Obviously, this kind of shrinking alters the topology of the original picture, since holes and cavities are eliminated.

As far as we know, the only 3D topology preserving shrinking algorithm has been proposed by Bertrand and Aktouf [2]. Their thinning algorithm can extract the topological kernel of an object if no end–point condition is applied. The strategy which is used for deleting 1's in parallel without altering the topology of the picture is based upon subfields: the cubic grid $\mathbb{Z}^3$ is divided into 8 subfields which are

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successively activated in each iteration step. The parallel algorithm examines the $3 \times 3 \times 3$ neighbourhood of the object points.

Two 3D shrinking algorithms belonging to the second type are known: Arcelli and Levialdi [1] proposed a parallel algorithm capable of transforming any finite object to an isolated 1 in a finite number of iteration step. Only the $2 \times 2 \times 2$ neighbourhood of 1's are investigated and the object to be shrunk never leaves its circumscribing box. Hall and Küçük [4] developed the other algorithm which uses 2 subfields and examines the $3 \times 3 \times 3$ neighbourhood of the object points.

In this work, a new 3D parallel shrinking algorithm is proposed for extracting the topological kernel of a binary picture. Our strategy used to preserve the topology is called directional or border sequential: Iteration steps are divided into 6 successive parallel subiterations, where only border 1's of a certain kind can be deleted in each subiteration. The algorithm examines the $3 \times 3 \times 3$ neighbourhood of 1's and it is topology preserving for any $(26, 6)$ pictures.

2 Basic Notions and Results

Let p be a point in the 3D digital space $\mathbb{Z}^3$. Let us denote $N_j(p)$ (for $j = 6, 18, 26$) the set of points *j-adjacent* to a point p (see Fig. 1).

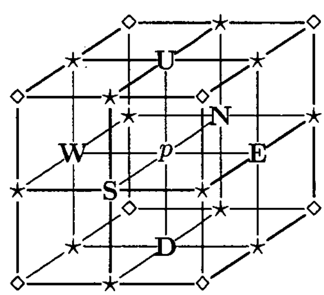

Figure 1: Frequently used adjacencies in $\mathbb{Z}^3$. The set $N_6(p)$ contains the central point p and points marked **U**, **N**, **E**, **S**, **W**, and **D**. The set of points $N_{18}(p)$ contains the set $N_6(p)$ and points marked "$\star$". The set of points $N_{26}(p)$ contains the set $N_{18}(p)$ and points marked "$\diamond$".

The sequence of distinct points $\langle x_0, x_1, \ldots, x_n \rangle$ is a *j-path* of length n from point x_0 to point x_n in a non-empty set of points X if each point of the sequence belongs to X and x_i is j-adjacent to x_{i-1} for each $1 \le i \le n$. Note that a single point is a j-path of length 0. Two points are *j-connected* in the set X if there is a j-path in X between them. A set of points X is *j-connected* in the set of points $Y \supseteq X$ if any two points in X are j-connected in Y.

The *3D binary (m,n) digital picture* $\mathcal{P}$ is a quadruple $\mathcal{P} = (\mathbb{Z}^3, m, n, B)$ [6]. Each element of $\mathbb{Z}^3$ is called a *point* of $\mathcal{P}$. Each point in $B \subseteq \mathbb{Z}^3$ is called a *black point* and value 1 is assigned to it. Each point in $\mathbb{Z}^3 \backslash B$ is called a *white point* and value 0 is assigned to it. Adjacency m corresponds to the black points and adjacency n corresponds to the white points. A *black component* or an *object* is a maximal m–connected set of points in B. A singleton black component is called *isolated point*. A *white component* is a maximal n–connected set of points in $\mathbb{Z}^3 \setminus B$.

We are dealing with (26,6) pictures. It is assumed that any picture contains finitely many black points. In a finite picture, there is a unique infinite white component, which is called the *background*. A finite white component (surrounded by an object) is called a *cavity*.

A black point p is called a *border point* if the set $N_6(p)$ contains at least one white point. A border point p is called a **U**–*border point* if the point marked by **U** in Fig. 1 is white. We can define **N**–, **E**–, **S**–, **W**–, and **D**–border points in the same way.

Our shrinking algorithm can be regarded as a reduction operation that changes some black points to white ones but does not alter white points.

A reduction operation does *not* preserve topology if

- any object in the input picture is split (into two or more parts) or completely deleted (i.e., each point belonging to the object is deleted),
- any cavity in the input picture is merged with the background or with another cavity, or
- a cavity is created where there was no cavity in the input picture.

There is an additional concept called *hole* or *tunnel* in 3D pictures. A hole (that a doughnut has) is formed by white points, but it is not a cavity [6]. Topology preservation implies that eliminating or creating holes is not allowed.

A black point is called a *simple point* if its deletion does not alter the topology of the picture [10]. We make use of the following result for (26,6) pictures:

Criterion 1. [9, 13]
Black point p is simple in picture $(\mathbb{Z}^3, 26, 6, B)$ if and only if all of the following conditions hold:

1. *the set $(B \setminus \{p\}) \cap N_{26}(p)$ contains exactly one 26–component; and*
2. *the set $(\mathbb{Z}^3 \setminus B) \cap N_6(p)$ is not empty and it is 6–connected in the set $(\mathbb{Z}^3 \setminus B) \cap N_{18}(p)$.*

The *topological kernel* of an object is topologically equivalent to the original object and there is no simple point in it. (In other words, the topological kernel can be get by the sequential deletion of simple points.)

Parallel reduction operations delete a set of black points and not only a single simple point (and each white point remains the same). We need to consider what is meant by topology preservation when a number of black points are deleted simultaneously. Some sufficient (but not necessary) conditions have been stated for parallel reduction operations [5, 7, 8]. We make use of the following result for (26,6) pictures:

Theorem 2. [11, 12]
Let $\mathcal{T}$ be a parallel reduction operation. Let p be any black point in any picture

$\mathcal{P} = (\mathbb{Z}^3, 26, 6, B)$ so that p is deleted by $\mathcal{T}$. Let $Q \subseteq (N_{18}(p) \backslash \{p\}) \cap B$ be any set of black points in picture $\mathcal{P}$. Operation $\mathcal{T}$ is topology preserving for (26,6) pictures if all of the following conditions hold:

1. *p is simple in the picture $(\mathbb{Z}^3, 26, 6, B \backslash Q)$; and*
2. *no black component contained entirely in a unit lattice cube (i.e., a $2 \times 2 \times 2$ configuration in $\mathbb{Z}^3$) can be deleted completely by operation $\mathcal{T}$ (i.e., at least one point in such an object must be preserved).*

3 The New Shrinking Algorithm

The proposed directional 6–subiteration shrinking algorithm can be sketched by the following program:

Input: picture $\mathcal{P} = (\mathbb{Z}^3, 26, 6, B)$;
Output: picture $\mathcal{P}' = (\mathbb{Z}^3, 26, 6, B')$.

```
6_subiteration_shrinking_algorithm(B,B')
  begin
    B' = B;
    repeat
      B' = deletion_from_U(B');
      B' = deletion_from_D(B');
      B' = deletion_from_N(B');
      B' = deletion_from_S(B');
      B' = deletion_from_E(B');
      B' = deletion_from_W(B');
    until no points are deleted;
  end.
```

The first subiteration (`deletion_from_U`) corresponding to the deletion direction **U** can delete certain **U**–border points; the second subiteration associated with the deletion direction **D** attempts to delete **D**–border points, and so on.

Our algorithm terminates when there are no more black points to be deleted. Since all considered input pictures are finite, the shrinking algorithm will terminate.

Deletable points in a subiteration are given by a set of $3 \times 3 \times 3$ matching templates. Templates are described by three kinds of elements, "black", "white", and "don't care". Each template defines a predicate: a black point p satisfies the predicate if the given template matches $N_{26}(p)$, where each black template element coincides with a black point, each white template element coincides with a white point, and a "don't care" element in the template coincides with either a white point or a black point. Note that no reflection or rotation is allowed in matching a template to $N_{26}(p)$. A black point is deletable if at least one template in the given set of templates matches it.

The set of templates assigned to the first subiteration is given by Fig. 2. Note that Fig. 2 shows only the five base templates **T1**–**T5**. Additionally, all their rotations around the vertical axis belong to $\mathcal{T}_U$, where the rotation angles are 90°, 180°, and 270°. The deletable points of the other five subiterations can be obtained by proper rotations and/or reflections of the templates in $\mathcal{T}_U$.

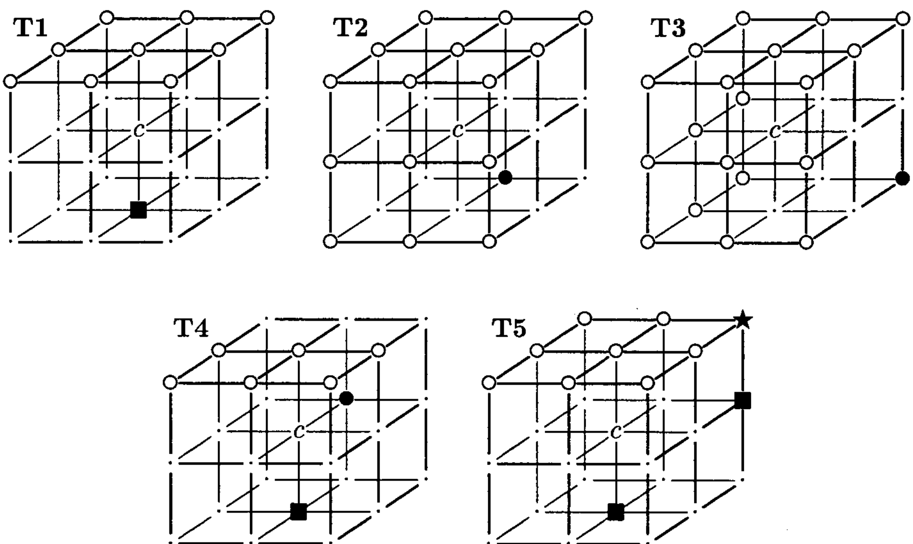

Figure 2: Base templates **T1**–**T5** and their rotations around the vertical axis form the set of templates $\mathcal{T}_U$ assigned to the first subiteration of the proposed algorithm. These templates can delete certain U–border points. Notations: each position marked "c", "●", "■", and "★" matches a black point; each position marked "○" matches a white point; every "·" ("don't care") matches either a black or a white point. (Note that using four different symbols for black template positions helps us to prove the topological correctness of the algorithm.)

Note that choosing another order of the deletion directions yields another algorithm, but it does not alter the topological correctness.

The templates of our algorithm can be regarded as a "careful" characterization of simple points. This Boolean characterization makes easy implementation possible.

4 Discussion

The proposed algorithm has been tested on objects of different shapes. Here we present only three examples (see Fig. 3).

An object is *simply–connected* if it has no holes nor cavities [6]. (In other words, a simply–connected object is topologically equivalent to a solid sphere.) The proposed algorithm is capable of transforming each simply–connected object to an isolated point. Each *multiply–connected object* (i.e., object having either holes

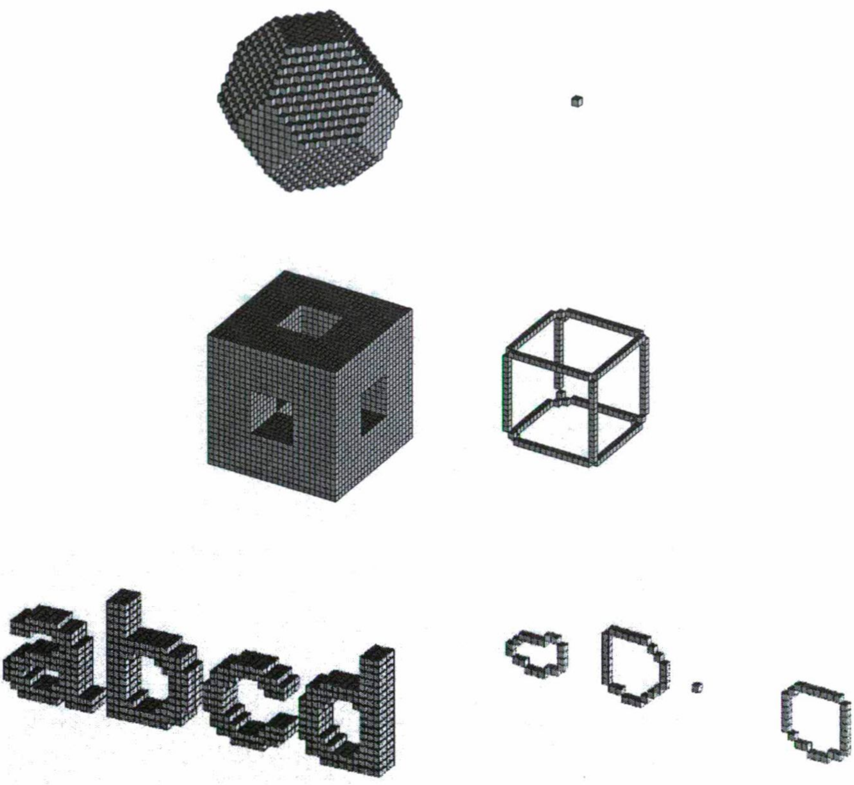

Figure 3: Three synthetic pictures (left) and their topological kernels produced by the proposed shrinking algorithm (right).

or cavities; e.g., a coffee cup or a cheese) is transformed into a closed "thin" curve or into a closed "thin" surface.

The proposed 6–subiteration shrinking algorithm is topology preserving for (26, 6) pictures. It is sufficient to prove that reduction operation given by the set of templates $\mathcal{T}_U$ is topology preserving. If the first subiteration of the algorithm is topology preserving, then the others are topology preserving, too, since the reflections and rotations of the deletion templates do not alter their topological properties. Therefore, the entire algorithm is topology preserving, since it is composed of topology preserving reductions.

In order to prove both conditions of Theorem 2, we classify the elements of templates and state some properties of the set of templates $\mathcal{T}_U$. The element in the very centre of a template is called *central* (marked by "*c*" in Fig. 2). A noncentral template element is called ***black*** if it is always black (marked by "●", "■", and "★" in Fig. 2). A noncentral template element is called *white* if it is always white (marked by "○" in Fig. 2). Any other noncentral template element which is not white and not black, is called *potentially black* (marked by "·" in Fig. 2). A black or a potentially black non-central template element is called *nonwhite*. A black point p is *deletable* if it can be deleted by at least one template in $\mathcal{T}_U$; p is *nondeletable* otherwise.

Observation 3. *Let us examine the configurations illustrated in Fig. 4.*

1. *If black point p in the configuration (a) is deletable then point q is white.*
2. *If both points p and q in the configuration (a) are black and point q is deletable then point p is nondeletable.*
3. *If black point p in the configuration (b) is deletable then at least one point marked q is black.*

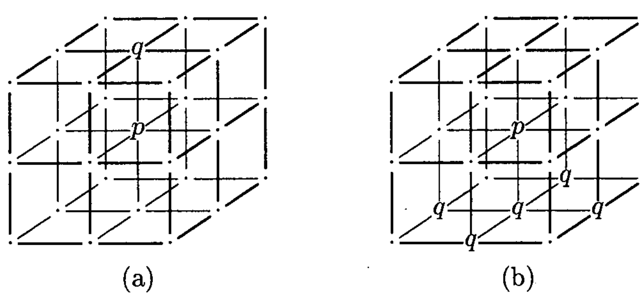

Figure 4: Configurations assigned to Observation 3.

The topological correctness of the first subiteration of the proposed algorithm is stated by the following theorem:

Theorem 4.
Reduction operation given by the set of templates $\mathcal{T}_U$ is topology preserving for (26, 6) *pictures.*

Proof.
It is to be proved that both conditions of Theorem 2 are satisfied.

It is easy to see that each template in $\mathcal{T}_U$ deletes only simple points of (26,6) pictures.

The first point is to verify that there exists a 26–path between any two nonwhite positions (condition 1 of Criterion 1). It is sufficient to show that any potentially black position is 26–adjacent to a black position and any black position is 26–adjacent to another black position. It is obvious by careful examination of the templates in $\mathcal{T}_U$.

To prove that condition 2 of Criterion 1 holds, it is sufficient to show for each template in $\mathcal{T}_U$ that:

1. there exists a white position 6–adjacent to the central position,

2. for any two white positions 6–adjacent to the central position p are 6–connected in the set of white positions 18–adjacent to p,
3. and for any potentially black position 6–adjacent to the central position p, there exists a 6–adjacent white 18–neighbour which is 6–adjacent to a white position 6–adjacent to p.

The first point holds by Observation 3/1. The remaining two points are obvious by a careful examination of the templates in $\mathcal{T}_U$.

We know that each deletable point p is simple. It can be stated that the value of any point coinciding with a potentially black template position does not alter the simplicity of p. We can state that the simplicity of a point p does not depend on the points that coincide with a template position marked "★" (see Fig. 2). In addition, black points that coincide with template positions marked "■" are nondeletable (by Observation 3/2). Therefore, it is sufficient to deal with deletable points that coincide with template positions marked "●". Note that base templates **T1** and **T5** (and their rotated versions) do not contain any positions marked "●". Therefore, only base templates **T2**, **T3**, and **T4** (and their rotated versions) are to be investigated.

Let us consider a black point p which can be deleted by template **T2**, **T3**, or **T4** (or their rotated versions) and let $Q \subseteq (N_{18}(p)\backslash\{p\}) \cap B$ be a nonempty set of deletable points. (If $Q = \emptyset$ then p remains simple after the deletion of Q, since each deletable point is simple.) Two cases are to be distinguished:

1. A black point $q \in Q$ does not coincide with the only element marked "●" of the template that may delete p.
   In this case, each point in Q must coincide with a potentially black position, therefore, the simplicity of p is not altered by the deletion of the set Q.
2. A black point $q \in Q$ coincides with the only element marked "●" of the template that may delete p.
   (a) Suppose that p is deleted by template **T2** (or one of its rotated versions) and let us consider the configurations in Fig. 5. We can state that point q may be deleted only by a rotated version of template **T4**. In this case, point r must be black and nondeletable (see 5/b), therefore, point p can be deleted by template **T1** (see 5/a) after the deletion of Q. Consequently, point p remains simple.
   (b) Suppose that p is deleted by template **T3** (or one of its rotated versions) and let us consider the configurations in Fig. 6. We can state that point q may be deleted only by a rotated version of template **T5**. In this case, point r must be black and nondeletable (see 6/b), therefore, point p can be deleted by template **T1** (see 6/a) after the deletion of Q. Consequently, point p remains simple.
   (c) Suppose that p is deleted by template **T4** (or one of its rotated versions) and let us consider the configurations in Fig. 7.

i. If point q may be deleted by one of the templates **T1**, **T2**, **T3**, **T5** (or by one of their rotated versions), or two rotated versions of template **T4** (where rotation angles are 0° and 180°) then all points r, s, and t are white (see 7/b), therefore, point p can be deleted by template **T1** (see 7/a) after the deletion of Q. Consequently, point p remains simple.

ii. If point q may be deleted by the rotated version of template **T4** (where the rotation angle is 90°) then $r = s = 0$ and $v = 1$ (see 7/b). If $t = 0$ then p can be deleted by template **T1**. If $t = 1$ then p can be deleted by template **T5** (and v is nondeletable, therefore, $v \notin Q$). Consequently, point p remains simple after the deletion of Q.

iii. If point q may be deleted by the rotated version of template **T4** (where the rotation angle is 270°) then $s = t = 0$ and $w = 1$ (see 7/b). If $r = 0$ then p can be deleted by template **T1**. If $r = 1$ then p can be deleted by a rotated version of template **T5** (and w is nondeletable, therefore, $w \notin Q$). Consequently, point p remains simple after the deletion of Q.

Therefore, condition 1 of Theorem 2 is satisfied.

Condition 2 of Theorem 2 can be seen with the help of Observation 3, too. Let us consider a unit lattice cube containing an upper set of four points $U = \{u_1, u_2, u_3, u_4\}$ and a lower set of four points $L = \{l_1, l_2, l_3, l_4\}$ (see Fig. 8). Let $C \subseteq U \cup L$ be a black component contained entirely in the unit lattice cube. If $C \cap U$ contains a deletable point then $C \cap L \neq \emptyset$ by Observation 3/3. It is easy to see, that any point in $C \cap L$ is nondeletable by Observation 3/3. Therefore, black component C cannot be deleted completely.

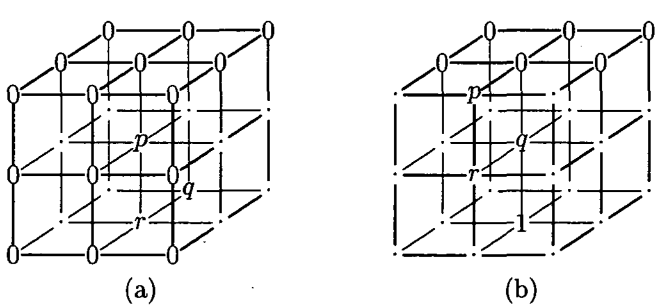

Figure 5: The $3 \times 3 \times 3$ configuration if point p is deleted by template **T2** (a) and the only configuration if point q is deletable (b).

5 Conclusions

Shrinking is fundamental operation in image processing. For example, shrinking can be used as a preprocessing step for counting distinct objects in a binary picture or to perform object labeling. An efficient 3D parallel shrinking algorithm for reducing a binary object to its topological kernel is presented. The applied directional strategy allows isotropic erosion, therefore, the residue is in the "middle" of an object. The topological correctness of the proposed algorithm is proved.

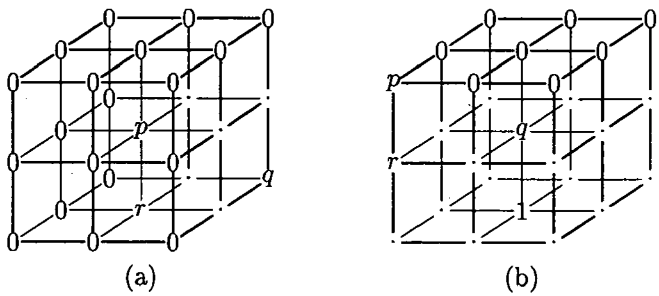

Figure 6: The $3 \times 3 \times 3$ configuration if point p is deleted by template **T3** (a) and the only configuration if point q is deletable (b).

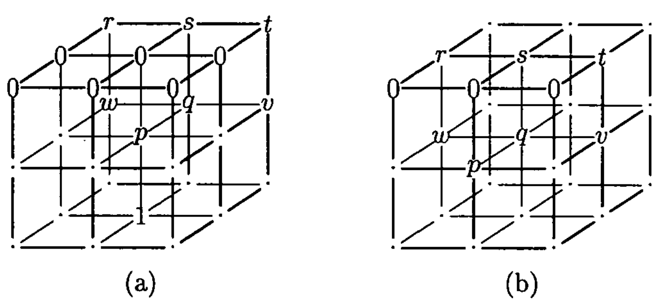

Figure 7: The $3 \times 3 \times 3$ configuration if point p is deleted by template **T4** (a) and the corresponding configuration of $N_{26}q$ (b).

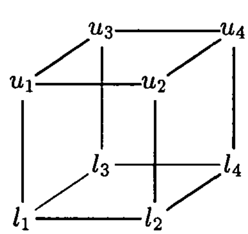

Figure 8: A unit lattice cube divided into two sets of four points $U = \{u_1, u_2, u_3, u_4\}$ and $L = \{l_1, l_2, l_3, l_4\}$.

Acknowledgment

The author is grateful to László G. Nyúl for his valuable suggestions. This work was supported by the FKFP 0908/1997 Grant.

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